Homework #8

The following relations allow the stresses and displacements to be found around fractures:

Mode I: stresses and plane strain displacements	For mode II: stresses and plane strain displacements	For mode III: stresses and anti-plane strain displacements
$\sigma_{xx} = \operatorname{Re} Z - y \operatorname{Im} Z'$	$\sigma_{xx} = 2\operatorname{Im} Z - y \operatorname{Re} Z'$	$\sigma_{yz} = \operatorname{Re} Z$ $\sigma_{xz} = \operatorname{Im} Z$
$\sigma_{yy} = \operatorname{Re} Z + y \operatorname{Im} Z'$	$\sigma_{yy} = -y \operatorname{Re} Z'$	
$\sigma_{xy} = -y \operatorname{Re} Z'$	$\sigma_{xy} = \operatorname{Re} Z - y \operatorname{Im} Z'$	
$u_x = \left[(1-2\nu)\operatorname{Re}\bar{Z} - y \operatorname{Im} Z\right]/2G$	$u_x = \left[2(1-\nu)\operatorname{Im}\bar{Z} + y \operatorname{Re} Z\right]/2G$	$u_z = \operatorname{Im}\bar{Z}/G$
$u_y = \left[2(1-\nu)\operatorname{Im}\bar{Z} - y \operatorname{Re} Z\right]/2G$	$u_y = \left[-(1-2\nu)\operatorname{Re}\bar{Z} - y \operatorname{Im} Z\right]/2G$	

The correct stress functions need to be found to match the desired boundary conditions.

The stress function below and its derivatives Z and Z' should yield the stresses and displacements for mode I, mode II, and mode III cracks that extend along the x-axis from x= -a to x = +a, with the boundary conditions being a constant traction along the walls of the fracture and a special uniform stress state at an infinite distance from the fracture.

$$\begin{Bmatrix} \bar{Z}_I(z) \\ \bar{Z}_{II}(z) \\ \bar{Z}_{III}(z) \end{Bmatrix} = \begin{Bmatrix} \sigma \\ \tau \\ \tau_\ell \end{Bmatrix} \left[\sqrt{z^2 - a^2} - z\right] = \begin{Bmatrix} \sigma \\ \tau \\ \tau_\ell \end{Bmatrix} \left[\sqrt{z+a}\sqrt{z-a} - z\right]$$

1 Find the derivative of the above function with respect to z; call this derivative Z.
SHOW YOUR WORK! **(4 pts)**

$$\begin{Bmatrix} Z_I(z) \\ Z_{II}(z) \\ Z_{III}(z) \end{Bmatrix} =$$

2 Find the derivative of the function Z with respect to z; call this derivative Z'.
SHOW YOUR WORK! **(4 pts)**

$$\begin{Bmatrix} Z_I(z) \\ Z_{II}(z) \\ Z_{III}(z) \end{Bmatrix} =$$

3 Solve below for the radical term on page 1 along the crack, where z = x, casting your answer such that the argument of the square root is positive. For example:

$$\sqrt{-4} = \sqrt{(-1)(4)} = i\sqrt{2}$$

Note that the positive square root is used, not the negative square root, and the result is not written as a fraction with i or –i in the denominator. **(2 pts)**

