THERMODYNAMICS OF FRACTURE GROWTH (18)

I Main topics

A Griffith energy balance and the fracture energy release rate (G)

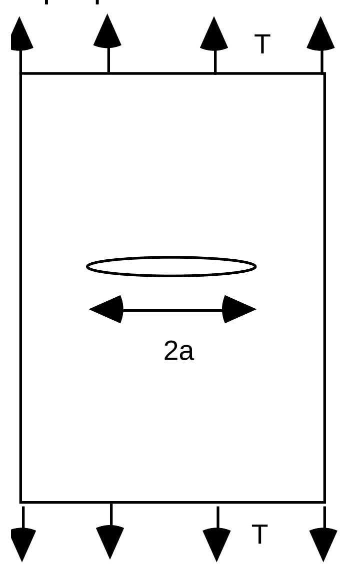
B Energy partitioning in a cracked solid & independence of G on loading conditions

C Fracture growth in terms of the near-tip stress field

D Fracture propagation criteria

II Griffith energy balance and the fracture energy release rate (G)

Consider a plate with a crack in it (we do not ask how the crack got there) under uniaxial tension perpendicular to the crack.



Fracture growth can occur if sufficient energy is available, and if the growth process causes the total energy in the body to remain the same or to decrease. Consider the energy distribution in a plate (per unit thickness).

Work done by loading system =

Internal Strain Energy (U) + Kinetic energy (K)+ Surface Energy (U_S)

$$W = U + K + U_s \tag{18.1}$$

The internal strain energy includes elastic and nonelastic deformation. For (a) a perfectly elastic solid (i.e., all the internal strain energy is stored elastically), and (b) cases where the kinetic energy is small, then

$$W = (U_{elastic}) + U_s \quad (18.2)$$

Now we define some terms. First, we define the total free energy of the whole system, including the loading system, as follows:

$$U_{total} \equiv (U_{elastic}) + U_s - W = (U_{elastic} - W) + U_s \quad (18.3)$$

Because the plate already had a crack to start with, $U_s>0$ and hence $U_{total}>0$. Second, we define the potential mechanical energy Π of an elastic body as:

$$\Pi \equiv U_e - W = U_{mechanical} \quad (18.4)$$

The -W term reflects the decrease in energy of the loading system as the elastic body deforms and stores energy. So from (18.3) and (18.4):

$$U_{total} = \Pi + U_s \quad (18.5)$$

Under equilibrium conditions (no change in the total free energy), the rate of change of total free energy per unit area of crack front advance is:

$$\frac{dU_{total}}{da} = \frac{d(\Pi + U_s)}{da} = 0 \quad (18.6)$$

or by separating the Π and U_s terms:

$$\frac{-d\Pi}{da} = \frac{dU_s}{da} \quad (18.7)$$

The term da is an incremental area of crack advance because the crack growth increment has length *and* depth. We define the energy release rate (per unit area of crack advance) as:

$$G \equiv \frac{-d\Pi}{da} = \frac{d(W_L - U_e)}{da} \quad (18.8)$$

Surface energy is gained as a crack propagates, so the mechanical potential energy must decrease as a crack propagates under equilibrium conditions. The energy required for a crack to increase its length by an amount da (i.e., to produce two faces of length da) in a brittle material is twice the free surface energy (γ) needed for a single surface

$$G_c da = 2\gamma \ \ da \quad . \qquad (18.9)$$

If $G_{crit} = 2\gamma$, the materials are brittle. If $G_{crit} >> 2\gamma$, the materials are tough. To account for non-elastic deformation as well as elastic deformation, we say $G_{crit} = 2\Gamma$, where Γ is the fracture toughness.

