MODE III FRACTURES (15)

I Main topics

A Modes of fracture

B Anti-plane strain

C Solution of the Laplace equation by functions of a complex variable

D Stress and displacement fields for a mode III fracture

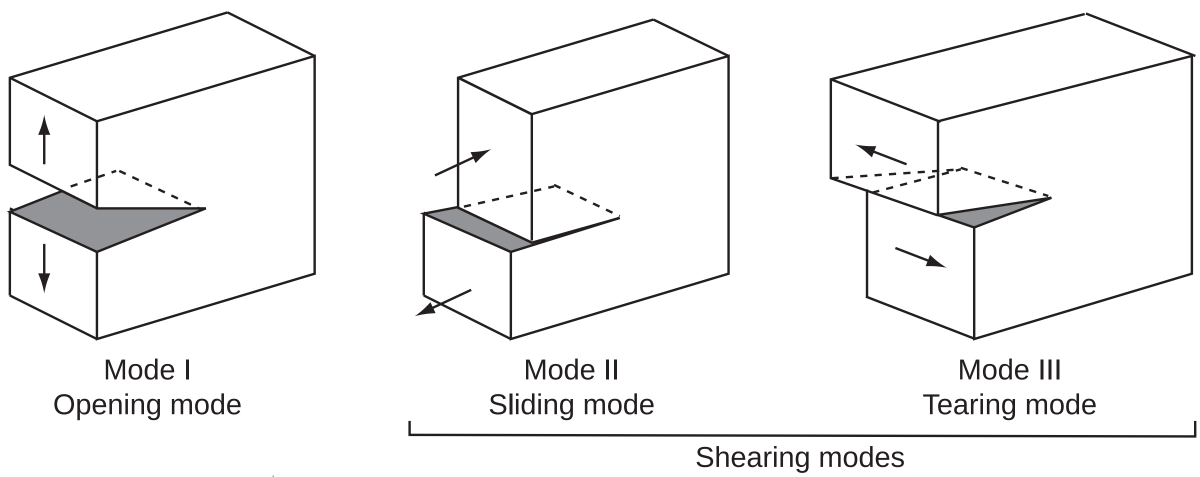
E Appendix on stress and displacement fields for all modes

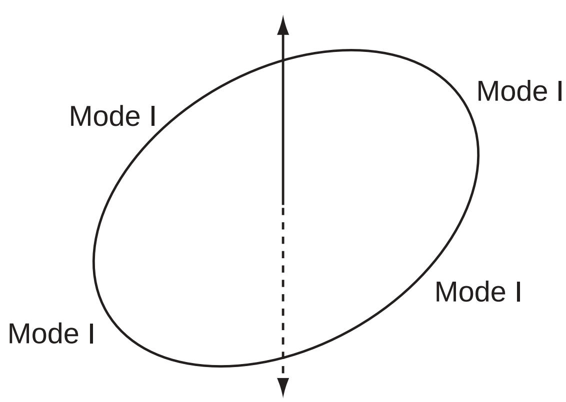
II Modes of fracture (i.e., modes of *relative* displacement)

A Mode I: Perpendicular to fracture and perpendicular to fracture front

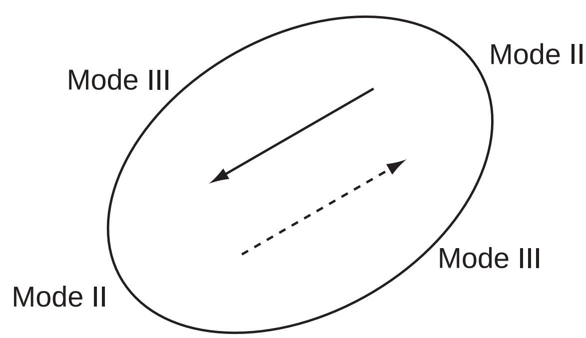
B Mode II: Parallel to fracture and perpendicular fracture front

C Mode III: Parallel to fracture and parallel to fracture front





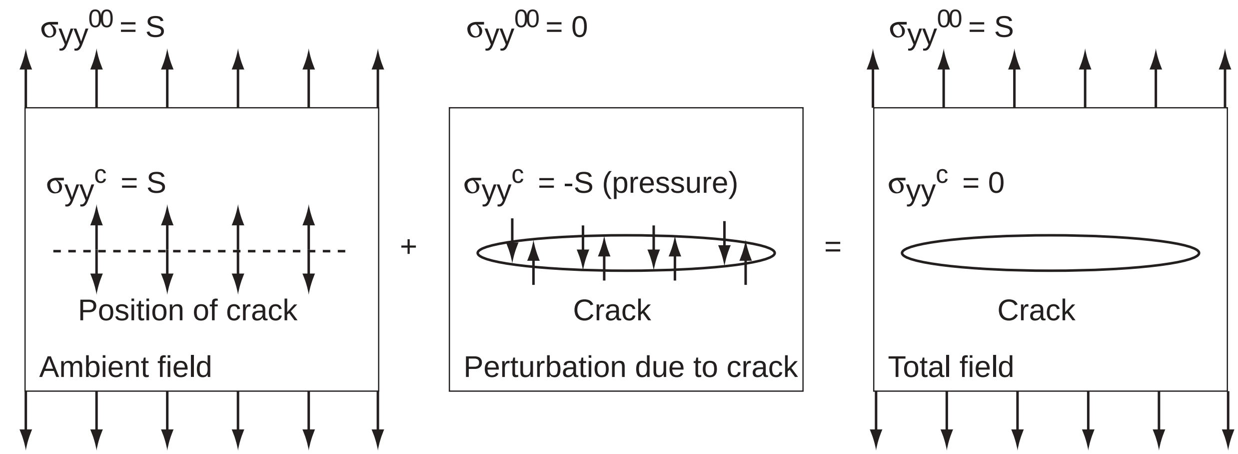
Single mode of fracture around an elliptical fracture under tension



