WESTERGAARD COMPLEX STRESS FUNCTIONS (16)

I Main topics

A Historical perspective

B Expression of the biharmonic function by harmonic functions

C Boundary conditions

D Use of symmetry and boundary conditions to simplify the stress function

E General solution for stresses and displacements in terms of a single complex stress function

F Stresses and displacements around mode I and mode II fractures

II Historical perspective

A Westergaard's 1939 paper presented a simple way to express the stresses and displacements around mode I fractures (later expanded to cover mode II fractures by Sih (1966)).

B One of the three most important early papers in fracture mechanics, the other two being by Inglis (1913) and Griffith (1921).

C Westergaard's stress function is given, not derived.

1 Solution constraints a bit hard to appreciate

2 Solution appears "like magic"

III Expression of the biharmonic function F by harmonic functions Θ_i

A Harmonic functions $\left(\nabla^2\Theta = 0\right)$ are better understood than biharmonic functions $\left(\nabla^4 F = 0\right)$ because of their greater use in many fields

B All harmonic functions are biharmonic, but not all biharmonic functions are harmonic

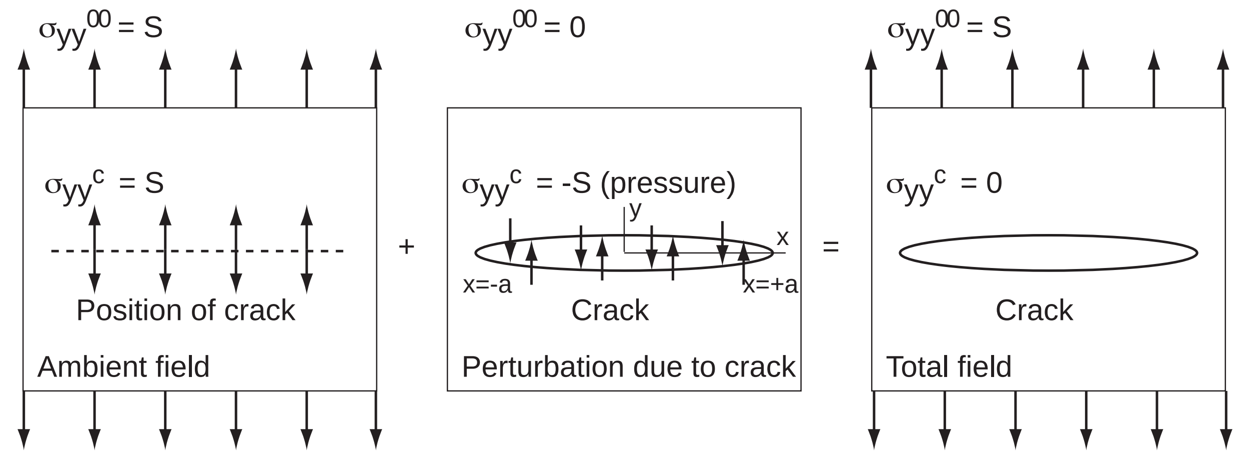
C The biharmonic function F can be expressed in terms of harmonic functions (see lecture 16 appendix and MacGregor, 1935)

If Θ, Θ_0, Θ_1, Θ_2, Θ_3, and Θ_4 are harmonic functions (i.e., $\left(\nabla^2\Theta_i = 0\right)$, then

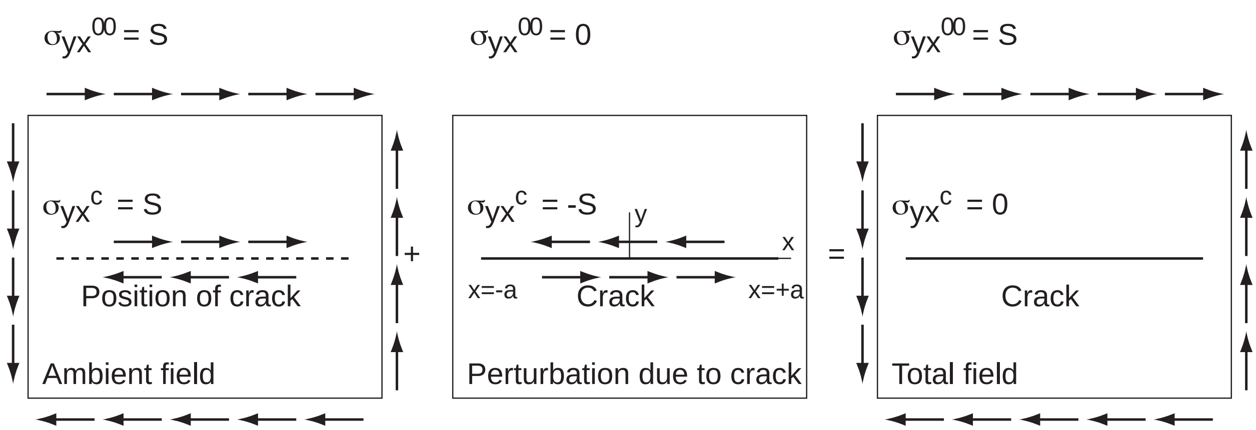
$$F = y\Theta + \Theta_0 = x\Theta_2 + \Theta_1 = \left(x^2 + y^2\right)\Theta_4 + \Theta_3 \quad (16.1)$$