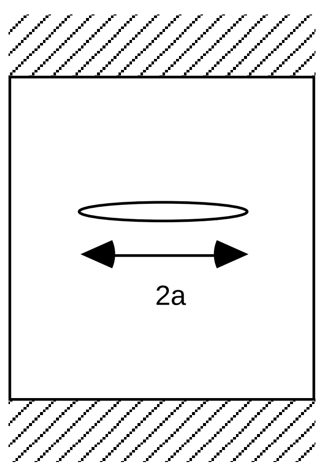
III Energy partitioning in a cracked solid & independence of G on loading conditions

Consider crack growth under two loading conditions on the plate boundary:

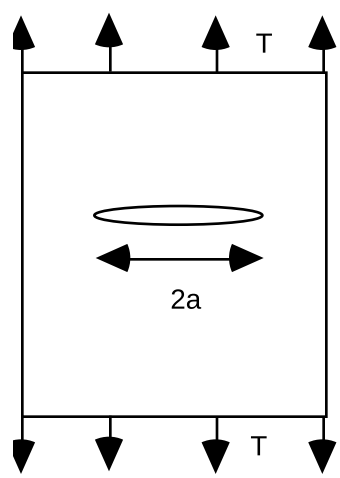
Fixed Grips: no displacement by external forces, so no work is done; fracture energy comes from strain energy

Constant Stress: driving stresses are held constant, so as boundary of cracked body is displaced, the external forces do work

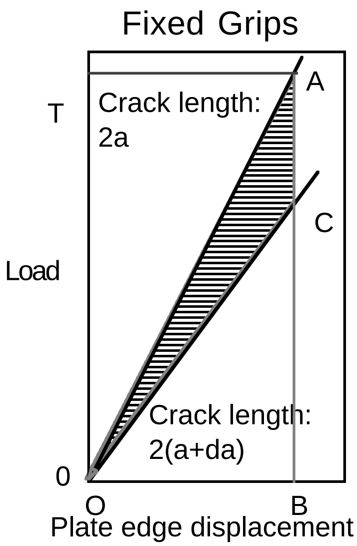
Fixed Grips

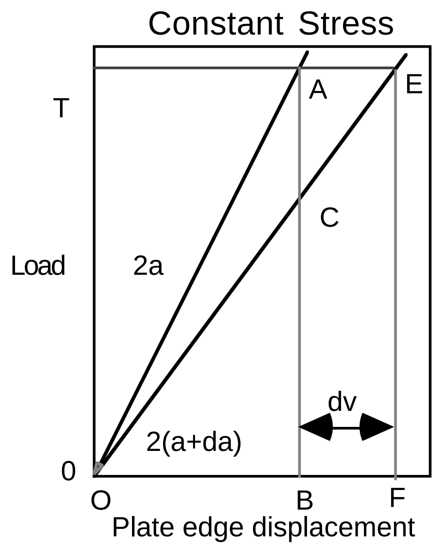


Constant Stress



Now suppose the crack grows by an amount 2(da)



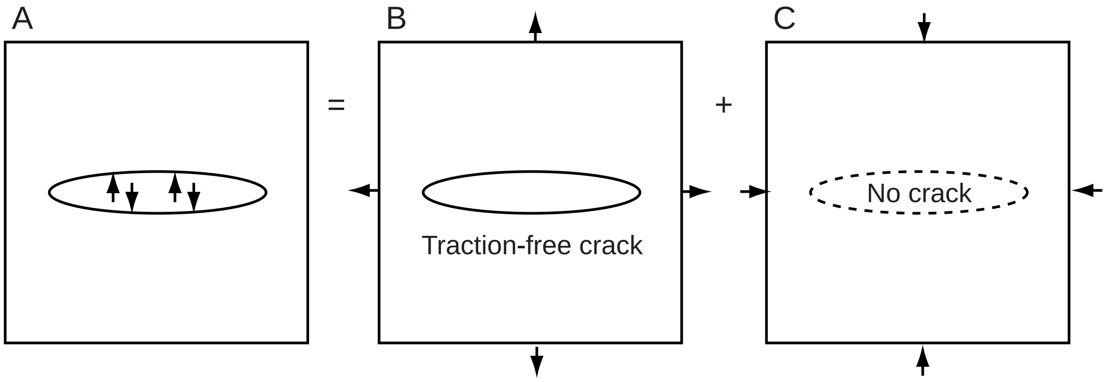


		Fixed Grips	Constant Stress
U_{e0}	Initial elastic energy	+OAB	+OAB
ΔW	Work during fracture	0	(T)(dv) = +AEFB
U_{efinal}	Final elastic energy	+OCB	+OEF
$\Delta U_e = U_{efinal} - U_{e0}$	Elastic energy change	-OAC	OEF-OAB = +OAE =AEFB/2
$\Delta W - \Delta U_e$	Energy used in fracture	+OAC	AEFB-OAE = OAE ≈ OAC

