DISPLACEMENT DISCONTINUITIES AND FRACTURES (14)

I Main topics

A Road map

B Displacement discontinuities

C Dislocation derivatives (see Barber 13.2.3-13.3.2)

D Solution for an opening mode fracture (see Barber, 13.3.2)

II Road map

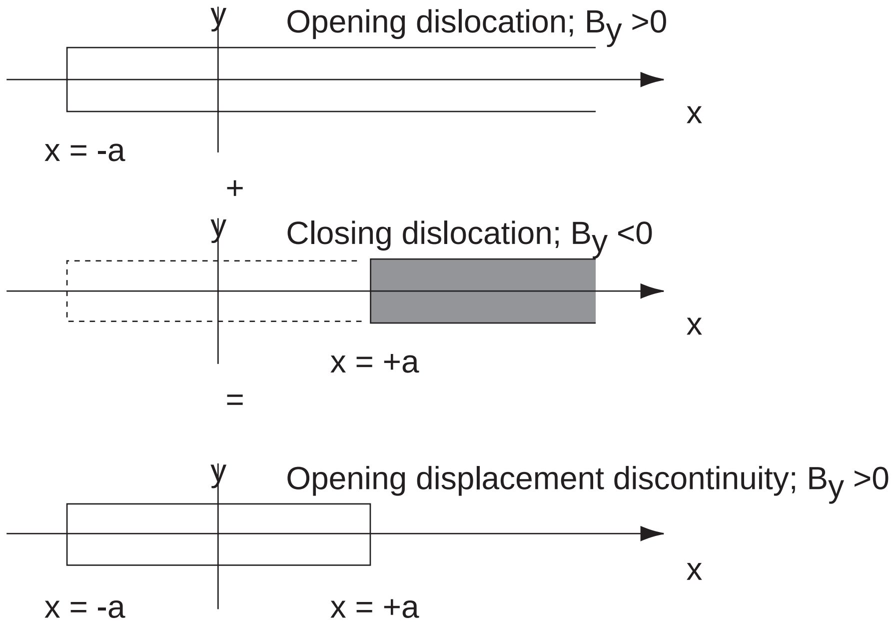
At this point we have solutions for 2-D dislocations, fundamental building blocks for fractures. Before we can build fractures with them, we need to address how a cut that is infinitely long can be used to construct fractures of finite dimensions. We now address two different but closely related ways to “build” fractures. The former is best suited for numerical solutions and is more intuitive. The latter we use for an analytical solution.

III Displacement discontinuities

In the solution for a dislocation (lecture 13), tractions were applied to make the dislocation open. We could have applied tractions of an opposite sign to make the walls of the dislocation interpenetrate; this is mathematically permitted. Although this may seem physically impossible, this sort of solution has been applied to predict the mechanical response of rock around stylolites or dissolution surfaces (see Pollard and Segall, 1989, in the book "Fracture Mechanics of Rock").

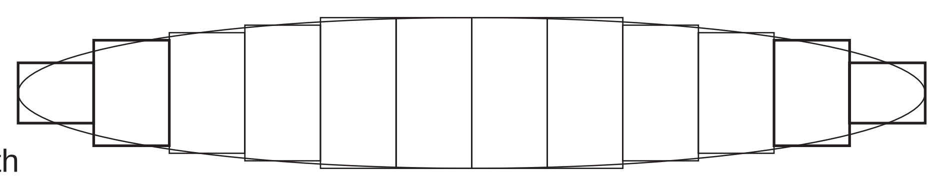
The solution of lecture 13 for a dislocation might be used to model the stress and displacement fields near the tip of a very long dike, but suppose we are interested in the displacement field around a dike of finite length; what do we do? Consider the deformation fields around the two dislocations on the top of the next page, one being an opening dislocation with its end at x= -a, the other being an interpenetrating dislocation of opposite strength with its end at x= +a.

