Homework #4.5

This homework assignment asks you to solve the biharmonic equation for an axisymmetric problem, and then to interpret the meaning of the solution for the stress function $\phi = T \ln r$.

The Laplace equation is one of the most common equations in physics. In Cartesian (x,y) coordinates it is

(1) $\nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = (\partial/\partial x^2 + \partial/\partial y^2)\ \phi = 0.$

In cylindrical (polar) coordinates it is

(2) $\nabla^2\phi = (\partial^2/\partial r^2 + (1/r)\ (\partial/\partial r) + (1/r^2)\ (\partial^2/\partial\theta^2)\ )\ \phi = 0.$

This simplifies if ϕ is not a function of θ (e.g., if the problem is axisymmetric):

(3) $\nabla^2\phi = (d^2/dr^2 + (1/r)\ (d/dr)\ )\ \phi = 0.$

The plus sign in (3) can be eliminated by recasting the equation in a different form

(4) $\nabla^2\phi = (1/r)\ [d/dr\ (r\ d\phi/dr)\ ] = 0.$

The proof that (3) and (4) are equivalent is straightforward. First, take the derivative of the product (r dϕ/dr) in (4) with respect to r

(5) $(d/dr\ (r\ d\phi/dr)\ ) = (r)\ (d^2\phi/dr^2) + (d\phi/dr)\ (dr/dr) = (r)\ (d^2\phi/dr^2) + (d\phi/dr).$

Now multiply (5) by 1/r

(6) $(1/r)\ (d/dr\ (r\ d\phi/dr)\ ) = (d^2\phi/dr^2) + (1/r)\ (d\phi/dr) = (d^2/dr^2 + (1/r)\ (d/dr)\ )\ \phi.$

The general expression for axisymmetric problems described by (6) can be found in straightforward fashion by integration from the "outside in" (i.e., from left to right). First multiply both sides of (6) by r:

(7) $[d/dr\ (r\ d\phi/dr)\ ] = 0.$

Now integrate both sides with respect to r

(8) $(r\ d\phi/dr) = C_1.$

where C_1 is a constant of integration. Now divide both sides by r

(9) $d\phi/dr = C_1/r.$

Now integrate both sides with respect to r:

(10) $\phi = C_1 \ln r + C_2,$

where ln is the natural log and C_2 is another constant of integration. <u>Note that the form of equation (4) lends itself to this kind of solution procedure, but the form of (3) does not</u>.

A) Using the approach here, obtain the general solution of the <u>biharmonic</u> equation ($\nabla^4\phi = 0$) for the case where ϕ is just a function of r (ϕ does not depend on θ). Show all your work. **(9 pts)**
B) Solve for the stresses in a polar reference frame if $\phi = Ta^2 \ln r$ using equations 10.27, 10.29, and 10.30 in the course notes, where *T* and *a* are constants with dimensions of stress and length, respectively. Show all your work. **(6 pts)**
C) Draw a labeled diagram with an inner boundary at $r = a$, and show the normal and shear tractions that act on it for the solution in (B); draw the whole inner boundary. **(3 pts)**
D) Imagine that the body is an infinitely large plate. Based on your solutions in (B), what are the tractions at the "outer boundary" of the infinitely large plate? **(1 pt)**
E) Based on your solutions for parts (B) - (D), describe in words what problem is solved by the stress function $\phi = Ta^2 \ln r$. You can consider T to be positive or negative. **(3 pts)**