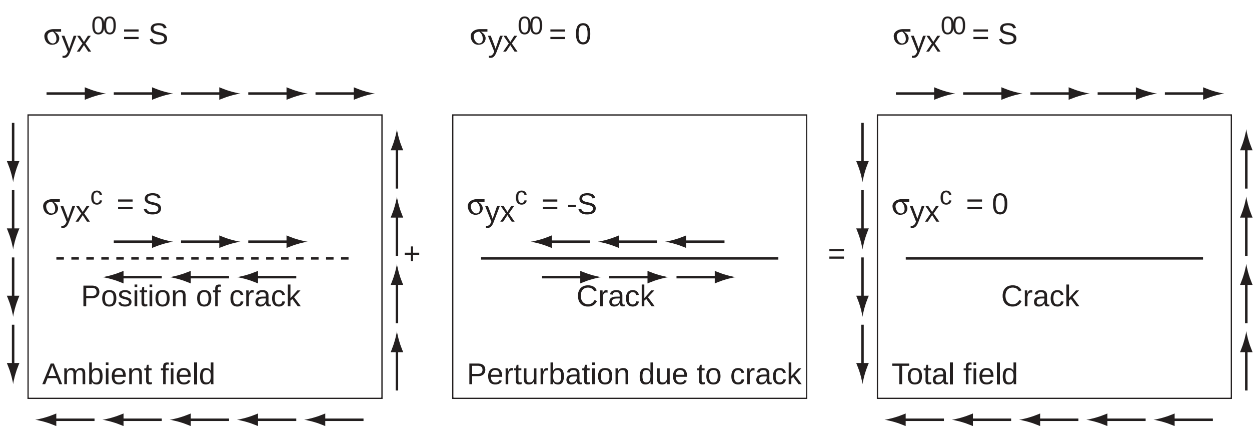
Variation in mode of fracture around an elliptical fracture under shear

Set up of Crack Problems by Perturbation of an Ambient Stress Field

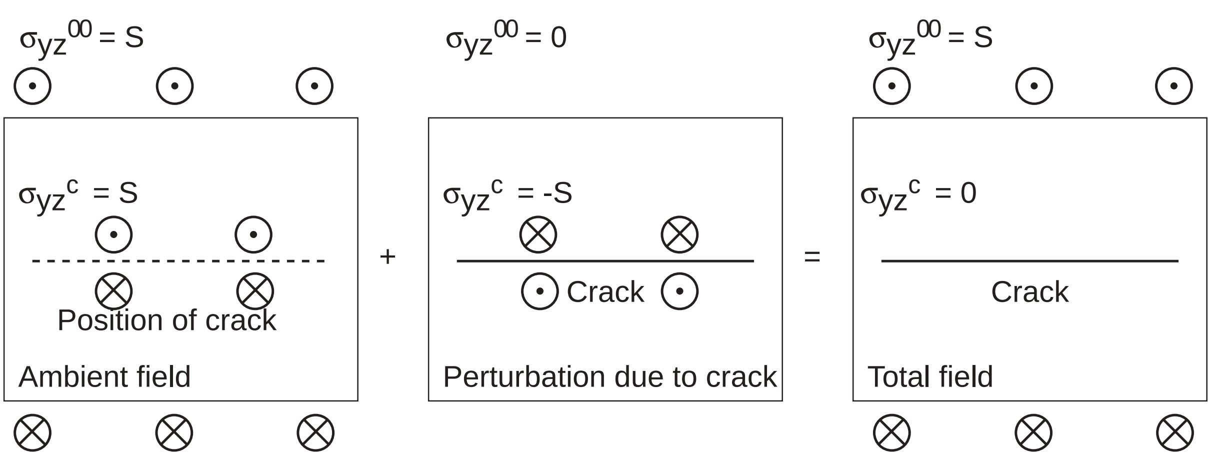
Mode I



Mode II



Mode III



III Anti-plane strain (for mode III fractures)

From course notes, ME238, Stanford, Prof. Nix, 1984

A A two-dimensional treatment where neither stresses nor displacements change in on direction (here, the z-direction).

$$\frac{\partial \sigma_{ij}}{\partial z} = 0 \tag{15.1}$$

$$\frac{\partial u_i}{\partial z} = 0 \tag{15.2}$$

B Displacements

1 $u_x = u_y = 0$

2 $u_z = f(x,y)$

C Equilibrium

According to equation (7.1c), the equation of equilibrium is

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_3 = 0 \tag{15.3}$$

In light of (15.1), if body forces are absent, then for anti-plane strain

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \tag{15.4}$$

These shear stress are related to the displacements as follows:

$$\sigma_{zx} = 2\mu\varepsilon_{zx} = 2\mu\frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \mu\frac{\partial u_z}{\partial x} \tag{15.5}$$

$$\sigma_{zy} = 2\mu\varepsilon_{zy} = 2\mu\frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \mu\frac{\partial u_z}{\partial y} \tag{15.6}$$