For infinitesimal values of dv, the area of triangle OCE in the righthand diagram becomes negligible, and the area of triangle OAC approaches that of OAC. So the energy used in a small increment of fracture growth is the same under the two loading conditions (i.e., G is independent of the loading). Under a constant load, half the work during fracture goes into elastic strain, and half goes into fracture, so $W = 2\Delta U_E$. Under a constant displacement, all the fracture energy comes from elastic strain energy stored in the body (it has to; there is no other energy source!). The appendix derives the same result algebraically.

IV Fracture growth in terms of the near-tip stress field

A Work to open a crack



The strain energy due to the opening of the crack in illustration B is also captured by the strain energy of the crack in A (the strain energy in C has nothing to do with a crack: no crack is present)

$$W = \int F \bullet d = \int_{-a}^{+a} (\sigma dx) \left\{ \left(\frac{4\sigma}{E}\right) \left(a^2 - x^2\right)^{1/2} \right\} = \sigma \int_{-a}^{a} \left(\frac{4\sigma}{E}\right) \left(a^2 - x^2\right)^{1/2} dx \qquad (18.10)$$

The opening profile above is for plane stress. Also, both the upper and lower surfaces of the crack are displaced so F•d = F•d_{u+} + F•d_{u-}. The maximum opening of *2b* is at x=0: $\Delta u_y = 4a\sigma/E$.

The rightmost integral is simply the area of the ellipse that the slit-like crack opens into. The length of the ellipse (twice the semi-major axis) is 2a. The height of the ellipse (twice the semi-minor axis) is 2b, which equals the maximum opening.

$$2b = 4a\frac{\sigma}{E}. \qquad (18.11)$$

The area A of the ellipse, and hence the value of the integral, is:

$$A = \pi ab = \pi a\left(2a\frac{\sigma}{E}\right) = 2\pi a^2\frac{\sigma}{E}. \qquad (18.12)$$

The work done, which in this case will equal the elastic strain energy, is obtained from (18.10), or by multiplying (18.12) by σ:

$$W = \sigma A = \frac{2\pi a^2 \sigma^2}{E} \tag{18.13}$$

If the crack does not lengthen, all the work goes into elastic strain energy. However, for a crack growing under constant far-field stresses, half this work will go into elastic energy and half this work will go into crack surface energy.

$$U_e = U_s = \frac{\pi a^2 \sigma^2}{E} \tag{18.14}$$

B Equilibrium length of a crack

Under equilibrium conditions, the total free energy is at a (local) minimum, so

$$\frac{dU_{total}}{da} = \frac{d(\Pi + U_s)}{da} = 0 \tag{18.15}$$

$$\frac{d\Pi}{da} = \frac{-dU_s}{da} \tag{18.16}$$

Using expression (18.14) for U_S, and assuming all the energy devoted to fracture goes into creating new surface area on <u>both</u> ends of the crack of length 2a:

$$\frac{d\left(\frac{-\pi a^2 \sigma^2}{E}\right)}{da} = \frac{-d(4\gamma a)}{da} \tag{18.17}$$

