# THE NEAR TIP FIELD AND STRESS INTENSITY FACTORS (17)

I Main topics

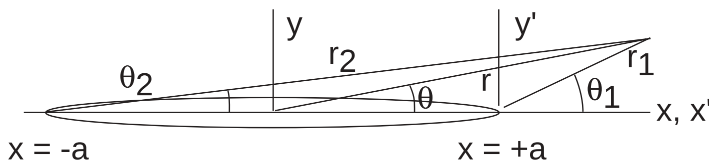
A The stress field near the tip of a fracture

B The stress intensity factor (K)

C Strain energy density

D Review of chapter by Pollard and Segall (1987)

II The stress field near the tip of a fracture




First we will consider a mode III fracture because it is the simplest. The stresses and displacements around the fracture are (see lecture 15):

$$\sigma_{zy} = \mathrm{Re}\frac{dZ_{III}}{d\zeta} = S\frac{r}{\sqrt{r_1 r_2}}\cos\left(\theta - \frac{\theta_1+\theta_2}{2}\right) \quad (15.35)$$

$$\sigma_{zx} = \mathrm{Im}\frac{dZ_{III}}{d\zeta} = S\frac{r}{\sqrt{r_1 r_2}}\sin\left(\theta - \frac{\theta_1+\theta_2}{2}\right) \quad (15.36)$$

$$u_z = \frac{1}{\mu}\mathrm{Im}Z_{III} = \frac{1}{\mu}\mathrm{Im}S\sqrt{r_1 r_2}e^{i\frac{\theta_1+\theta_2}{2}} = \frac{S}{\mu}\sqrt{r_1 r_2}\sin\frac{\theta_1+\theta_2}{2} \quad (15.37)$$

Consider the region near the right-hand crack tip. In the limit as $r_1 \to 0$, $r \to a$, $r_2 \to 2a$, $\theta \to 0$, and $\theta_1 \to 0$, and the above equations become:

$$\sigma_{zy} = S\frac{a}{\sqrt{r_1 2a}}\cos\left(0 - \frac{\theta_1+0}{2}\right) = S\frac{\sqrt{a}}{\sqrt{2r_1}}\cos\left(-\frac{\theta_1}{2}\right) = S\frac{\sqrt{\pi a}}{\sqrt{2\pi r_1}}\cos\left(\frac{\theta_1}{2}\right) = \frac{K_{III}}{\sqrt{2\pi r_1}}\cos\left(\frac{\theta_1}{2}\right) \quad (17.1)$$

$$\sigma_{zx} = S\frac{a}{\sqrt{r_1 2a}}\sin\left(0 - \frac{\theta_1+0}{2}\right) = S\frac{\sqrt{a}}{\sqrt{2r_1}}\sin\left(-\frac{\theta_1}{2}\right) = -S\frac{\sqrt{\pi a}}{\sqrt{2\pi r_1}}\sin\left(\frac{\theta_1}{2}\right) = -\frac{K_{III}}{\sqrt{2\pi r_1}}\sin\left(\frac{\theta_1}{2}\right) \quad (17.2)$$

$$u_z = \frac{S}{\mu}\sqrt{r_1 2a}\sin\left(\frac{\theta_1+0}{2}\right) = \frac{S\sqrt{\pi a}}{\mu}\sqrt{\frac{2r_1}{\pi}}\sin\frac{\theta_1}{2} = \frac{K_{III}}{\mu}\sqrt{\frac{2r_1}{\pi}}\sin\frac{\theta_1}{2} \quad (17.3)$$

Note that as the distance from the fracture tip goes to zero, the stresses approach infinite levels (the stresses vary as $r^{-1/2}$) – this is a consequence of the infinitely sharp fracture tip in our model. The near-tip stresses thus are said to be *singular*.

## III The stress intensity factor (K)

The stress intensity factor (SIF) $K_{III}$ is a parameter that characterizes the strength of the near-tip stress field. It is a function of the loading of the fractured body and the key dimensions (or length scales) of the fractured system. In the example of the previous page, the relevant load (S) is uniform, and the only length scale is that of the fracture (we pick the half-length a as opposed to the whole-length 2a). Thus

$$K_{III} = S\sqrt{\pi a} \tag{17.4}$$

In general the stress intensity factor will have the form (Tada et al, 1973):

$$\{K_m\}_{\pm a} = \int_{-a}^{+a} \frac{1}{\sqrt{\pi a}} S(x) \sqrt{\frac{a \pm x}{a \mp x}} dx \tag{17.5}$$

where S(x) is the driving stress distribution associated with the relative displacement of the fracture walls.

Regardless of how the fracture is loaded, the stresses near the fracture tip will in general have the form

$$\sigma_{ij} = \frac{K_m}{(2\pi r)^{1/2}} f_{ij}(\theta) \tag{17.6}$$

where r (rather than $r_1$) is the distance from the fracture tip, not from the fracture center, and with m = III for mode III. The near-tip stresses thus can be described in terms of the angular position and the radial distance from the fracture tip, with the angular position and the radial distance being separable. The near-tip displacements, measured relative to the fracture tip, are

$$u_i = \frac{K_m}{\mu} \sqrt{\frac{r}{2\pi}} f_i(\theta) \tag{17.7}$$

The near-tip displacements have a parabolic form. As the distance goes to zero, the displacements, relative to the position of the fracture tip, also go to zero.

These same comments hold for modes I and II, with $K_{III}$ being replaced by $K_I$ and $K_{II}$, respectively.

IV Strain energy density

The strain energy density (strain energy per unit area per unit depth) is (Chou and pagano, eq. 7.7) varies with the square of the stresses:

$$U_0 = \frac{1}{2}\left(\frac{1}{E}\left(\sigma_{xx}{}^2 + \sigma_{yy}{}^2 + \sigma_{zz}{}^2\right) - \frac{2\nu}{E}\left(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}\right) + \frac{1}{G}\left(\sigma_{xy}{}^2 + \sigma_{yz}{}^2 + \sigma_{zx}{}^2\right)\right) \tag{17.8}$$

Suppose that the stresses near the tip of a fracture vary as $r^n$, and suppose n were unknown. The strain energy density in a small volume per unit depth rdrdθ near the fracture tip can be integrated to obtain the strain energy in a region near the fracture tip (Barber, eq. 11.34); the solution will have the following form:

$$U = \int_0^{2\pi} \int_0^r U_0 r dr d\theta = \int_0^{2\pi} \int_0^r (Cr^n)^2 r dr d\theta = C^* \int_0^r r^{2n+1} dr = C^* r^{2n+2} \tag{17.9}$$

For the strain energy to be bounded (i.e., finite) as r→0, the exponent on the integrand must not be negative (i.e., 2n+1≥0), otherwise the strain energy . This means that n must be greater than or equal to –1. So the crack-tip singularity where the stresses vary as $r^{-1/2}$ is allowed on energetic terms, even if the stresses approach infinite levels near the fracture tip.