The term μ is the shear modulus (also designated as G). Inserting these expressions into (15.3) yields

$$\frac{\partial\left(\mu\frac{u_z}{\partial x}\right)}{\partial x} + \frac{\partial\left(\mu\frac{u_z}{\partial y}\right)}{\partial y} = 0 \tag{15.7}$$

$$\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right) = \nabla^2 u_z = 0 \tag{15.8}$$

So the displacement u_z in anti-plane strain must satisfy the Laplace eqn.

IV Solution of the Laplace equation by functions of a complex variable

The Laplace equation in two dimensions can by solved easily using functions of a complex variable. This also allows stresses and displacements in two-dimensional fracture problems (either plane strain, plane stress, or anti-plane strain) to be expressed compactly. We will consider a complex stress function (Z) of the complex variable ζ, where

$$\zeta = x + iy \tag{15.9}$$

The parameter x is the real part of ζ, and y is the imaginary part. We will use the term ζ (zeta, the Greek z) rather than z, as is customary, so as not to confuse the z-direction with the complex number x+iy.

Any <u>analytic</u> function $u_z(\zeta)$ (i.e., any function that has a derivative in the neighborhood of any point in the region of interest) of the complex variable ζ solves the Laplace equation. Here is the proof:

$$\frac{\partial u_z}{\partial x} = \frac{\partial \zeta}{\partial x}\frac{du_z}{d\zeta} = \frac{du_z}{d\zeta} \qquad Note: \; u_z = u_z(\zeta), \; \zeta = \zeta(x,y), \; and \; \frac{\partial \zeta}{\partial x} = 1 \tag{15.10}$$

$$\frac{\partial^2 u_z}{\partial x^2} = \frac{d}{d\zeta}\frac{\partial \zeta}{\partial x}\left(\frac{\partial u_z}{\partial x}\right) = \frac{d}{d\zeta}\frac{\partial \zeta}{\partial x}\left(\frac{du_z}{d\zeta}\right) = \frac{d}{d\zeta}\left(\frac{du_z}{d\zeta}\right) = \frac{d^2 u_z}{d\zeta^2} \tag{15.11}$$

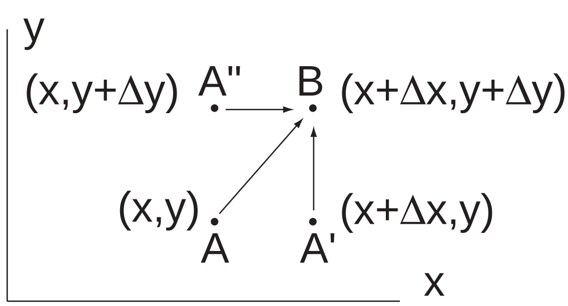
$$\frac{\partial u_z}{\partial y} = \frac{\partial \zeta}{\partial y}\frac{du_z}{d\zeta} = i\frac{du_z}{d\zeta} \qquad \frac{\partial \zeta}{\partial y} = i \tag{15.12}$$

$$\frac{\partial^2 u_z}{\partial y^2} = \frac{d}{d\zeta}\frac{\partial \zeta}{\partial y}\left(\frac{\partial u_z}{\partial y}\right) = \frac{d}{d\zeta}\frac{\partial \zeta}{\partial y}\left(i\frac{du_z}{d\zeta}\right) = \frac{d}{d\zeta}i\left(i\frac{du_z}{d\zeta}\right) = \frac{-d^2 u_z}{d\zeta^2} \tag{15.13}$$

Summing (15.13) and (15.15) proves that $\nabla^2 u_z(\zeta) = 0$, so any displacement field written as a function of ζ satisfies equilibrium conditions.

Derivatives of a complex function and the Cauchy-Riemann conditions

To obtain the strains (and hence the stresses) from the displacements, we need to know how to take derivatives of the displacement function. For a unique derivative to exist at a point, it must have the same value no matter what direction the point is approached from.



The complex variable ζ can be separated into real and imaginary components:

$\mathrm{Re}\,\zeta = x$ (15.14a) $\mathrm{Im}\,\zeta = y$ (15.14b)

And so can the complex function $\bar{Z}_{III}(\zeta)$

$$\bar{Z}_{III} = \bar{Z}_{III}(\zeta) = \mathrm{Re}\,\bar{Z}_{III} + i\,\mathrm{Im}\,\bar{Z}_{III} = \alpha(x,y) + i\beta(x,y) \quad (15.15)$$

Now we consider the derivative $d\bar{Z}_{III}(\zeta)/d\zeta$.

$$\frac{d\bar{Z}_{III}}{d\zeta} = \lim_{\Delta\zeta = \Delta x + i\Delta y \to 0} \frac{\alpha(x+\Delta x, y+\Delta y) - \alpha(x,y)}{\Delta x + i\Delta y} + i\frac{\beta(x+\Delta x, y+\Delta y) - \beta(x,y)}{\Delta x + i\Delta y} \quad (15.16)$$