$$\frac{-2\pi a \sigma^2}{E} = -4\gamma \tag{18.18}$$

Solving for the equilibrium length at a given load

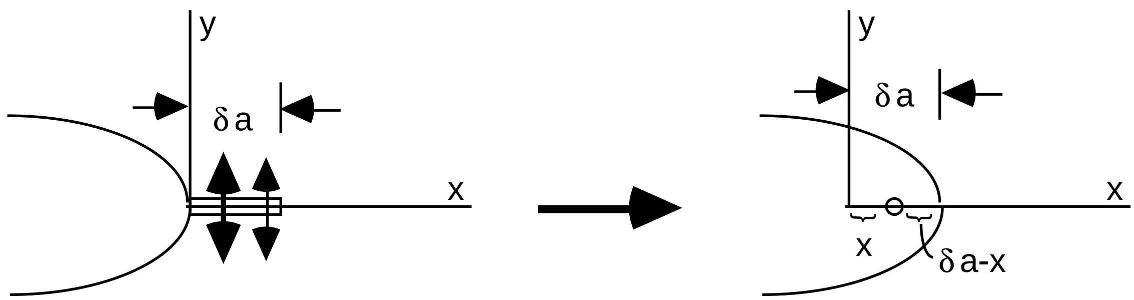
$$a = \frac{2\gamma E}{\pi \sigma^2} \tag{18.19}$$

The maximum far-field tensile load (σ) that a rock with an optimally oriented crack can sustain is

$$\sigma = \sqrt{\frac{2\gamma E}{\pi a}} \tag{18.20}$$

C Relationship between G and K

Consider a crack of half-length *a* propagating along plane to a new half-length a+δa. We can imagine this happening if the forces ahead of the crack tip are sufficient to open the crack into its ultimate shape:



The change in elastic energy is most easily examined by dealing with fixed grips, so no work is done on the sample during fracture growth. In this case the strain energy change just ahead of the crack tip is:

$$G_I = \lim_{\delta a \to 0} \frac{-\Delta U_e}{\delta a} = 2 \lim_{\delta a \to 0} \frac{1}{\delta a} \int_0^{\delta a} \frac{1}{2}\left(\sigma_{yy} u_y\right) dx \tag{18.21}$$

The factor of 2 arises because two surfaces are displaced, and the factor of 1/2 arises in the same way as in the equation for strain energy density. Because the change in elastic energy is negative, G is positive. The stresses and displacements we are interested in are those in front of the original fracture along the fracture plane. No work is done by the displacement of the wall behind the crack tip because they are traction-free. The near-tip stresses a distance *r* ahead of the crack tip in the left-hand diagram are given by Lawn and Wilshaw (p. 53):

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \left\{ \cos\frac{\theta}{2} \left[1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right] \right\} \tag{18.22}$$

Directly ahead of the crack θ = 0 and r = x, so this simplifies to:

$$\sigma_{yy}\Big|_{\theta=0} = \frac{K_I}{\sqrt{2\pi x}} \tag{18.23}$$

The displacements these stresses will are those that occur at a distance *r* <u>behind</u> the crack tip in the <u>right-hand</u> diagram (i.e., at $\theta = \pi$). From Lawn and Wilshaw (p. 53):

$$u_y = \frac{K_I}{2E}\left(\frac{r}{2\pi}\right)^{1/2}\left\{(1+\nu)\left[(2\kappa+1)\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right]\right\}; \quad (18.24)$$

For $\theta=\pi$, r = δa-x, so this reduces to

$$u_y = \frac{K_I}{2E}\left(\frac{\delta a - x}{2\pi}\right)^{1/2}\{(1+\nu)(2\kappa+2)\} \quad (18.25)$$

Now we substitute the expressions for σ_{yy} and u_y into (18.21)

$$G_I = 2\lim_{\delta c\to 0}\frac{1}{\delta a}\int_0^{\delta a}\frac{1}{2}\left(\frac{K_I}{\sqrt{2\pi x}}\right)\left(\frac{K_I}{2E}\left(\frac{\delta a - x}{2\pi}\right)^{1/2}\{(1+\nu)(2\kappa+2)\}\right)dx \quad (18.26)$$