4 Solve for the following terms at the "top" of the crack (i.e., z=x for –a<x<+a, y = 0+).

Mode I **(11 pts)**	Mode II **(11 pts)**	Mode III **(9 pts)**
$\mathrm{Re}\,\overline{\overline{Z}}$	$\mathrm{Re}\,\overline{\overline{Z}}$	$\mathrm{Re}\,\overline{\overline{Z}}$
$\mathrm{Im}\,\overline{\overline{Z}}$	$\mathrm{Im}\,\overline{\overline{Z}}$	$\mathrm{Im}\,\overline{\overline{Z}}$
ReZ	ReZ	ReZ
ImZ	ImZ	ImZ
ReZ'	ReZ'	ReZ'
ImZ'	ImZ'	ImZ'
$\sigma_{xx} = \mathrm{Re}\,Z - y\,\mathrm{Im}\,Z'$	$\sigma_{xx} = 2\,\mathrm{Im}\,Z - y\,\mathrm{Re}\,Z'$	$\sigma_{yz} = \mathrm{Re}\,Z$
$\sigma_{yy} = \mathrm{Re}\,Z + y\,\mathrm{Im}\,Z'$	$\sigma_{yy} = -y\,\mathrm{Re}\,Z'$	$\sigma_{xz} = \mathrm{Im}\,Z$
$\sigma_{xy} = -y\,\mathrm{Re}\,Z'$	$\sigma_{xy} = \mathrm{Re}\,Z - y\,\mathrm{Im}\,Z'$	
$u_x = \left[(1-2\nu)\mathrm{Re}\,\overline{Z} - y\,\mathrm{Im}\,Z\right]/2G$	$u_x = \left[2(1-\nu)\mathrm{Im}\,\overline{Z} + y\,\mathrm{Re}\,Z\right]/2G$	$u_z = \mathrm{Im}\,\overline{Z}\,/G$
$u_y = \left[2(1-\nu)\mathrm{Im}\,\overline{\overline{Z}} - y\,\mathrm{Re}\,Z\right]/2G$	$u_y = \left[-(1-2\nu)\mathrm{Re}\,\overline{\overline{Z}} - y\,\mathrm{Im}\,Z\right]/2G$	

5 Solve for the following terms as z goes to infinity

Mode I **(11 pts)**	Mode II **(11 pts)**	Mode III **(9 pts)**
$\mathrm{Re}\,\overline{Z}$	$\mathrm{Re}\,\overline{Z}$	$\mathrm{Re}\,\overline{Z}$
$\mathrm{Im}\,\overline{Z}$	$\mathrm{Im}\,\overline{Z}$	$\mathrm{Im}\,\overline{Z}$
ReZ	ReZ	ReZ
ImZ	ImZ	ImZ
ReZ'	ReZ'	ReZ'
ImZ'	ImZ'	ImZ'
$\sigma_{xx} = \mathrm{Re}\,Z - y\,\mathrm{Im}\,Z'$	$\sigma_{xx} = 2\,\mathrm{Im}\,Z - y\,\mathrm{Re}\,Z'$	$\sigma_{yz} = \mathrm{Re}\,Z$
$\sigma_{yy} = \mathrm{Re}\,Z + y\,\mathrm{Im}\,Z'$	$\sigma_{yy} = -y\,\mathrm{Re}\,Z'$	$\sigma_{xz} = \mathrm{Im}\,Z$
$\sigma_{xy} = -y\,\mathrm{Re}\,Z'$	$\sigma_{xy} = \mathrm{Re}\,Z - y\,\mathrm{Im}\,Z'$	
$u_x = \left[(1-2\nu)\mathrm{Re}\,\overline{Z} - y\,\mathrm{Im}\,Z\right]/2G$	$u_x = \left[2(1-\nu)\mathrm{Im}\,\overline{Z} + y\,\mathrm{Re}\,Z\right]/2G$	$u_z = \mathrm{Im}\,\overline{Z}\,/G$
$u_y = \left[2(1-\nu)\mathrm{Im}\,\overline{Z} - y\,\mathrm{Re}\,Z\right]/2G$	$u_y = \left[-(1-2\nu)\mathrm{Re}\,\overline{Z} - y\,\mathrm{Im}\,Z\right]/2G$	

6 On the figure on the following page, show the tractions (not the stresses) acting on all the sides of all three boxes for each figure – this is intended to help you think about the meaning of your solutions and what the associated boundary conditions are. The skinny horizontal rectangle represents the crack. **Show the non-zero tractions in the direction in which they act** – for example, if σxx is negative, then draw arrows to show compression (i.e., the arrows should point towards the center of the box), and if σyx is negative, then draw an arrow on the positive y-face that points in the negative x-direction. The tractions should be consistent with the crack-wall displacements that you calculated. The uppermost box should be used to represent the conditions at an infinite distance from a crack. Note that boxes are on the top (y=0+)and bottom (y=0-) sides of the crack near the right end of the crack, and that your solutions above are just for the top sides of the cracks (y=0+), so you will have to think a bit as to how to handle the box on the bottom side of the crack (y=0-). Interpreting the stress solutions to obtain the correct tractions is somewhat challenging.

