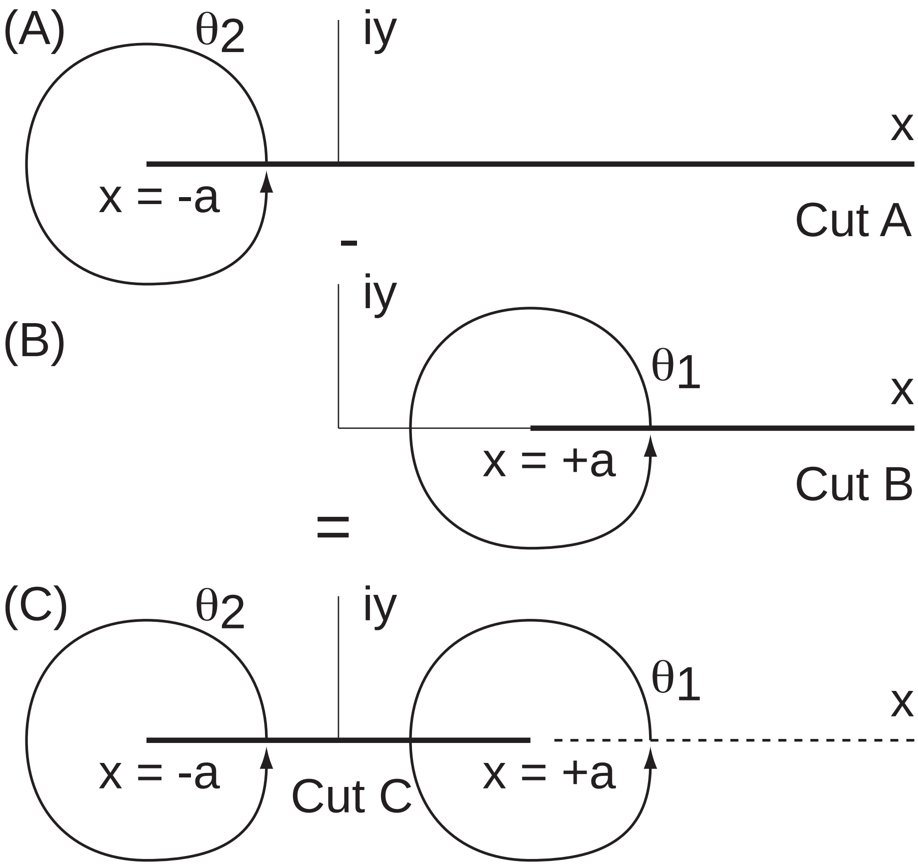
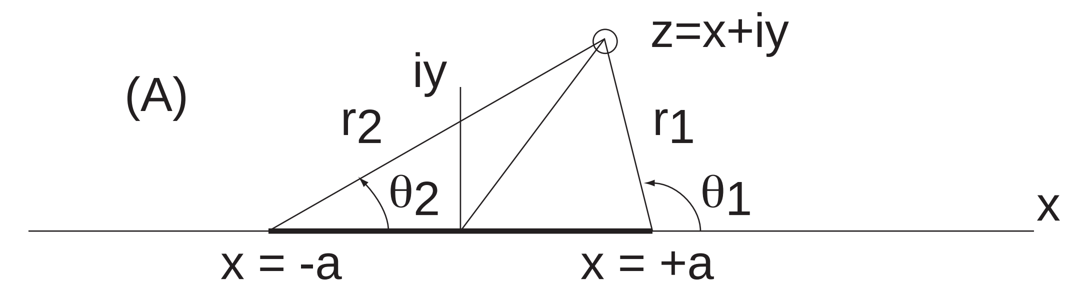
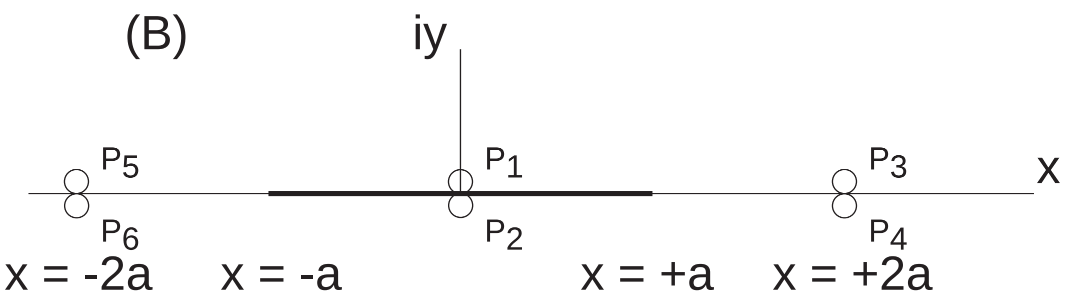
Branch Cuts and Interpretation of $\sqrt{z^2 - a^2}$