Set up of Crack Problems by Perturbation of an Ambient Stress Field

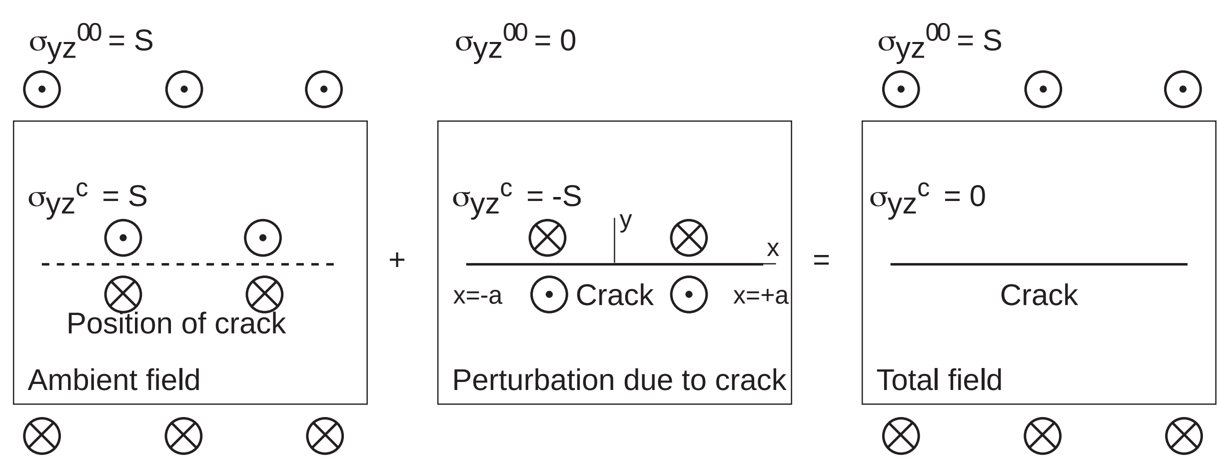
Mode I



Mode II



Mode III



IV Boundary conditions

A Mode I fracture with a uniform normal traction **along y=0**

$$\sigma_{yy}{}^{c} = S \quad for \ |x| < a \tag{16.2a}$$

By analogy with the solution for a mode III fracture, we could also imagine that this first boundary condition could be replaced by

$$u_y = C_I\left(a^2 - z^2\right) = C_I\left(a^2 - x^2\right) \quad for \ |x| < a, y = 0 \tag{16.2b}$$

The other boundary conditions <u>along the fracture plane</u> are

$$u_y = 0 \quad for \ |x| > a \tag{16.3}$$

$$\sigma_{yx} = 0 \quad for \ |x| \le a \tag{16.4}$$

$$\sigma_{yx} = 0 \quad for \ |x| > a \tag{16.5}$$

The second and fourth boundary conditions follow by symmetry.

B Mode II fracture with a uniform normal traction **along y=0**

$$\sigma_{yx}{}^{c} = S \quad for \ |x| < a \tag{16.6a}$$

By analogy with the solution for a mode III fracture, we can anticipate that this first boundary condition could be replaced by

$$u_x = C_{II}\left(a^2 - z^2\right) = C_{II}\left(a^2 - x^2\right) for \ |x| < a, y = 0 \tag{16.6b}$$

The other boundary conditions <u>along the fracture plane</u> are

$$u_x = 0 \quad for \ |x| > a \tag{16.7}$$

$$\sigma_{yy} = 0 \quad for \ |x| \le a \tag{16.8}$$

$$\sigma_{yy} = 0 \quad for \ |x| > a \tag{16.9}$$

The second and fourth boundary conditions follow by symmetry.

V Use of symmetry and boundary conditions to simplify the stress function

A First simplification of the stress function

The plane of the crack (i.e., the plane of the boundary conditions) is y=0. An inspection of (16.1) shows that the from of the biharmonic function can be chosen as

$$F = y\Theta + \Theta_0 \tag{16.10}$$

Our boundary conditions are along y=0, so even though both Θ and Θ_0 must be found to determine F, we can anticipate that choosing a solution in the form of (16.10) will be helpful for finding F.

B Second simplification of the stress function

We need to know what Θ and Θ_0 are and how they are related. We use the boundary conditions to do this. This will lead us to expressions for stresses (and displacements) in terms of "regular" derivatives of a complex function (i.e., its real and imaginary parts solve the Laplace equation) instead of partial derivatives of a biharmonic function.

We start by expressing the stresses in terms of the biharmonic function F

$$\sigma_{xx} = \frac{\partial F^2}{\partial^2 y} \qquad \sigma_{yy} = \frac{\partial F^2}{\partial^2 x} \qquad \sigma_{yx} = \frac{-\partial F^2}{\partial x \partial y} \qquad (16.11)$$

The first partial derivatives of F with respect to x and y are, respectively:

$$\frac{\partial F}{\partial x} = \frac{\partial(y\Theta + \Theta_0)}{\partial x} = y\frac{\partial \Theta}{\partial x} + \Theta\frac{\partial y}{\partial x} + \frac{\partial \Theta_0}{\partial x} = y\frac{\partial \Theta}{\partial x} + \frac{\partial \Theta_0}{\partial x} \qquad (16.12a)$$

$$\frac{\partial F}{\partial y} = \frac{\partial(y\Theta + \Theta_0)}{\partial y} = y\frac{\partial \Theta}{\partial y} + \Theta\frac{\partial y}{\partial y} + \frac{\partial \Theta_0}{\partial y} = y\frac{\partial \Theta}{\partial y} + \Theta + \frac{\partial \Theta_0}{\partial y} \qquad (16.12b)$$

To get the second partial derivatives we will make some substitutions. Let us refer to the partial derivatives in (16.12) by the following terms:

$$\Phi = \frac{\partial \Theta}{\partial y} \qquad \Psi = \frac{\partial \Theta}{\partial x} \qquad \psi = \frac{\partial \Theta_0}{\partial y} \qquad \chi = \frac{\partial \Theta_0}{\partial x} \qquad (16.13a)$$