Construction of a displacement discontinuity

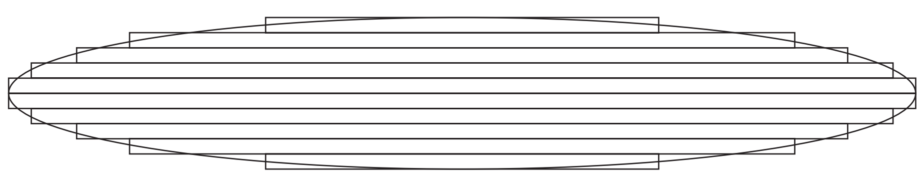


For x>+a, the dislocations annihilate each other, leaving a displacement discontinuity of strength B_y of finite length that extends from x = -a to x = +a. For |x| > a, there is no displacement discontinuity; stresses, strains, and displacements are continuous across the plane of the discontinuity for |x| > a. In contrast, these terms are neither continuous nor defined for x>a, y=0 for the two individual dislocations. Numerical round-off errors can cause solutions with dislocations to “blow up” along the plane of a crack built with dislocation pairs. The appendix for this lecture has plane strain solutions for stresses and displacements around a displacement discontinuity in terms of an x,y reference frame. The diagram below shows how a fracture can be built from displacement discontinuities.

A Displacement discontiuities of constant length but varying strength



B Displacement discontiuities of constant strength but varying length



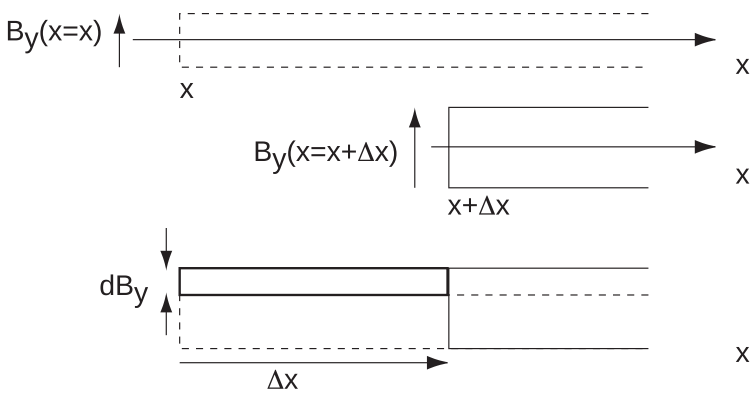
IV Dislocation derivatives (see Barber 13.2.3-13.3.2)

The dislocation derivative is the rate at which a dislocation with a potentially variable relative displacement changes value with respect to position along the dislocation.

$$B'_y = \frac{dB_y}{dx} \tag{14.1}$$

The dislocation strength B_y is the gap of an opening mode dislocation and has dimensions of length. The dislocation derivative magnitude B'_y is dimensionless. Barber uses B_y to represent both the strength of an opening mode dislocation (p. 169) *and* the dislocation derivative magnitude (see his equation 13.27, and examine the dimensions of the terms). To avoid potential confusion, the dislocation derivative will be given a prime symbol here: B'_y.

Construction of a dislocation derivative (B'_y)



$$B'_y = \lim_{x \to 0} \frac{B_y(x+\Delta x) - B_y(x)}{\Delta x} = \frac{dB_y}{dx}$$

or

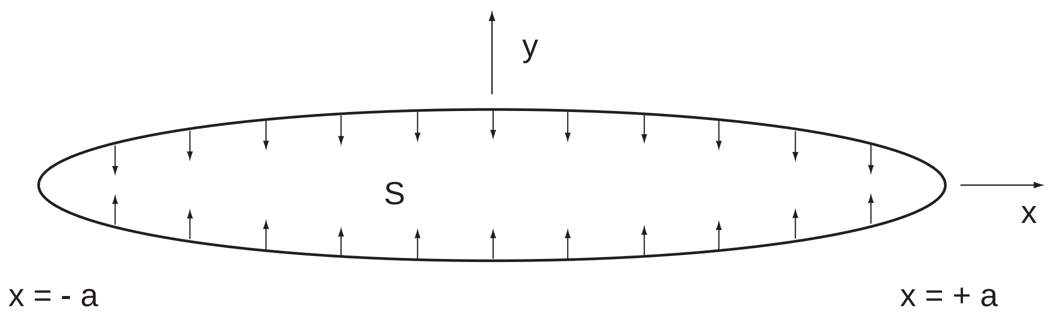
$$B'_y\, dx = B_y(x+dx) - B_y(x) = dB_y$$

In the limit, a distribution of the dislocation derivatives over the interval dx is like an infinitesimal displacement discontintuity of strength dB_y and length dx.