Along the path from A’ to B, Δx = 0, so

$$\frac{d\bar{Z}_{III}}{d\zeta} = \lim_{\Delta\zeta = i\Delta y \to 0} \frac{\alpha(x, y+\Delta y) - \alpha(x,y)}{i\Delta y} + i\frac{\beta(x, y+\Delta y) - \beta(x,y)}{i\Delta y} = \frac{1}{i}\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial y} \quad (15.17)$$

or alternatively

$$\frac{d\bar{Z}_{III}}{d\zeta} = \frac{i}{i}\frac{1}{i}\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial y} = -i\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial y} \quad (15.18)$$

Along the path from A” to B, Δy = 0, so

$$\frac{d\bar{Z}_{III}}{d\zeta} = \lim_{\Delta\zeta = \Delta x \to 0} \frac{\alpha(x+\Delta x, y) - \alpha(x,y)}{\Delta x} + i\frac{\beta(x+\Delta x, y) - \beta(x,y)}{\Delta x} = \frac{\partial\alpha}{\partial x} + i\frac{\partial\beta}{\partial x} \quad (15.19)$$

Equating the real and imaginary terms in (15.19) and (15.20) yields the Cauchy-Riemann conditions:

$$\frac{\partial\alpha}{\partial x} = \frac{\partial\beta}{\partial y} \quad (15.20a) \qquad \frac{\partial\alpha}{\partial y} = \frac{-\partial\beta}{\partial x} \quad (15.20b)$$

Differentiating (15.20a) with respect to x and of (15.20b) with respect to y and summing shows that α satisfies the Laplace equation too. Similarly, β also satisfies the Laplace equation. So possible solutions for the Laplace equation (and hence the displacement field) can come from $\bar{Z}_{III}$, $\mathrm{Re}\,\bar{Z}_{III}$ (*i.e.*, α), and $\mathrm{Im}\,\bar{Z}_{III}$ (*i.e.*, β).

Solution for the displacement and stress field around a mode III fracture

We now consider the following displacement function

$$u_z = \frac{1}{\mu}\mathrm{Im}\bar{Z}_{III} \tag{15.21}$$

The displacement here is a shear displacement, and it should be inversely related to the shear modulus μ (see Table 9.1 in Barber). We now need to find a function that satisfies our boundary conditions. The solution for the relative displacement across a mode III fracture (see the Appendix) under a driving stress S is

$$\Delta u_{III} = \frac{2S}{\mu}\sqrt{a^2 - x^2}\,. \tag{15.22}$$

This solution is found by the method of lecture 14. By symmetry, half the relative displacement occurs on one side of the fracture, and half on the other. So

$$u_{III}(|x| < a, y = 0^+) = S\frac{\sqrt{a^2 - x^2}}{\mu} \tag{15.23}$$

This means that

$$\mathrm{Im}\bar{Z}_{III} = \beta = S\sqrt{a^2 - \zeta^2}\,. \tag{15.24}$$

This leads directly to the following solution

$$\bar{Z}_{III} = iS\sqrt{a^2 - \zeta^2} = S\sqrt{-1}\sqrt{a^2 - \zeta^2} = S\sqrt{\zeta^2 - a^2} \tag{15.25}$$

The displacements throughout the body with a mode III crack are then

$$u_Z = \frac{S}{\mu}\mathrm{Im}\sqrt{\zeta^2 - a^2} = \frac{S}{\mu}\beta \tag{15.26}$$

The nonzero stresses throughout the body come from the strains.

$$\sigma_{zx} = \mu\frac{\partial u_z}{\partial x} = \mu\frac{\partial\left(\frac{S}{\mu}\beta\right)}{\partial x} = \frac{\partial\beta}{\partial x} \tag{15.27a}$$

$$\sigma_{yz} = \mu\frac{\partial u_z}{\partial y} = \mu\frac{\partial\left(\frac{S}{\mu}\beta\right)}{\partial y} = \frac{\partial\beta}{\partial y} \tag{15.27b}$$

Applying (15.17) to (15.27a), and (15.19) to (15.27b) leads to

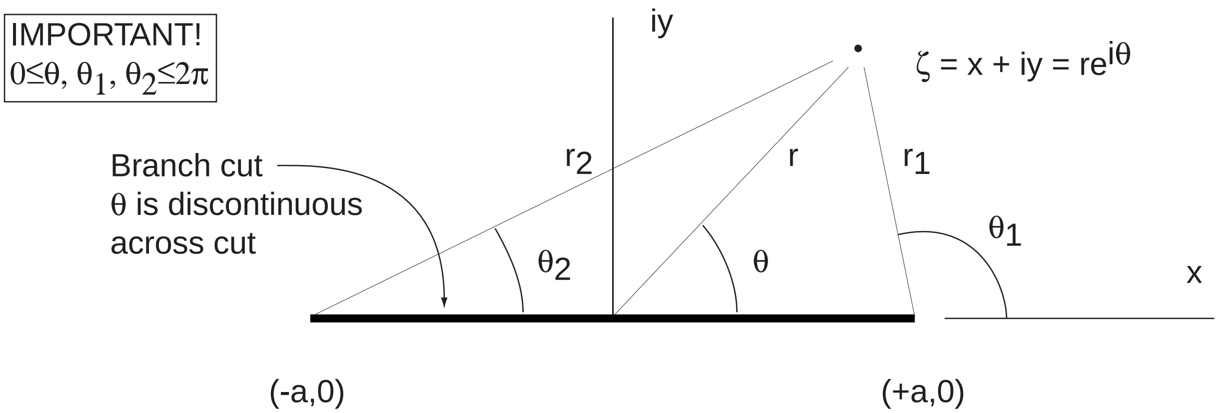
$$\sigma_{zx} = \frac{\partial\beta}{\partial x} = \mathrm{Im}\frac{d\bar{Z}_{III}}{d\zeta} \tag{15.28a}$$