In light of (16.3a), and because Θ and Θ_0 are harmonic functions

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad \frac{\partial \psi}{\partial x} = \frac{\partial \chi}{\partial y} \qquad (16.13b)$$

$$\frac{\partial^2 \Theta}{\partial x^2} = \frac{-\partial^2 \Theta}{\partial^2 y} \Rightarrow \frac{\partial \Psi}{\partial x} = \frac{-\partial \Phi}{\partial y} \quad \text{and} \quad \frac{\partial^2 \Theta_0}{\partial x^2} = \frac{-\partial^2 \Theta_0}{\partial^2 y} \Rightarrow \frac{\partial \chi}{\partial x} = \frac{-\partial \psi}{\partial y} \qquad (16.13c)$$

So (16.12) and (16.13) yield

$$\frac{\partial F}{\partial x} = y\Psi + \chi \quad (16.14a) \qquad \frac{\partial F}{\partial y} = y\Phi + \Theta + \psi \qquad (16.14b)$$

Now we return to the stresses. Using (16.12) and (16.13)

$$\sigma_{xx} = \frac{\partial\left(\frac{\partial F}{\partial y}\right)}{\partial y} = \frac{\partial(y\Phi + \Theta + \psi)}{\partial y} = y\frac{\partial \Phi}{\partial y} + \Phi\frac{\partial y}{\partial y} + \Phi + \frac{\partial \psi}{\partial y} = 2\Phi + y\frac{\partial \Phi}{\partial y} - \frac{\partial \chi}{\partial x} \qquad (16.15a)$$

$$\sigma_{yy} = \frac{\partial\left(\frac{\partial F}{\partial x}\right)}{\partial x} = \frac{\partial(y\Psi + \chi)}{\partial x} = y\frac{\partial \Psi}{\partial x} + \Psi\frac{\partial y}{\partial x} + \frac{\partial \chi}{\partial x} = y\frac{\partial \Psi}{\partial x} + \frac{\partial \chi}{\partial x} = -y\frac{\partial \Phi}{\partial x} + \frac{\partial \chi}{\partial x} \qquad (16.15b)$$

$$\sigma_{yx} = \frac{-\partial\left(\frac{\partial F}{\partial x}\right)}{\partial y} = \frac{-\partial(y\Psi + \chi)}{\partial y} = -y\frac{\partial \Psi}{\partial y} - \Psi\frac{\partial y}{\partial y} + \frac{\partial \chi}{\partial y} = -\Psi - y\frac{\partial \Phi}{\partial x} - \frac{\partial \chi}{\partial y} \qquad (16.15c)$$

1 Mode I constraints and mode I stresses

The mode I boundary conditions allow (16.15) to be simplified.

For mode I, along y=0 the shear stress is zero, so from (16.15c)

$$\Psi = \frac{-\partial\chi}{\partial y} \quad \text{or using (16.13b)} \quad \Psi = \frac{-\partial\psi}{\partial x} \qquad (16.16)$$

This relates Θ and Θ_0 for mode I. Substituting this back into (16.15c), the shear stress becomes a function of only y and Φ.

$$\sigma_{yx} = -y\frac{\partial\Phi}{\partial x} \qquad (16.17)$$

Now we turn our attention to the normal stresses in (16.15). We let the derivatives of Θ be the real and imaginary parts of the following complex function; the imaginary part can be expressed using (16.16) for Ψ.

$$Z_I(\zeta) = \Phi + i\Psi = \Phi + i\frac{\partial\chi}{\partial y} \qquad (16.18)$$

For this to be worthy of consideration as a stress function (i.e., for it to have derivatives that are defined) the two Cauchy-Riemann conditions (15.20) must apply. The first Cauchy-Riemann condition, applied to (16.18), yields

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y} \qquad (16.19a)$$

An inspection of (16.13b) shows that Φ and Ψ always satisfy this.

Applying the second Cauchy-Riemann condition to (16.18), requires that

$$\frac{\partial\Phi}{\partial y} = \frac{-\partial\Psi}{\partial x} \quad \text{or using (16.16 to express } \Psi\text{)} \quad \frac{\partial\Phi}{\partial y} = \frac{-\partial\left(\frac{-\partial\chi}{\partial y}\right)}{\partial x} = \frac{\partial^2\chi}{\partial x\partial y} = \frac{\partial\left(\frac{\partial\chi}{\partial x}\right)}{\partial y} \qquad (16.19b)$$

The righthand part of (16.19b) is satisfied if

$$\Phi = \frac{\partial\chi}{\partial x} \qquad (16.20)$$

This key constraint results from symmetry across the fracture plane.

To obtain the mode I stresses we now substitute (16.20) into (16.15a)

$$\sigma_{xx} = 2\Phi + y\frac{\partial\Phi}{\partial y} - \frac{\partial\chi}{\partial x} = 2\Phi + y\frac{\partial\Phi}{\partial y} - \Phi = \Phi + y\frac{\partial\Phi}{\partial y}, \qquad (16.22a)$$

then substitute (16.20) into (16.15a), and apply the left side of (16.19b)

$$\sigma_{yy} = y\frac{\partial\Psi}{\partial x} + \frac{\partial\chi}{\partial x} = \Phi + y\frac{\partial\Psi}{\partial x} = \Phi - y\frac{\partial\Phi}{\partial x} \qquad (16.22b)$$

and finally reproduce (16.17)

$$\sigma_{yx} = -y\frac{\partial\Phi}{\partial x} \qquad (16.22c)$$

Only y and the partial derivatives of Θ are used here (see (16.10)).

2 Mode II constraints and mode II stresses

For mode II, the normal traction on the plane y=0 is zero, so (16.15b) gives

$$\frac{\partial \chi}{\partial x} = 0 \qquad (16.23)$$

The simplest way to enforce this condition is to set $\chi = 0$.