This simplifies greatly because many terms are constants

$$G_I = \frac{K_I^2\{(1+\nu)(2\kappa+2)\}}{4\pi E}\lim_{\delta c\to 0}\frac{1}{\delta a}\int_0^{\delta a}\left(\frac{\delta a - x}{x}\right)^{1/2}dx \quad (18.27)$$

or

$$G_I = \frac{K_I^2\{(1+\nu)(2\kappa+2)\}}{4\pi E}\lim_{\delta c\to 0}\frac{1}{\delta a}\int_{0a}^{\delta a}\left(\frac{1-x/\delta a}{x/\delta a}\right)^{1/2}dx \quad (18.28)$$

At this point we substitute $\sin^2\omega$ for x/δa, 2δa sinω cosω $\delta\omega$ for dx, and change the limits of integration to get

$$G_I = \frac{K_I^2\{(1+\nu)(2\kappa+2)\}}{4\pi E}\lim_{\delta a\to 0}\frac{\delta a}{\delta a}\int_0^{\pi/2}\left(\frac{1-\sin^2\omega}{\sin^2\omega}\right)^{1/2} 2\sin\omega\cos\omega d\omega \quad (18.29)$$

$$G_I = \frac{K_I^2\{(1+\nu)(2\kappa+2)\}}{4\pi E}\int_0^{\pi/2} 2\cos^2\omega d\omega \quad (18.30)$$

$$G_I = \frac{K_I^2\{(1+\nu)(2\kappa+2)\}}{4\pi E}\left|\sin\omega\cos\omega + \omega\right|_0^{\pi/2} \quad (18.31)$$

Expression (18.31) boils down to this key result:

$$G_I = \frac{K_I^2\{(1+\nu)(2\kappa+2)\}}{8E} \quad (18.32)$$

For plane stress:

$$\kappa = \frac{3-\nu}{1+\nu}\ , \tag{18.33}$$

$$G_I = \frac{{K_I}^2}{E} \tag{18.36}$$

For plane strain:

$$\kappa = 3 - 4\nu\ , \tag{18.34}$$

$$G_I = \frac{{K_I}^2\left(1-\nu^2\right)}{E} \tag{18.35}$$

So the fracture energy release rate is directly tied to the stresses and displacement in the neighborhood of the crack tip and hence to the stress intensity factor(s).

Now G is an energy term (and a scalar), and energy terms can be added, so the expression for G when all three modes are present is:

$$G = G_I + G_{II} + G_{III} \tag{18.37}$$

For plane stress

$$G = \frac{1}{E}\left({K_I}^2 + {K_{II}}^2 + {K_{III}}^2 /[1+\nu]\right)\ , \tag{18.38}$$

and for plane strain

$$G = \frac{(1-\nu)^2}{E}\left({K_I}^2 + {K_{II}}^2 + {K_{III}}^2 /[1-\nu]\right)\ . \tag{18.39}$$

If fracture toughness is a material parameter, then a critical value of G (i.e., G_{crit}), and hence a critical stress intensity factor (K_{crit}) is needed for a fracture to propagate.

V Three common different criteria for fracture propagation directions

A The direction perpendicular to the most tensile near-tip hoop stress $\sigma_{\theta\theta}$ (Erdogan and Sih, 1963)

1 Advantage: Has intuitive appeal

2 Disadvantage: A bit problematic with all stresses $\to\infty$

B The direction which maximizes the energy release rate (Gell and Smith, 1967)

1 Advantage: Has energy basis

2 Disadvantage: A bit more complicated to calculate than $\sigma_{\theta\theta}$

C The direction of the minimum strain energy density (Sih, 1974)

1 Advantages: Has energy basis

2 Disadvantage: Not clear why A crack “should” propagate in the direction of the minimum (as opposed to maximum) strain energy density, and Sih does not provide insight into this choice.

All yield similar predictions for crack propagation directions.

References

Broek, D.B., Elemetary engineering fracture mechanics: Martinus Nijhoff, The Hague, 469 p.

Erdogan, F., and Sih, G.C., 1963, On the crack extension in plates under plane loading and transverse shear: Journal of Basic Engineering, v. 85, p. 519-527.