This term is used to represent a discontinuity in the displacement field across a cut. A useful way to remember how to interpret this comes from recalling how we constructed a displacement discontinuity:



In (A), the cut extends an infinite distance to the right, and $\theta 2$ is not allowed to extend continuously across the cut. On the upper side of the cut $\theta 2 = 0$, and on the lower side $\theta 2 = 2\pi$. The same kind of argument applies in (B) to $\theta 1$. When we subtract cut B from Cut A to obtain Cut C, we do not change the allowable ranges of $\theta 1$ and $\theta 2$: both can range from 0 to 2π. Allowing $\theta 1$ to range from 0 to 2π might seem strange in (C) because the angle extends continuously across the cut, but considering how Cut C was created, $\theta 1$ has to extend from 0 to 2π. These restrictions in θ occur at *branch cuts*. By allowing $\theta 1$ and $\theta 2$ to range from 0 to 2π in (C), the angles are





Consider how $\sqrt{z^2 - a^2} = \sqrt{z+a}\sqrt{z-a}$ is evaluated at the sets of points above

Point P_1 (x = 0, y = 0+).

$$\sqrt{z+a}\sqrt{z-a} = \sqrt{ae^{i0}}\sqrt{ae^{i\pi}} = a\sqrt{e^{i0}e^{i\pi}} = a\sqrt{e^{i\pi}} = ae^{i\pi/2} = ia$$

Point P_2 (x = 0, y = 0-).

$$\sqrt{z+a}\sqrt{z-a} = \sqrt{ae^{i2\pi}}\sqrt{ae^{i\pi}} = a\sqrt{e^{i2\pi}e^{i\pi}} = a\sqrt{e^{i3\pi}} = ae^{i3\pi/2} = -ia$$

Point P_3 (x = 2a, y = 0+).

$$\sqrt{z+a}\sqrt{z-a} = \sqrt{3ae^{i0}}\sqrt{ae^{i0}} = \sqrt{3}a\sqrt{e^{i0}e^{i0}} = \sqrt{3}a\sqrt{e^{i0}} = \sqrt{3}ae^{i0/2} = \sqrt{3}a$$

Point P_4 (x = 2a, y = 0-).

$$\sqrt{z+a}\sqrt{z-a} = \sqrt{3ae^{i2\pi}}\sqrt{ae^{i2\pi}} = \sqrt{3}a\sqrt{e^{i2\pi}e^{i2\pi}} = \sqrt{3}a\sqrt{e^{i4\pi}} = \sqrt{3}ae^{i2\pi} = \sqrt{3}a$$

Point P_5 (x = -2a, y = 0+).

$$\sqrt{z+a}\sqrt{z-a} = \sqrt{ae^{i\pi}}\sqrt{3ae^{i\pi}} = \sqrt{3}a\sqrt{e^{i\pi}e^{i\pi}} = \sqrt{3}a\sqrt{e^{i2\pi}} = \sqrt{3}ae^{i\pi} = -\sqrt{3}a$$

Point P_6 (x = -2a, y = 0-).

$$\sqrt{z+a}\sqrt{z-a} = \sqrt{ae^{i\pi}}\sqrt{3ae^{i\pi}} = \sqrt{3}a\sqrt{e^{i\pi}e^{i\pi}} = \sqrt{3}a\sqrt{e^{i2\pi}} = \sqrt{3}ae^{i\pi} = -\sqrt{3}a$$

The only discontinuity in the function is for points on opposing sides of the cut, as desired.