$$\sigma_{yz} = \frac{\partial\beta}{\partial x} = \mathrm{Re}\frac{d\bar{Z}_{III}}{d\zeta} \tag{15.28b}$$

The solutions for the displacements and the stresses still do not lend themselves to tremendous insight, so we pursue this a bit more. The next step is to take the derivative of the function of (15.25):

$$\frac{d\overline{Z}_{III}}{d\zeta} = \frac{d\left(S\left(\zeta^2 - a^2\right)^{1/2}\right)}{d\zeta} = \frac{1}{2}S\left(\zeta^2 - a^2\right)^{-1/2} 2\zeta = \frac{S\zeta}{(\zeta + a)^{1/2}(\zeta - a)^{1/2}} \qquad (15.29)$$

At this point we switch to polar coordinates



$$\zeta = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta} \qquad (15.30)$$

$$\zeta - a = r_1(\cos\theta_1 + i\sin\theta_1) = re^{i\theta_1} \qquad (15.31)$$

$$\zeta + a = r_2(\cos\theta_2 + i\sin\theta_2) = re^{i\theta_2} \qquad (15.32)$$

Equation (15.29) can be re-written in either of the two following ways

$$\frac{d\overline{Z}_{III}}{d\zeta} = \frac{Sre^{i\theta}}{\left(r_1 e^{i\theta_1}\right)^{1/2}\left(r_2 e^{i\theta_2}\right)^{1/2}} = S\frac{r}{\sqrt{r_1 r_2}}\, e^{i\left(\theta - \frac{\theta_1+\theta_2}{2}\right)} \qquad (15.33)$$

$$\frac{d\overline{Z}_{III}}{d\zeta} = S\frac{r}{\sqrt{r_1 r_2}}\cos\left(\theta - \frac{\theta_1+\theta_2}{2}\right) + iS\frac{r}{\sqrt{r_1 r_2}}\sin\left(\theta - \frac{\theta_1+\theta_2}{2}\right) \qquad (15.34)$$

So the nonzero stresses and displacements are

$$\sigma_{zy} = \mathrm{Re}\frac{d\overline{Z}_{III}}{d\zeta} = S\frac{r}{\sqrt{r_1 r_2}}\cos\left(\theta - \frac{\theta_1+\theta_2}{2}\right) \qquad (15.35)$$

$$\sigma_{zx} = \mathrm{Im}\frac{d\overline{Z}_{III}}{d\zeta} = S\frac{r}{\sqrt{r_1 r_2}}\sin\left(\theta - \frac{\theta_1+\theta_2}{2}\right) \qquad (15.36)$$

$$u_z = \frac{1}{\mu}\mathrm{Im}\overline{Z}_{III} = \frac{1}{\mu}\mathrm{Im} S\sqrt{r_1 r_2}\, e^{i\frac{\theta_1+\theta_2}{2}} = \frac{S}{\mu}\sqrt{r_1 r_2}\sin\frac{\theta_1+\theta_2}{2} \qquad (15.37)$$

Lecture 15 Appendix

I Main topics

A Displacement and stress fields for a screw dislocation (mode III)

B Evaluating “deformation fields” for fractures of modes I, II, and III

II Displacement and stress fields for a screw dislocation (mode III)

A Displacement parallel to the dislocation edge increases uniformly along a spiral-like circuit from one side of the dislocation to the other (for a right-handed screw dislocation, point your right thumb along the dislocation edge; displacement parallel to the edge increases in the direction your fingers curl.

B Angular position: $\theta = \tan^{-1}(y/x)$

C Expressions for displacements and strains, Cartesian ref. frame

1 Cartesian displacements: $u = u_x$ $v = u_y$ $w = u_z$

2 Normal strains: $\varepsilon_{xx} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)$ $\varepsilon_{yy} = \frac{1}{2}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right)$ $\varepsilon_{zz} = \frac{1}{2}\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\right)$

3 Shear strains: $\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$ $\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$ $\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$

D Expressions for displacements and strains, cylindrical ref. frame

1 Cylindrical displacements: u_r u_θ $u_z = w$

2 Normal strains: $\varepsilon_{rr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial r} + \frac{\partial u_r}{\partial r}\right)$ $\varepsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta}\right)$ $\varepsilon_{zz} = \frac{1}{2}\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\right)$

3 Shear strains: $\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$ $\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)$ $\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$