Gell, M., and Smith, E., 1967, The propagation of cracks though grain boundaries in polycrystalline 3% silicon-iron: Acta Metallurgica, v. 15, p. 253.

Lawn, B.R., and Wilshaw, T.R., 1975, Fracture of brittle solids: Cambridge University Press, London, 204 p.

Sih, G.C., 1974, Strain energy density factor applied to mixed mode crack problems: International Journal of Fracture, v. 11, p. 305-322.

Lecture 18 Appendix

Suppose we load an elastic body with a crack but the crack can't grow. The body will obey Hooke's Law and act like a spring:

$F = ku_y$ (18.A1a) or $u_y = F/k$ (18.A1b)

where u_y is the length change of the body in response to force F, and k is the spring constant. Alternatively,

$F = u_y/C$ (18.A2a) or $u_y = FC$ (18.A2b)

where C is the compliance (C=1/k).

The initial strain energy at the onset of crack growth will equal the work done on the body by external forces:

$$U_e = \int_0^u F \bullet du_y = \int_0^u ku \;\; du_y = \frac{1}{2} k u_y^2 \quad . \qquad (18.A3)$$

If we substitute expression (18.A11b) into (18.A12) we get

$$U_e = \frac{1}{2}\frac{1}{C}(FC)^2 = \frac{1}{2}F^2 C \qquad (18.A4)$$

If we substitute expression (18.A10b) into (18.A12) we get

$$U_e = \frac{1}{2}\frac{1}{C}(u_y)^2 \qquad (18.A5)$$

We can differentiate equations (18.A4) and (18.A5) to see what controls the way the elastic energy changes. If we know how the <u>force</u> changes during fracture we can differentiate expression (18.A4):

$$dU_e = \frac{1}{2}\left(F^2 dC + Cd[F^2]\right) = \frac{1}{2}\left(F^2 dC + 2CFdF\right) \qquad (18.A6)$$

If we know how the <u>displacement</u> changes during fracture we use expression (18.A5).

$$dU_e = \frac{1}{2}\left(\frac{1}{C} d[u_y^2] + u_y^2 d[\frac{1}{C}]\right) = \frac{1}{2}\left(\frac{2}{C} u_y d[u_y] - \frac{u_y^2}{C^2} dC\right) \qquad (18.A7)$$

Now suppose the crack starts to grow. From (18.A11b) we relate displacement of the plate edges to changes in the loads on the plate and changes in the compliance of the plate:

$$du_y = CdF + FdC \qquad (18.A8)$$

Under constant stresses, $dF = 0$, so $du_y = FdC$, so:

$$dW = F(du_y) = F(FdC) = F^2 dC \qquad (18.A9)$$

Because we know that $dF=0$, we use (18.A15) to get

$$dU_e = (1/2)\, F^2 dC \qquad (18.A10)$$

Subtracting (18.A18) from (18.A19), and comparing with equation (18.4) yields

$$d\Pi = d(-W_L + dU_e) = -(1/2)\, F^2 dC \qquad (18.A11)$$

Now consider a different case: constant displacement conditions. Here, no work is done by external forces, so

$$dW_L = Fdu_y = 0 \qquad (18.A12)$$

Because we know that $du_y=0$, we use (18.A16) to get

$$dU_e = \frac{-1}{2}\left(\frac{{u_y}^2}{C^2}\, dC\right) \qquad (18.A13)$$

By using equation (18.A11b) we can express u_y in terms of F:

$$dU_e = \frac{-1}{2}\left(\frac{(FC)^2}{C^2}\, dC\right) = \frac{-1}{2} F^2 dC \qquad (18.A14)$$

Subtracting (18.A21) from (18.A23), and comparing with equation (18.4) yields

$$d\Pi = (-dW_L + dU_e) = -(1/2)\, F^2 dC \qquad (18.A15)$$

This was the same result as for the constant displacement condition (18.A20). Hence

$$G = -d(-W_L + dU_e)/dc \qquad (18.A16)$$

holds independent of the loading configuration