For the mode I case, indicate the direction of the normal and shear non-zero tractions in the plane of the page. There should be a maximum of two arrows on each side of each box. **(24 pts).**

For the mode II case, indicate the direction of the normal and shear tractions in the plane of the page. There should be a maximum of two arrows on each side of each box. **(24 pts).**

For the mode III case, indicate the direction of the normal and shear tractions that act in and out of the page by the following symbols, respectively:

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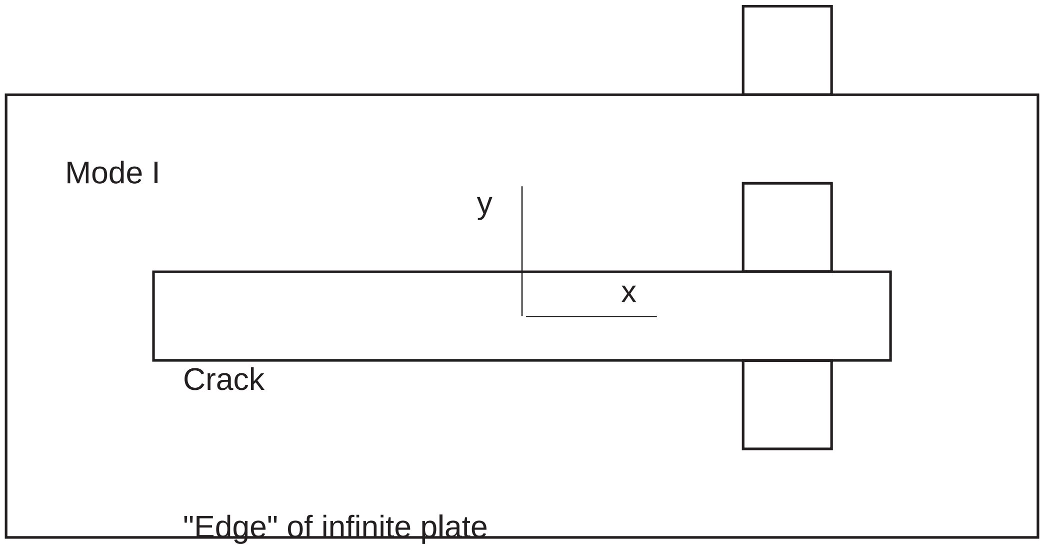
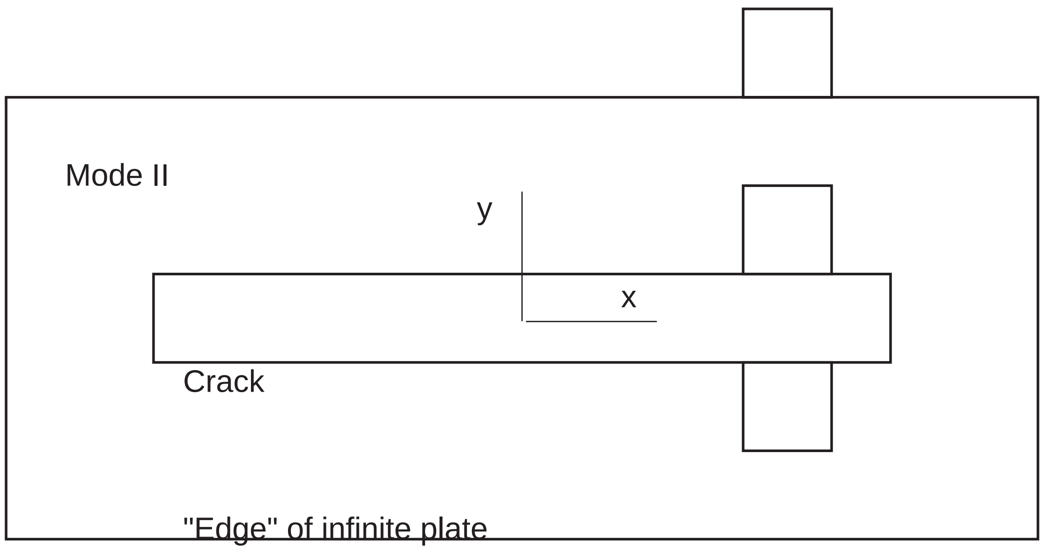
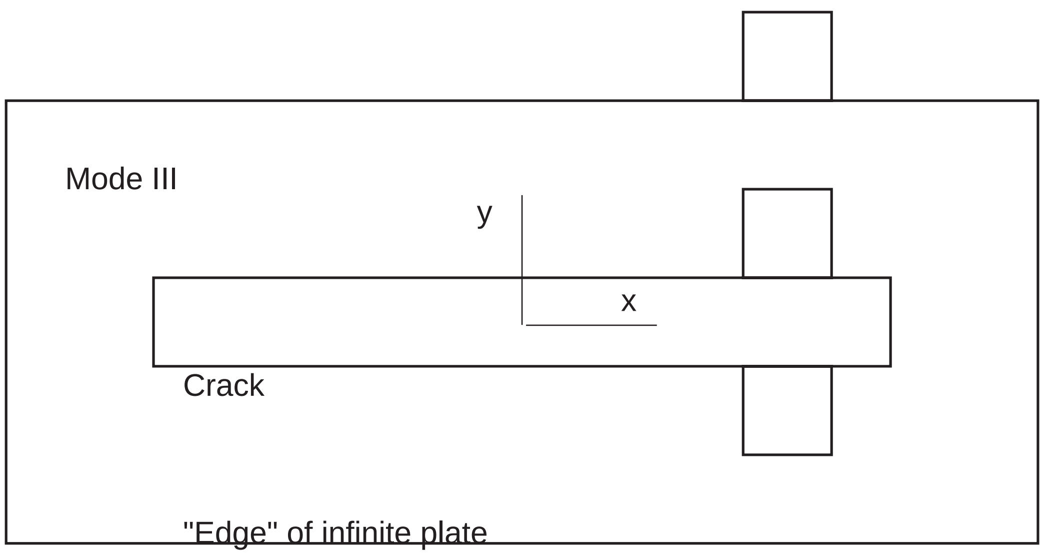
There should be a maximum of one arrow on each side of each box. **(12 pts).**