E Comparison of screw dislocation expressions

1 Displacement

Polar coordinates	Cartesian coordinates
a $u_r = 0$	$u = 0$
b $u_\theta = 0$	$v = 0$
c $w = b\frac{\theta_z}{2\pi}$	$w = \frac{b}{2\pi}\tan^{-1}\frac{y}{x}$

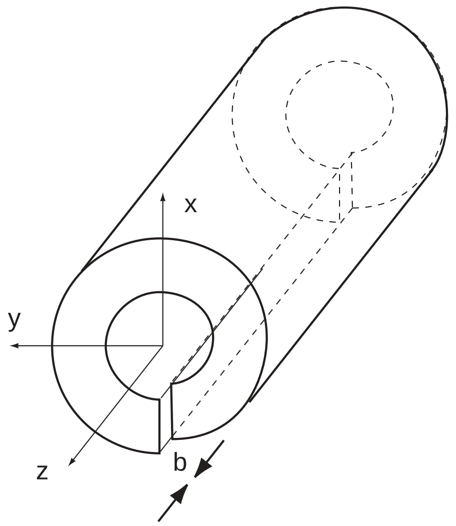
2 Strain

Polar coordinates	Cartesian coordinates
a $\varepsilon_{r\theta} = \varepsilon_{\theta r} = 0$	$\varepsilon_{xy} = \varepsilon_{yx} = 0$
b $\varepsilon_{\theta z} = \varepsilon_{z\theta} = \frac{1}{2}\frac{b}{2\pi r}$	$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\frac{b}{2\pi}\frac{x}{x^2+y^2} = \frac{1}{2}\frac{b}{2\pi}\frac{x}{r^2}$
c $u_{rz} = u_{zr} = 0$	$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\frac{-b}{2\pi}\frac{y}{x^2+y^2} = \frac{1}{2}\frac{-b}{2\pi}\frac{y}{r^2}$
d $\varepsilon_{rr} = 0$	$\varepsilon_{xx} = 0$
e $\varepsilon_{\theta\theta} = 0$	$\varepsilon_{yy} = 0$
f $\varepsilon_{zz} = 0$	$\varepsilon_{zz} = 0$

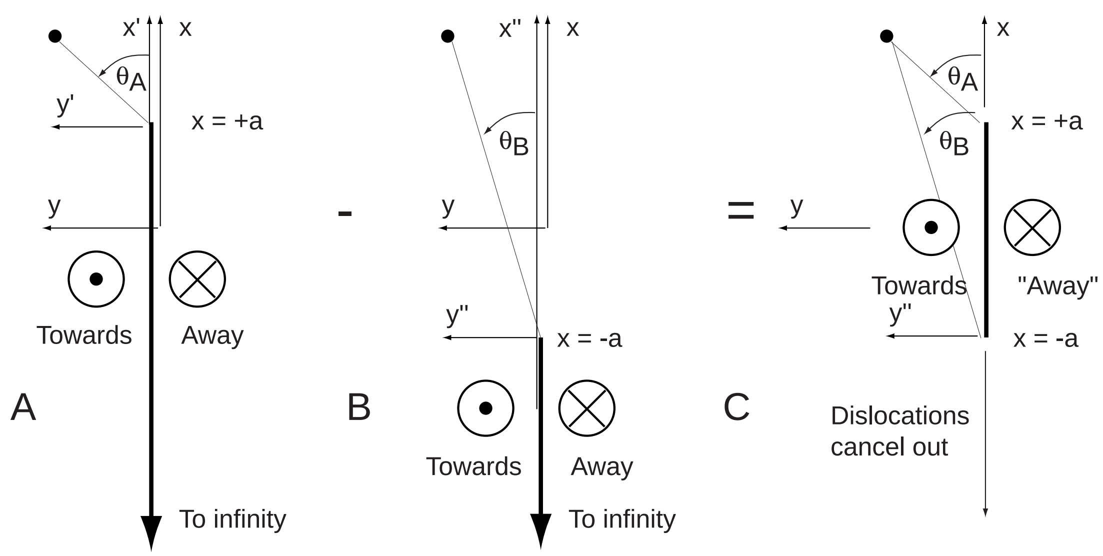
3 Stress (G = shear modulus) $\sigma_{ij} = 2G\varepsilon_{ij} \;\; if \;\; i \neq j$

Polar coordinates	Cartesian coordinates
a $\sigma_{r\theta} = \sigma_{\theta r} = 0$	$\sigma_{xy} = \sigma_{yx} = 0$
b $\sigma_{\theta z} = \sigma_{z\theta} = \frac{Gb}{2\pi r}$	$\sigma_{yz} = \sigma_{zy} = \frac{Gb}{2\pi}\frac{x}{x^2+y^2} = \frac{Gb}{2\pi}\frac{x}{r^2}$
c $\sigma_{rz} = \sigma_{zr} = 0$	$\sigma_{xz} = \sigma_{zx} = \frac{-Gb}{2\pi}\frac{y}{x^2+y^2} = \frac{-Gb}{2\pi}\frac{y}{r^2}$
d $\sigma_{rr} = 0$	$\sigma_{xx} = 0$
e $\sigma_{\theta\theta} = 0$	$\sigma_{yy} = 0$
f $\sigma_{zz} = 0$	$\sigma_{zz} = 0$

