Homework #3.5

This homework assignment deals with equilibrium, compatibility, reference frame changes, and superposition. Read the full assignment before you start to work on it (except for drawing any labeled pictures you need). You will find a total stress field by superposing two stress fields, one for an ambient (i.e., original) stress field, and the other for a stress perturbation.

A relatively recent article in a technical journal addressed the 2-D, plane strain elastic solution for stresses in the Earth near a long straight valley with a semi-circular cross section of radius R. Here we consider the valley in cross section, with the positive x-axis horizontal and at the Earth's surface, the positive y-direction vertical and pointing up, and the positive z-direction parallel to the axis of the valley and out of the page. The ambient stress state (σ^0) in the plane of the cross section (i.e., the stress state before the valley existed) is hydrostatic <u>owing to the vertical gravitational body force;</u> tensile stresses as positive:

(1) $\sigma_{xx}^{\ 0} = \sigma_{yy}^{\ 0} = \rho g y$
(2) $\sigma_{xy}^{\ 0} = \sigma_{yx}^{\ 0} = 0$

(A)   Show that these conditions obey the equations of equilibrium for plane strain.
(B)   Show that these conditions obey the compatibility equation as expressed in terms of stresses for plane strain (eq. 7.15 from notes).
(C)   Show that these equations yield a traction-free surface for the surface of the Earth, which is the plane y = 0.
(D)   Recast the stresses in polar form (i.e., solve for $\sigma_{rr}^{\ 0}$, $\sigma_{\theta\theta}^{\ 0}$, and $\sigma_{r\theta}^{\ 0}$).

The stress perturbation due to the valley (σ^1) is found using the Flamant solution, which we will visit again in the coming weeks. The Flamant solution is for the stresses in a half-space with no body forces but with a vertical tension F at the surface applied at a point at the origin:

(3) $\sigma_{rr}^{\ 1} = -2F\ \sin\theta/\pi r$ (Note that F must have dimension of stress/unit depth)
(4) $\sigma_{\theta\theta}^{\ 1} = \sigma_{r\theta}^{\ 1} = 0.$ (Note that $\theta = 0$ along the x-axis)

(E)   Recast the stresses in Cartesian form (i.e., solve for $\sigma_{xx}^{\ 1}$, $\sigma_{xy}^{\ 1}$, $\sigma_{yx}^{\ 1}$, and $\sigma_{yy}^{\ 1}$ as a function of x and y).
(F)   ~~Show that these conditions obey the equations of equilibrium for plane strain.~~
(G)   ~~Show that these conditions obey the compatibility equation as expressed in terms of stresses for plane strain (eq. 7.15 from notes).~~
(H)   Show that these equations yield a traction-free surface for the surface of the Earth, which is the plane y = 0, except at the origin.

(I)   Now superpose these two stress states to solve for the total 2-D stress state ($\sigma^0 + \sigma^1 = \sigma^T$) around a valley with a semi-circular cross section, <u>solving for F such that at the surface of the semi-circular valley walls (of radius R) the total tractions are zero</u>. You should use your judgment as to which reference frame you should use to express the stresses such that you can easily see that the tractions on the semi-circular valley walls the tractions are zero. In this superposition we "throw away" the part of the solution that is not in the Earth (i.e., the part of the solution where the radial distance from the origin is less than R).