Mode I
y
x
Crack
"Edge" of infinite plate


Mode II
y
x
Crack
"Edge" of infinite plate


Mode III
y
x
Crack
"Edge" of infinite plate

7 Use your solutions above to complete the Matlab script below, and run the script to get the displacement and stress fields around cracks for modes I, II, and III . The 15 lines to complete are marked by question marks (????????). Submit your completed Matlab script as well as the plots it produces.

For the plots, use a driving stress S=1, a shear modulus G = 1, and a Poisson's ratio PR = 0.25. For the observation point array, let [x,y] = meshgrid(-1.95:0.1:1.95).

It is absolutely essential that the square root term be evaluated in the following way:

$$\sqrt{z^2 - a^2} = \sqrt{z+a}\sqrt{z-a}$$

so I have inserted a line in the code that does this:

```
sqrtz2_a2 = sqrt(z+a).*sqrt(z-a);
```

After you obtain the correct solutions might want to see what happens when the term is evaluated in the following manner:

```
sqrtz2_a2 = sqrt(z.*z-a.*a);
```

(15 pts)

```
function [ux1,uy1,sxx1,sxy1,syy1,ux2,uy2,sxx2,sxy2,syy2,...
     uz3,syz3,sxz3] = GG711_HW8(x,y,S,G,PR)
% Calculates the stresses and displacements at grid points
% about cracks of mode I, mode II, and mode III
% using coplex stress functions, where the cracks
% experience a uniform driving stress S (see Pollard and
% Segall, 1987), and no far-field stress exists.
% The crack extends along the x-axis
% from x = -a to x = +a.
% ux = displacement in the z-direction
% uy = displacement in the y-direction
% sxx = sigma xx
% sxy = sigma xy
% syy = sigma yy
% Parameters G, k, and PR are elastic parameters (see Barber, 1992)
% S is the driving stress.
% x,y = coordinates of observation gridpoints,
%  [x,y] = meshgrid(-1.95:0.1:1.95);
%  [ux1,uy1,sxx1,sxy1,syy1,ux2,uy2,sxx2,sxy2,syy2,...
%      uz3,syz3,sxz3] = tada_2000_5_1a(x,y,S,G,PR);
% Last revised on 3/12/03%

z = x + i*y;
a = 1;             % Unit half-length crack
a2 = a.*a;
z2 = z.*z;
sqrtz2_a2 = sqrt(z+a).*sqrt(z-a);
% Calculate stress functions and derivative
Zbar   =  S*( sqrtz2_a2  - z );
Z      =  ?????;
Zprime  =  ?????;

% Calculate the stresses for mode I
sxx1 = ?????;
syy1 = ?????;
sxy1 = ?????;
% Calculate the plane strain displacements for mode I
ux1 = ?????;
uy1 = ?????;

% Calculate the stresses for mode II
sxx2 = ?????;
syy2 = ?????;
sxy2 = ?????;
% Calculate the plane strain displacements for mode II
ux2 = ?????;
uy2 = ?????;
```

```
% Calculate the anti-plane strain stresses for mode III
syz3 = ?????;
sxz3 = ?????;
% Calculate the anti-plane strain displacements for mode III
uz3 = ?????;


% Plots
figure(1);     quiver(x,y,ux1,uy1);
   xlabel('x');  ylabel('y');    axis('equal');   title ('Displacement');
figure(2);     v1 = -0.8:0.2:0.8;   c1 = contour(x,y,sxx1,v1);  clabel(c1);
   xlabel('x');   ylabel('y');    axis('equal');   title('sxx')
figure(3);     v2 = -0.3:0.1:0.3;   c2 = contour(x,y,sxy1,v2);  clabel(c2);
   xlabel('x');   ylabel('y');    axis('equal');   title('sxy')
figure(4);     v3 = -1.0:0.2:1.0;   c3 = contour(x,y,syy1,v3);  clabel(c3);
   xlabel('x');   ylabel('y');    axis('equal');   title('syy')

figure(5);     quiver(x,y,ux2,uy2);
   xlabel('x');  ylabel('y');    axis('equal');   title ('Displacement');
figure(6);     v1 = -0.6:0.2:0.6;   c1 = contour(x,y,sxx2,v1);  clabel(c1);
   xlabel('x');   ylabel('y');    axis('equal');   title('sxx')
figure(7);     v2 = -0.3:0.1:0.3;   c2 = contour(x,y,sxy2,v2);  clabel(c2);
   xlabel('x');   ylabel('y');    axis('equal');   title('sxy')
figure(8);     v3 = -0.3:0.1:0.3;  c3 = contour(x,y,syy2,v3);  clabel(c3);
   xlabel('x');   ylabel('y');    axis('equal');   title('syy')

figure(9);     v1 = -0.6:0.2:0.6;     c1 = contour(x,y,uz3);  clabel(c1);
   xlabel('x');  ylabel('y');    axis('equal');   title ('Displacement');
figure(10);    v2 = -0.6:0.2:0.6;   c2 = contour(x,y,syz3,v2);  clabel(c1);
   xlabel('x');   ylabel('y');    axis('equal');   title('sxx')
figure(11);    v3 = -0.4:0.1:0.4;   c3 = contour(x,y,sxz3,v3);  clabel(c2);
   xlabel('x');   ylabel('y');    axis('equal');   title('sxy')
```