The dislocation derivatives effectively convert the dislocations in (B) on the previous page to displacement discontinuities in (A). A distribution of dislocation derivatives along the infinitesimal interval dx is like an infinitesimally long displacement discontinuity of strength dB_y. We will integrate these derivatives to obtain the stress field around a crack.

V Solution for an opening mode fracture (see Barber, 13.3.2)

Now suppose we wish to investigate the stresses and displacements around an opening mode fracture with a variable aperture. We will consider the case of a fracture with a normal stress S acting on the wall of the fracture that extends from x = -a to x = +a. The following figure illustrates the geometry of the problem:



Under the application of a positive (tensile) stress S along the walls, the fracture walls would interpenetrate. For negative S, the walls would open.

The boundary conditions are

$$\sigma_{yy} = S; \ |x| < a, y = 0 \quad (14.2)$$

$$\sigma_{yx} = 0; \ |x| < a, y = 0 \quad (14.3)$$

$$\sigma_{yy} = 0; \ r \to \infty \quad (14.4)$$

$$\sigma_{xx} = 0; \ r \to \infty \quad (14.5)$$

$$\sigma_{yx} = 0; \ r \to \infty \quad (14.6)$$

The stresses due to a line load die off as 1/r, so summing the contributions of tractions over the finite length of the crack will automatically satisfy (14.4)-(14.6). We focus attention then on the boundary conditions along the crack.

We start with a dislocation building block. From (13.31) and (13.36) the normal stress acting on the plane of the dislocation, where r = x and θ = 0, is:

$$\sigma_{yy}(y=0) = \frac{-2\mu B_y}{\pi(\kappa+1)x} \quad (14.7)$$

$$\sigma_{yx}(y=0) = 0 \quad (14.8)$$

where

$\kappa = 3 - 4\nu$ (=2 for ν=0.25) for plane strain, and

$\kappa = (3-\nu)/(1+\nu)$ (=2.2 for ν=0.25) for plane stress.

For a dislocation with its end at x = x*, the stresses at "x" are

$$\sigma_{yy}(y=0) = \frac{-2\mu B_y}{\pi(\kappa+1)(x-x^*)} \tag{14.9}$$

$$\sigma_{yx}(y=0) = 0 \tag{14.10}$$

The incremental stress contribution $d\sigma_{yy}$ associated with an incremental distribution of dislocation derivatives over the interval from x = ξ to x = ξ+Δξ is equivalent to a displacement discontinuity of strength dB_y along this interval

$$d\sigma_{yy}(y=0) = \frac{-2\mu dB_y}{\pi(\kappa+1)(x-\xi)} \tag{14.11}$$

$$d\sigma_{yx}(y=0) = 0 \tag{14.12}$$

or

$$d\sigma_{yy}(y=0) = \frac{-2\mu B_y'(\xi)d\xi}{\pi(\kappa+1)(x-\xi)} \tag{14.13}$$

$$d\sigma_{yx}(y=0) = 0 \tag{14.14}$$

By integrating all these contributions along the length of the crack (i.e., from ξ = -a to ξ = +a) the total shear stress will be zero, and the normal stress acting on the crack plane at position "x" is:

$$\sigma_{yy}(x) = \int_{-a}^{+a} \frac{\left[\frac{-2\mu}{\pi(\kappa+1)}\right] B_y'(\xi)d\xi}{(x-\xi)} = S \tag{14.15}$$

This is an integral equation: the value of the integral is known (it is S for |x|<a), but the dislocation derivative distribution in the integrand is unknown. Some imaginative moves let us solve this. We start by switching the constants around in (14.15) to solve for -π rather than S.

$$\sigma_{yy} = \int_{-a}^{+a} \frac{\left[\frac{2\mu}{S(\kappa+1)}\right] B_y'(\xi)d\xi}{(x-\xi)} = -\pi \qquad \text{for } -a<x<+a \tag{14.16}$$

A trigonometric substitution makes this easier to solve

$$x = a\cos\phi; \quad \xi = a\cos\theta; \qquad d\xi = -a\sin\theta d\theta; \tag{14.17}$$