So (16.15), when specialized for mode II, becomes

$$\sigma_{xx} = 2\Phi + y\frac{\partial \Phi}{\partial y} \qquad (16.24a)$$

$$\sigma_{yy} = y\frac{\partial \Psi}{\partial x} \qquad (16.24b)$$

$$\sigma_{yx} = -\Psi - y\frac{\partial \Psi}{\partial y} - \frac{\partial \chi}{\partial y} = -\Psi - y\frac{\partial \Psi}{\partial y} = -\Psi - y\frac{\partial \Phi}{\partial x} \qquad (16.24c)$$

3 Closing comments on this section

So the functions Φ and Ψ, along y = 0, yield the boundary conditions on the fracture. MacGregor (1935) notes that these quantities are proportional to the dilation (volumetric strain) and rotation in the body.

We have shown that the stress can be obtained by two harmonic functions that are partial derivatives of the same higher order function. Similarly, these two harmonic functions can also be taken to be the real and imaginary components of a single complex function.

VI General solution for stresses and displacements in terms of a single complex stress function

A Road map

The work so far yields stresses in terms of the partial derivatives of some harmonic functions with respect to x and y. The associated stress functions (and stresses and displacements) can be cast in terms of the derivatives of a single complex function in terms of the complex variable ζ, as (16.18 suggests). The following functions are needed:

$\bar{\bar{Z}}, \bar{Z}, Z, \text{ and } Z'$, where

$$\bar{Z} = \frac{d\bar{\bar{Z}}}{d\zeta} \qquad Z = \frac{d\bar{Z}}{d\zeta} \qquad Z' = \frac{dZ}{d\zeta}$$

These functions can be found directly from the boundary conditions, but owing to time constraints we will not do this. Instead we will rely upon analogies with the solutions that we have derived so far and show that the solutions for the stress functions satisfy the boundary conditions. The stresses will come from Z and Z'. The strains are proportional to the stresses, and the displacements come from integration of the strains, so the displacements must depend on the integrals of those terms. From our previous results we know that the relative displacements along a fracture vary as $(a^2-x^2)^{1/2}$, provided that the stresses promoting relative displacement of the fracture walls are uniform. We thus can anticipate a term like this being in the complex functions.

B Derivatives of a complex function

We apply the Cauchy-Riemann conditions (see (15.8) and (15.9)) several times in taking derivatives of complex functions with respect to y:

$$\frac{dZ}{d\zeta} = Z' = \frac{\partial \operatorname{Re} Z}{\partial x} + i\frac{\partial \operatorname{Im} Z}{\partial x} = \frac{\partial \operatorname{Im} Z}{\partial y} - i\frac{\partial \operatorname{Re} Z}{\partial y} = \operatorname{Re} Z' + i \operatorname{Im} Z' \tag{16.25}$$

C Mode I

Consider a candidate mode I stress function to have the form of (16.10)

$$F_I = \operatorname{Re} \overline{\overline{Z}}_I + y \operatorname{Im} \overline{Z}_I \tag{16.26}$$

The first derivative of F_I with respect to y is:

$$\frac{\partial F_I}{\partial y} = \left(\frac{\partial \operatorname{Re} \overline{\overline{Z}}_I}{\partial y}\right) + y\frac{\partial \operatorname{Im} \overline{Z}_I}{\partial y} + \operatorname{Im} \overline{Z}_I \frac{\partial y}{\partial y} = \left(\frac{\partial \operatorname{Re} \overline{\overline{Z}}_I}{\partial y}\right) + y\frac{\partial \operatorname{Im} \overline{Z}_I}{\partial y} + \operatorname{Im} \overline{Z}_I \tag{16.27}$$

Using the Cauchy-Riemann relations, this can be rewritten

$$\frac{\partial F_I}{\partial y} = \left(-\operatorname{Im} \overline{Z}_I\right) + y \operatorname{Re} Z_I + \operatorname{Im} \overline{Z}_I = y \operatorname{Re} Z_I \tag{16.28}$$

And then, noting that (16.28) feeds into (16.29a) and 16.29b),

$$\sigma_{xx} = \frac{\partial^2 F_I}{\partial y^2} = \frac{\partial\left(y \operatorname{Re} Z_I\right)}{\partial y} = y\frac{\partial \operatorname{Re} Z_I}{\partial y} + \operatorname{Re} Z_I \frac{\partial y}{\partial y} = \operatorname{Re} Z_I - y \operatorname{Im} Z_I' \tag{16.29a}$$

$$\sigma_{yy} = \frac{\partial^2 F_I}{\partial x^2} = \frac{\partial^2\left(\operatorname{Re} \overline{\overline{Z}}_I + y \operatorname{Im} \overline{Z}_I\right)}{\partial x^2} = \operatorname{Re} Z_I + y \operatorname{Im} Z_I' \tag{16.29b}$$

$$\sigma_{xy} = \frac{-\partial^2 F_I}{\partial x \partial y} = \frac{-\partial\left(\frac{\partial F_I}{\partial y}\right)}{\partial x} = \frac{-\partial\left(y \operatorname{Re} Z_I\right)}{\partial x} = -y \operatorname{Re} Z_I' \tag{16.29c}$$

Comparing (16.29) with (16.22) term by term

$$\sigma_{xx} = \Phi + y\frac{\partial \Phi}{\partial y}, \tag{16.22a}$$

$$\sigma_{yy} = \Phi - y\frac{\partial \Phi}{\partial x} \tag{16.22b}$$

$$\sigma_{yx} = -y\frac{\partial \Phi}{\partial x} \tag{16.22c}$$

shows that $\Phi = \operatorname{Re} Z_I$ and $\Psi = y\operatorname{Re} Z_I$ in the expression below (16.18)

$$Z_I(\zeta) = \Phi + i\Psi = \Phi + i\frac{\partial \chi}{\partial y} \tag{16.18}$$

So the harmonic functions of section V are represented in the complex function Z_I.