SCREW DISLOCATIONS



SUPERPOSTION OF TWO (INFINITE) SCREW DISLOCATIONS (A,B)
TO FORM A FINITE *DISPLACEMENT DISCONTINTUITY (C)*
(View along the -z direction)



$w_A = b\theta_A/(2\pi)$

$= (b/2\pi)\tan^{-1}(y'/x')$

$= (b/2\pi)\tan^{-1}(y/[x-a])$

$w_B = b\theta_B/(2\pi)$

$= (b/2\pi)\tan^{-1}(y''/x'')$

$= (b/2\pi)\tan^{-1}(y/[x+a])$

$-\pi \le \theta \le \pi$

$w_C = b(\theta_A - \theta_B)/(2\pi)$

$= (b/2\pi)\,[\tan^{-1}(y/(x-a)) - \tan^{-1}(y/(x+a))]$

$w_C\ (\theta_A = -\pi, \theta_B = 0) = -B/2$
$w_C\ (\theta_A = 0, \theta_B = 0) = 0$
$w_C\ (\theta_A = \pi, \theta_B = 0) = B/2$
$w_C\ (\theta_A = \pi, \theta_B = \pi) = 0$

III Evaluating "deformation fields" for fractures of modes I, II, and III

The solutions for the stresses, displacements, and relative displacement around fractures of modes II and III are entirely homologous to the corresponding solutions for an opening mode (mode I) fracture. Rather than rederive the entire solutions, we can demonstrate that the starting points, key steps, and results for the solutions are identical in form (see the tables on the following pages). The key differences are in the multiplicative constants (i.e., C_m values) that accompany the solutions. We will consider for each case an *isolated* 2-D fracture in a body with no stresses at an infinite distance from the fracture. The relative displacement of the fracture walls is driven by a uniform stress +S acting on the walls of a fracture. The fracture extends from x = -a to x = +a, so the length *2a* is the *short* dimension of the fracture (the longest dimension is infinite!). The diagram below shows the case for mode III.

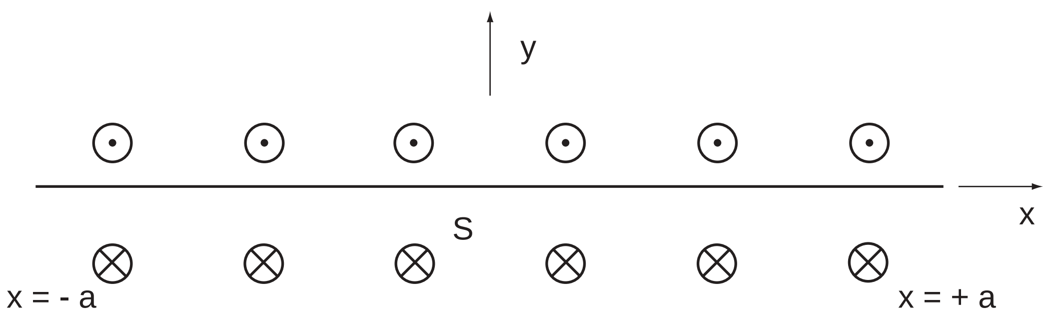


Table 1. Stresses around a 2-D dislocation (critical steps are boxed)

Mode I	Mode II	Mode III
$\sigma_{rr} = \frac{C_I \cos\theta}{r}$	$\sigma_{rr} = \frac{C_{II} \sin\theta}{r}$	$**\sigma_{rr} = 0$
$\boxed{\sigma_{\theta\theta} = \frac{C_I \cos\theta}{r}}$	$\sigma_{\theta\theta} = \frac{C_{II} \sin\theta}{r}$	$**\sigma_{\theta\theta} = 0$
$\sigma_{zz}^{plane\ \ stress} = 0$ $\sigma_{zz}^{plane\ \ strain} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$	$\sigma_{zz}^{plane\ \ stress} = 0$ $\sigma_{zz}^{plane\ \ strain} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$	$**\sigma_{zz} = 0$
$\sigma_{\theta r} = \frac{C_I \sin\theta}{r}$	$\boxed{\sigma_{\theta r} = \frac{-C_{II} \cos\theta}{r}}$	$**\sigma_{\theta r} = 0$
$\sigma_{\theta z} = 0$	$\sigma_{\theta z} = 0$	$\boxed{\sigma_{\theta z} = \frac{C_{III}}{r}}$
$\sigma_{zr} = 0$	$\sigma_{zr} = 0$	$**\sigma_{zr} = 0$
$\boxed{\sigma_{yy}(\theta = 0) = \frac{C_I}{x}}$	$\sigma_{yy}(\theta = 0) = 0$	$*\sigma_{yy}(\theta = 0) = 0$
$\sigma_{yx}(\theta = 0) = 0$	$\boxed{\sigma_{yx}(\theta = 0) = \frac{-C_{II}}{x}}$	$**\sigma_{yx}(\theta = 0) = 0$
$*\sigma_{yz}(\theta = 0) = 0$	$*\sigma_{yz}(\theta = 0) = 0$	$\boxed{\sigma_{yz}(\theta = 0) = \frac{C_{III}}{x}}$
$\boxed{C_I = \frac{-2\mu B_y}{\pi(\kappa + 1)}}$	$\boxed{C_{II} = \frac{-2\mu B_x}{\pi(\kappa + 1)}}$	$\boxed{C_{III} = \frac{\mu B_z}{2\pi}}$

* From plane strain or plane stress constraint

** From anti-plane strain constraint

The C-values in the last row produce a positive displacement discontinuity (i.e., u(y=0+) - u(y=0-) > 0)

$\kappa = 3 - 4\nu$ for plane strain (κ=2 for ν=0.25),

and

$\kappa = (3 - \nu)/(1 + \nu)$ for plane stress (κ=2.2 for ν=0.25).