<u>The stress function for mode I</u>

The elliptical relative displacement profile across a mode III fracture are given by the stress function of (15.25)

$$\bar{Z}_{III} = S\sqrt{\zeta^2 - a^2} \qquad (15.25)$$

A mode I fracture has an elliptical relative displacement profile too (14.25), so by analogy we try the above stress function and its derivatives

$$\bar{Z}_I = S\left(\zeta^2 - a^2\right)^{1/2} \qquad (16.30a)$$

$$Z_I = \frac{d\bar{Z}_I}{d\zeta} = S\left(\frac{1}{2}\right)\left(\zeta^2 - a^2\right)^{-1/2} 2\zeta = S\zeta\left(\zeta^2 - a^2\right)^{-1/2} \qquad (16.30b)$$

$$Z_I{}' = \frac{dZ_I}{d\zeta} = S\left\{\zeta\frac{d\left(\left(\zeta^2 - a^2\right)^{-1/2}\right)}{d\zeta} + \left(\zeta^2 - a^2\right)^{-1/2}\frac{d(\zeta)}{d\zeta}\right\} \qquad (16.30c)$$

$$= S\left\{\zeta\left(-\frac{1}{2}\right)\left(\zeta^2 - a^2\right)^{-3/2}(2\zeta) + \left(\zeta^2 - a^2\right)^{-1/2}\right\} = S\left\{-\zeta^2\left(\zeta^2 - a^2\right)^{-3/2} + \left(\zeta^2 - a^2\right)^{-1/2}\right\}$$

Substituting (16.30) into (16.29) gives the stresses, here in tri-polar form

$$\sigma_{xx} = S\left[\frac{r}{(r_1 r_2)^{1/2}}\cos\left(\theta - \frac{\theta_1 + \theta_2}{2}\right) - \frac{ya^2}{(r_1 r_2)^{3/2}}\sin\left(3\frac{\theta_1 + \theta_2}{2}\right)\right] \qquad (16.31a)$$

$$\sigma_{yy} = S\left[\frac{r}{(r_1 r_2)^{1/2}}\cos\left(\theta - \frac{\theta_1 + \theta_2}{2}\right) + \frac{ya^2}{(r_1 r_2)^{3/2}}\sin\left(3\frac{\theta_1 + \theta_2}{2}\right)\right] \qquad (16.31b)$$

$$\sigma_{xy} = S\left[\frac{ya^2}{(r_1 r_2)^{3/2}}\cos\left(3\frac{\theta_1 + \theta_2}{2}\right)\right] \qquad (16.31c)$$

Do these recover our boundary conditions?
For y=0, 0<x<a

$$\sigma_{yy} = S\left[\frac{x}{((a-x)(a+x))^{1/2}}\cos\left(0-\frac{\pi+0}{2}\right)\right] = 0 \tag{16.32a}$$

$$\sigma_{xy} = S[0] = 0 \tag{16.32b}$$

For y=0, -a<x<0

$$\sigma_{yy} = S\left[\frac{x}{((a-x)(a+x))^{1/2}}\cos\left(\pi-\frac{\pi+0}{2}\right)\right] = 0 \tag{16.32c}$$

$$\sigma_{xy} = S[0] = 0 \tag{16.32d}$$

As r, r_1, $r_2 \to \infty$, and $\theta \to \theta_1$, $\to \theta_2$

$$\sigma_{xx} \to S\left[\frac{1}{1}\cos(0) - 0\sin(3\theta)\right] = S \tag{16.33a}$$

$$\sigma_{xx} \to S\left[\frac{1}{1}\cos(0) - 0\sin(3\theta)\right] = S \tag{16.33b}$$

$$\sigma_{xy} = S[(0)\cos(3\theta)] = 0 \tag{16.33c}$$

So the solution is for a plate under biaxial tension of magnitude S with a crack.

A uniaxial compression <u>parallel</u> to the crack (σ_{xx} = -S) can be added to yield the solution for a traction-free crack under a unaxial tension perpendicular to the crack. This corresponds to the right column of page 16-2.

To obtain the solution for the central columns of page 16-2, and biaxial stress of magnitude -S must be added to the solutions of (16.31). The associated stress function and relevant derivatives are

$$\bar{Z}_I = S\left[\left(\zeta^2 - a^2\right)^{1/2} - \zeta\right] \tag{16.34a}$$

$$Z_I = S\left[\zeta\left(\zeta^2 - a^2\right)^{-1/2} - 1\right] \tag{16.34b}$$

$$Z_I{}' = -S\left(\frac{a^2}{\zeta^3}\right)\left(1-\left(\frac{a^2}{\zeta^2}\right)\right)^{-3/2} \tag{16.34c}$$

Westergaard (1939) gives the following expression for the displacements (and MacGregor (1935) shows where they come from) for mode I, where G is the shear modulus:

$$2Gu_x = (1-2\nu)\mathrm{Re}\,\bar{Z}_I - y\,\mathrm{Im}\,Z_I \qquad (16.35a)$$

$$2Gu_y = 2(1-\nu)\mathrm{Im}\,\bar{Z}_I - y\,\mathrm{Re}\,Z_I \qquad (16.35a)$$

Substituting (16.34)) these expressions into (16.32) gives the displacements. Segall and Pollard give the results. A point Segall and Pollard highlight is that geologists and geophysicists commonly are the most interested in the displacements associated with the relative motion of the fracture walls (e.g., during an earthquake), and not the displacements due to the far-field stresses.

For mode II (Tada et al., 1976)

$$\Phi_{II} = \text{-yRe}\bar{\text{Z}} \qquad (16.36)$$

$$\sigma_{xx} = 2\,\mathrm{Im}\,Z + y\,\mathrm{Re}\,Z' \qquad (16.37a)$$

$$\sigma_{yy} = -y\,\mathrm{Re}\,Z' \qquad (16.37b)$$

$$\sigma_{xy} = \mathrm{Re}\,Z - y\,\mathrm{Im}\,Z' \qquad (16.37c)$$

$$u_x = \frac{\kappa+5}{8G}\mathrm{Im}\,\bar{Z} + y\,\mathrm{Re}\,Z \qquad (16.38a)$$

$$u_y = \frac{1-\kappa}{4G}\mathrm{Re}\,\bar{Z} - y\,\mathrm{Im}\,Z \qquad (16.38b)$$

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Westergaard, H.M., 1939, Bearing pressures and cracks: Journal of Applied mechanics, v. 6, p. A49-A53.

Comparison of solutions of MacGregor, Westergaard, Tada et al., and Pollard and Segall

	MacGregor	Westergaard	Tada et al.	P & S
Airy stress function	F	F	Φ	U
σxx	$2\Phi + y\frac{\partial\Phi}{\partial y} + \frac{\partial\chi}{\partial x}$			$\mathrm{Re}\,\phi_I' + y\,\mathrm{Im}\left[\phi_I'' + \phi_{II}''\right]$
σyy	$-y\frac{\partial\Phi}{\partial y} - \frac{\partial\chi}{\partial x}$			$\mathrm{Re}\left[\phi_I' + 2\phi_{II}'\right] - y\,\mathrm{Im}\left[\phi_I'' + \phi_{II}''\right]$
σxy	$-\Psi - y\frac{\partial\Phi}{\partial x} + \frac{\partial\chi}{\partial x}$			$-\mathrm{Im}\left[\phi_{II}'\right] - y\,\mathrm{Re}\left[\phi_I'' + \phi_{II}''\right]$
2μux	$-2(1-\nu)\Omega - y\frac{\partial\Omega}{\partial y}$ $-2(1-\nu)\Omega - y\frac{\partial\Theta}{\partial x}$			$2(1-\nu)\,\mathrm{Im}\,\phi_I' + (1-2\nu)\,\mathrm{Im}\,\phi_{II} - y\,\mathrm{Re}\left[\phi_I' + \phi_{II}'\right]$
2μux				
2μuy				$2(1-\nu)\,\mathrm{Re}\,\phi_{II} + (1-2\nu)\,\mathrm{Re}\,\phi_I - y\,\mathrm{Im}\left[\phi_I' + \phi_{II}'\right]$
2μuy				

	MacGregor	Westergaard	Tada et al.	P & S
Airy stress function I	F	F	Φ	$\frac{1}{2}\mathrm{Re}[\bar{z}\phi+\chi]$
	$\Phi=-\mathrm{Im}W$	$\mathrm{Re}\,Z$	$\mathrm{Re}\,Z$	
	$\frac{\partial\Phi}{\partial x}$	$\mathrm{Re}\,Z'$	$\mathrm{Re}\,Z'$	
	$\frac{\partial\Phi}{\partial y}$	$-\mathrm{Im}\,Z'$	$-\mathrm{Im}\,Z'$	
	$\Psi=\mathrm{Re}W$	$\mathrm{Im}\,Z$	$\mathrm{Im}\,Z$	
	Ω	$-\mathrm{Re}\,\bar{Z}$	$-\mathrm{Re}\,\bar{Z}$	
	$\frac{\partial\Omega}{\partial y}$	$\mathrm{Im}\,Z$	$\mathrm{Im}\,Z$	
	Θ	$\mathrm{Im}\,\bar{Z}$	$\mathrm{Im}\,\bar{Z}$	
	$\frac{\partial\Theta}{\partial y}$	$\mathrm{Re}\,Z$	$\mathrm{Re}\,Z$	
	Θ_0	$\mathrm{Re}\,\bar{\bar{Z}}$	$\mathrm{Re}\,\bar{\bar{Z}}$	
	$\mathrm{Re}W$	$\mathrm{Im}\,Z$		
	$\mathrm{Im}W$	$-\mathrm{Re}\,Z$		
	$\mathrm{Re}W'$	$\mathrm{Im}\,Z'$		
	$\mathrm{Im}W'$	$-\mathrm{Re}\,Z'$		
σxx	$\Phi+y\frac{\partial\Phi}{\partial y}$	$\mathrm{Re}\,Z-y\,\mathrm{Im}\,Z'$	$\mathrm{Re}\,Z-y\,\mathrm{Im}\,Z'$	$\mathrm{Re}\,\phi_I'+y\,\mathrm{Im}\,\phi''$
	$-\mathrm{Im}W-y\,\mathrm{Re}W'$	$\mathrm{Re}\,Z-y\,\mathrm{Im}\,Z'$	$\mathrm{Re}\,Z-y\,\mathrm{Im}\,Z'$	$\mathrm{Re}\,\phi_I'+y\,\mathrm{Im}\,\phi''$
σyy	$\Phi-y\frac{\partial\Phi}{\partial y}$	$\mathrm{Re}\,Z+y\,\mathrm{Im}\,Z'$	$\mathrm{Re}\,Z+y\,\mathrm{Im}\,Z'$	$\mathrm{Re}\,\phi_I'-y\,\mathrm{Im}\,\phi_I''$
	$-\mathrm{Im}W+y\,\mathrm{Re}W'$	$\mathrm{Re}\,Z+y\,\mathrm{Im}\,Z'$	$\mathrm{Re}\,Z+y\,\mathrm{Im}\,Z'$	$\mathrm{Re}\,\phi_I'-y\,\mathrm{Im}\,\phi_I''$
σxy	$-y\frac{\partial\Phi}{\partial x}$	$-y\,\mathrm{Re}\,Z'$	$-y\,\mathrm{Re}\,Z'$	$-y\,\mathrm{Re}\,\phi_I''$
	$y\,\mathrm{Im}W'$	$-y\,\mathrm{Re}\,Z'$	$-y\,\mathrm{Re}\,Z'$	$-y\,\mathrm{Re}\,\phi_I''$
2μux	$-(1-2\nu)\Omega-y\frac{\partial\Omega}{\partial y}$	$(1-2\nu)\mathrm{Re}\,\bar{Z}$ $-y\,\mathrm{Im}\,Z$	$(1-2\nu)\mathrm{Re}\,\bar{Z}$ $-y\,\mathrm{Im}\,Z$	$2(1-\nu)\mathrm{Im}\,\phi_I'$ $-y\,\mathrm{Re}\,\phi_I'$
2μux	$-(1-2\nu)\Omega-y\frac{\partial\Omega}{\partial y}$	$(1-2\nu)\mathrm{Re}\,\bar{Z}$ $-y\,\mathrm{Im}\,Z$	$(1-2\nu)\mathrm{Re}\,\bar{Z}$ $-y\,\mathrm{Im}\,Z$	$2(1-\nu)\mathrm{Im}\,\phi_I'$ $-y\,\mathrm{Re}\,\phi_I'$
2μuy	$2(1-\nu)\Theta-y\frac{\partial\Theta}{\partial y}$	$2(1-\nu)\mathrm{Im}\,\bar{Z}$ $-y\,\mathrm{Re}\,Z$	$2(1-\nu)\mathrm{Im}\,\bar{Z}$ $-y\,\mathrm{Re}\,Z$	$(1-2\nu)\mathrm{Re}\,\phi_I$ $-y\,\mathrm{Im}\,\phi_I'$
2μuy	$2(1-\nu)\Theta-y\frac{\partial\Theta}{\partial y}$	$2(1-\nu)\mathrm{Im}\,\bar{Z}$ $-y\,\mathrm{Re}\,Z$	$2(1-\nu)\mathrm{Im}\,\bar{Z}$ $-y\,\mathrm{Re}\,Z$	$(1-2\nu)\mathrm{Re}\,\phi_I$ $-y\,\mathrm{Im}\,\phi_I'$