Table 2. Boundary conditions on fracture walls

Mode I	Mode II	Mode III
$\boxed{\sigma_{yy}{}^c = S;\ \lvert x\rvert < a, y = 0}$	$\boxed{\sigma_{yx}{}^c = S;\ \lvert x\rvert < a, y = 0}$	$\boxed{\sigma_{yz}{}^c = S;\ \lvert x\rvert < a, y = 0}$
$\sigma_{yx}{}^c = 0;\ \lvert x\rvert < a, y = 0$	$\sigma_{yy}{}^c = 0;\ \lvert x\rvert < a, y = 0$	$**\sigma_{yy}{}^c = 0;\ \lvert x\rvert < a, y = 0$
$*\sigma_{yz}{}^c = 0;\ \lvert x\rvert < a, y = 0$	$*\sigma_{yz}{}^c = 0;\ \lvert x\rvert < a, y = 0$	$**\sigma_{yz}{}^c = 0;\ \lvert x\rvert < a, y = 0$

Table 3. Key steps in determining the relative displacement across a fracture

Mode I	Mode II	Mode III
Integral equation for boundary condition $S=\int_{-a}^{+a}\frac{C_I}{B_y}B'_y(\xi)d\xi$	Integral equation for boundary condition $S=\int_{-a}^{+a}\frac{C_{II}}{B_x}B'_x(\xi)d\xi$	Integral equation for boundary condition $S=\int_{-a}^{+a}\frac{C_{III}}{B_z}B'_z(\xi)d\xi$
Constant from dislocation solution $C_I=\frac{-2\mu B_y}{\pi(\kappa+1)}$	Constant from dislocation solution $C_{II}=\frac{-2\mu B_x}{\pi(\kappa+1)}$	Constant from dislocation solution $C_{III}=\frac{\mu B_z}{2\pi}$
Unit-strength constant $C_I{}^*=\frac{-2\mu}{\pi(\kappa+1)}$	Unit-strength constant $C_{II}{}^*=\frac{-2\mu B_x}{\pi(\kappa+1)}$	Unit-strength constant $C_{III}{}^*=\frac{\mu B_z}{2\pi}$
$S=\int_{-a}^{+a}C_I{}^*B'_y(\xi)d\xi$	$S=\int_{-a}^{+a}C_{II}{}^*B'_x(\xi)d\xi$	$S=\int_{-a}^{+a}C_{III}{}^*B'_z(\xi)d\xi$
Dislocation derivative distribution that yield boundary condition $B'_y=\frac{-\pi S\xi}{C_I{}^*\sqrt{a^2-\xi^2}}$	Dislocation derivative distribution that yield boundary condition $B'_x=\frac{-\pi S\xi}{C_{II}{}^*\sqrt{a^2-\xi^2}}$	Dislocation derivative distribution that yield boundary condition $B'_z=\frac{-\pi S\xi}{C_{III}{}^*\sqrt{a^2-\xi^2}}$
Equation to solve for relative displacement of fracture walls $\Delta u_y(x)=\int_{-a}^{x}B'_y(\xi)d\xi$	Equation to solve for relative displacement of fracture walls $\Delta u_x(x)=\int_{-a}^{x}B'_x(\xi)d\xi$	Equation to solve for relative displacement of fracture walls $\Delta u_z(x)=\int_{-a}^{x}B'_z(\xi)d\xi$
$\Delta u_I=\frac{-\pi S\sqrt{a^2-x^2}}{C_I{}^*}$	$\Delta u_{II}=\frac{-\pi S\sqrt{a^2-x^2}}{C_{II}{}^*}$	$\Delta u_{III}=\frac{-\pi S\sqrt{a^2-x^2}}{C_{III}{}^*}$
Relative displacement of fracture walls $\Delta u_I=\frac{-S(\kappa+1)\sqrt{a^2-x^2}}{2\mu}$	Relative displacement of fracture walls $\Delta u_{II}=\frac{-S(\kappa+1)\sqrt{a^2-x^2}}{2\mu}$	Relative displacement of fracture walls $\Delta u_{III}=\frac{2S\sqrt{a^2-x^2}}{\mu}$

The equations below yield the "deformation fields" around a 2-D fracture of any mode. The terms $\sigma_{ij}(z-\xi,y)$ and $u_i(z-\xi,y)$ are Green's functions.

For stresses	For displacements
$\sigma_{ij}(x,y)=\int_{-a}^{+a}B'(\xi)\sigma_{ij}(x-\xi,y)d\xi$	$u_i(x,y)=\int_{-a}^{+a}B'(\xi)u_i(x-\xi,y)d\xi$