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Mode II				
	$\Phi = -\mathrm{Im}W$	$\mathrm{Im}Z$	$\mathrm{Im}Z$	
	$\frac{\partial\Phi}{\partial x}$	$\mathrm{Im}Z'$	$\mathrm{Im}Z'$	
	$\frac{\partial\Phi}{\partial y}$	$\mathrm{Re}\,Z'$	$\mathrm{Re}\,Z'$	
	$\Psi = \mathrm{Re}W$	$-\mathrm{Re}\,Z$	$-\mathrm{Re}\,Z$	
	Ω	$-\mathrm{Im}\bar{Z}$	$-\mathrm{Im}\bar{Z}$	
	$\frac{\partial\Omega}{\partial y}$	$-\mathrm{Re}\,Z$	$-\mathrm{Re}\,Z$	
	Θ	$-\mathrm{Re}\,\bar{Z}$	$-\mathrm{Re}\,\bar{Z}$	
	$\frac{\partial\Theta}{\partial y}$	$\mathrm{Im}Z$	$\mathrm{Im}Z$	
	Θ_0			
	$\mathrm{Re}W$	$-\mathrm{Re}\,Z$	$-\mathrm{Re}\,Z$	
	$\mathrm{Im}W$	$-\mathrm{Im}Z$	$-\mathrm{Im}Z$	
	$\mathrm{Re}W'$	$-\mathrm{Re}\,Z'$	$-\mathrm{Re}\,Z'$	
	$\mathrm{Im}W'$	$-\mathrm{Im}Z'$	$-\mathrm{Im}Z'$	
σxx	$2\Phi + y\frac{\partial\Phi}{\partial y}$		$2\,\mathrm{Im}Z + y\,\mathrm{Re}\,Z'$	$y\,\mathrm{Im}\phi''_{II}$
	$-2\,\mathrm{Im}W - y\,\mathrm{Re}W'$		$2\,\mathrm{Im}Z + y\,\mathrm{Re}\,Z'$	$y\,\mathrm{Im}\phi''_{II}$
σyy	$-y\frac{\partial\Phi}{\partial y}$		$-y\,\mathrm{Re}\,Z'$	$2\,\mathrm{Re}\,\phi'_{II}$ $-y\,\mathrm{Im}\phi''_{II}$
	$y\,\mathrm{Re}W'$		$-y\,\mathrm{Re}\,Z'$	$2\,\mathrm{Re}\,\phi'_{II}$ $-y\,\mathrm{Im}\phi''_{II}$
σxy	$-\Psi - y\frac{\partial\Phi}{\partial x}$		$\mathrm{Re}\,Z - y\,\mathrm{Im}Z'$	$-\mathrm{Im}\phi'_{II}$ $-y\,\mathrm{Re}\,\phi''_{II}$
	$-\mathrm{Re}W + y\,\mathrm{Im}W'$		$\mathrm{Re}\,Z - y\,\mathrm{Im}Z'$	$-\mathrm{Im}\phi'_{II}$ $-y\,\mathrm{Re}\,\phi''_{II}$
2μux	$-2(1-\nu)\Omega - y\frac{\partial\Omega}{\partial y}$		$2(1-\nu)\,\mathrm{Im}\bar{Z}$ $+y\,\mathrm{Re}\,Z$	$(1-2\nu)\,\mathrm{Im}\phi_{II}$ $-y\,\mathrm{Re}\,\phi'_{II}$
2μux	$-2(1-\nu)\Omega - y\frac{\partial\Omega}{\partial y}$		$2(1-\nu)\,\mathrm{Im}\bar{Z}$ $+y\,\mathrm{Re}\,Z$	$(1-2\nu)\,\mathrm{Im}\phi_{II}$ $-y\,\mathrm{Re}\,\phi'_{II}$
2μuy	$(1-2\nu)\Theta - y\frac{\partial\Theta}{\partial y}$		$-(1-2\nu)\,\mathrm{Re}\,\bar{Z}$ $-y\,\mathrm{Im}Z$	$2(1-\nu)\,\mathrm{Re}\,\phi_{II}$ $-y\,\mathrm{Im}\phi'_{II}$
2μuy	$(1-2\nu)\Theta - y\frac{\partial\Theta}{\partial y}$		$-(1-2\nu)\,\mathrm{Re}\,\bar{Z}$ $-y\,\mathrm{Im}Z$	$2(1-\nu)\,\mathrm{Re}\,\phi_{II}$ $-y\,\mathrm{Im}\phi'_{II}$

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Mode III				

Appendix for Lecture 16

I Main topic:

Solution of biharmonic functions in terms of harmonic functions (see MacGregor, 1935, Timoshenko and Goodier, 1970, p. 172)

Consider the harmonic functions $\Theta = \Theta(x,y)$ and $\Theta_0 = \Theta_0(x,y)$. We want to show that $y\Theta + \Theta_0$ is biharmonic. Clearly if

$$\nabla^2\Theta_0 = 0 \tag{16.A1}$$

then

$$\nabla^4\Theta_0 = 0 \tag{16.A2}$$

So the crux of the issue is $y\Theta$.

$$\frac{\partial(y\Theta)}{\partial x} = y\frac{\partial\Theta}{\partial x} + \Theta\frac{\partial y}{\partial x} = y\frac{\partial\Theta}{\partial x} \tag{16.A3}$$

$$\frac{\partial^2(y\Theta)}{\partial x^2} = \frac{\partial\left(y\frac{\partial\Theta}{\partial x}\right)}{\partial x} = y\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial\Theta}{\partial x}\frac{\partial y}{\partial x} = y\frac{\partial^2\Theta}{\partial x^2} \tag{16.A4}$$

Similarly,

$$\frac{\partial(y\Theta)}{\partial y} = y\frac{\partial\Theta}{\partial y} + \Theta\frac{\partial y}{\partial y} = y\frac{\partial\Theta}{\partial y} + \Theta \tag{16.A5}$$

$$\frac{\partial^2(y\Theta)}{\partial y^2} = \frac{\partial\left(y\frac{\partial\Theta}{\partial y}\right)}{\partial y} = y\frac{\partial^2\Theta}{\partial y^2} + \frac{\partial\Theta}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial\Theta}{\partial y} = y\frac{\partial^2\Theta}{\partial y^2} + 2\frac{\partial\Theta}{\partial y} \tag{16.A6}$$

So $\nabla^2(y\Theta) = y\left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2}\right) + 2\frac{\partial\Theta}{\partial y}$ (16.A7)

The term in parentheses on the right side of the above equation is zero because Θ is harmonic (i.e., $\left(\nabla^2\Theta = 0\right)$), so

$$\nabla^2(y\Theta) = 2\frac{\partial\Theta}{\partial y} \tag{16.A8}$$

This means $\nabla^4(y\Theta) = \nabla^2\left(2\frac{\partial\Theta}{\partial y}\right)$ (16.A9)

Taking the second partial derivatives of $2\frac{\partial\Theta}{\partial y}$ yields

$$\frac{\partial\left(2\frac{\partial\Theta}{\partial y}\right)}{\partial y} = 2\frac{\partial^2\Theta}{\partial y^2} \tag{16.A10}$$

$$\frac{\partial^2\left(2\frac{\partial\Theta}{\partial y}\right)}{\partial y^2} = 2\frac{\partial^3\Theta}{\partial y^3} \qquad (16.A11)$$

and,

$$\frac{\partial\left(2\frac{\partial\Theta}{\partial y}\right)}{\partial x} = 2\frac{\partial^2\Theta}{\partial x\partial y} \qquad (16.A12)$$

$$\frac{\partial^2\left(2\frac{\partial\phi}{\partial y}\right)}{\partial x^2} = 2\frac{\partial^3\phi}{\partial y\partial x^2} \qquad (16.A13)$$

Using these results

$$\nabla^4(y\Theta) = \nabla^2\left(2\frac{\partial\Theta}{\partial y}\right) = 2\frac{\partial^3\Theta}{\partial y\partial y^2} + 2\frac{\partial^3\Theta}{\partial y\partial x^2} = 2\frac{\partial}{\partial y}\left(\frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial x^2}\right) = 0 \qquad (16.A14)$$

So $y\Theta$ is biharmonic if Θ is harmonic.

You might wonder why one would say if Θ is harmonic then $y\Theta + \Theta_0$ is biharmonic – why bother including Θ_0? We can draw an analogy with polynomial functions of a single variable. Consider the fourth derivative of $f=y^3$ with respect to y

$$\frac{d^4\left(y^3\right)}{dy^4} = \frac{d^3\left(3y^2\right)}{dy^3} = \frac{d^2\left(6y^1\right)}{dy^2} = \frac{d(6)}{dy} = 0 \qquad (16.A15)$$

If we integrated the second derivative (f"= 6y) twice to recover a general solution for $\frac{d^4 f}{dy^4}$, we would not get y^3 – we would get $y^3 + (Ay + B)$, with the term in brackets being a "function of integration". Similarly, even though $\nabla^2\left(\frac{\partial\Theta}{\partial y}\right)$ = 0, one can not just integrate $\nabla^2(y\Theta) = \frac{\partial\Theta}{\partial y}$ twice to obtain a

a general solution. A harmonic function needs to be added to obtain a general solution for $\nabla^4 f(x,y) = 0$.