So (14.16) becomes

$$\int_{\pi}^{0} \frac{\left[\frac{2\mu}{S(\kappa+1)}\right] B_y'(\theta)(-\sin\theta d\theta)}{(\cos\phi-\cos\theta)} = \int_{0}^{\pi} \frac{\left[\frac{2\mu}{S(\kappa+1)}\right] B_y'(\theta)(\sin\theta d\theta)}{(\cos\phi-\cos\theta)} = -\pi \qquad \text{for } 0<\phi<\pi \tag{14.18}$$

Now comes a subtle part. The -π term on the right side can be rewritten:

$$\int_{0}^{\pi} \frac{\cos\theta d\theta}{(\cos\phi-\cos\theta)} = -\pi \qquad \text{for } 0<\phi<\pi \tag{14.19}$$

so

$$\int_0^{\pi} \frac{\left[\frac{2\mu}{S(\kappa+1)}\right] B_y'(\theta)\sin\theta d\theta}{(\cos\phi - \cos\theta)} = \int_0^{\pi} \frac{\cos\theta d\theta}{(\cos\phi - \cos\theta)} \quad \text{for } 0<\phi<\pi \tag{14.20}$$

This equation is solved if the numerators of the integrands are equal, so

$$\left[\frac{2\mu}{S(\kappa+1)}\right] B_y'(\theta)\sin\theta = \cos\theta \tag{14.21}$$

Solving now for B'

$$B_y' = \frac{S(\kappa+1)\cos\theta}{2\mu\sin\theta} = \frac{S(\kappa+1)\cos\theta}{2\mu\sqrt{1-\cos^2\theta}} = \frac{S(\kappa+1)\left(\frac{\xi}{a}\right)}{2\mu\sqrt{1-\cos^2\left(\frac{\xi}{a}\right)}} = \frac{S(\kappa+1)\xi}{2\mu\sqrt{a^2-\xi^2}} \tag{14.22}$$

This can be integrated, in the direction of increasing x, starting at one end of the fracture where the aperture is zero, to give the aperture along the fracture:

$$\Delta u_y = \int_{-a}^{x} B'(\xi)d\xi = \int_{-a}^{x} \frac{S(\kappa+1)\xi d\xi}{2\mu\sqrt{a^2-\xi^2}} = \left.\frac{-S(\kappa+1)\sqrt{a^2-\xi^2}}{2\mu}\right|_{-a}^{x} = \frac{-S(\kappa+1)\sqrt{a^2-x^2}}{2\mu} \tag{14.23}$$

This is the equation of an ellipse. The aperture is zero at both ends of the fracture (x=±a), satisfying the closer criterion of (13.35) of Barber. If S is negative (i.e., a pressure) then (14.23) is the opening profile of a pressurized fracture.

Equation (14.21) can also be used to find the other stress components

$$\sigma_{ij}(x,y) = \int_{-a}^{+a} B'(\xi)\sigma_{ij}(x-\xi,y)d\xi \tag{14.24}$$

and the displacements

$$u_i(x,y) = \int_{-a}^{+a} B'(\xi)u_i(x-\xi,y)d\xi \tag{14.25}$$

where

$\sigma_{ij}(x-\xi,y)$

is the effect of a displacement discontinuity of unit strength at x = ξ and length $d\xi$ on the stress components at a point (x,y),

$u_i(x-\xi,y)$

is the effect of a dislocation of unit strength on the displacement components at a point (x,y), and

$B'(\xi)d\xi$

is the effective strength of the displacement discontinuity in terms of dislocation derivatives.

Appendix

The stress and displacement fields for the opening-mode displacement discontinuity are determined by superposition. To do this it will be convenient to convert the stresses and displacements from the polar frame of reference to an x,y frame. One way we could do it is to convert the stress function polar terms to xy terms

(14.A1) $\phi = C_3 r \ln r (\cos\theta)$

(14.A2) $\phi = C_3\left(x^2 + y^2\right)^{1/2} \ln\left(x^2 + y^2\right)^{1/2} (\cos\frac{x}{r})$

but taking the second derivatives of this would be a bit messy.

An easier alternative is to convert the stress solutions:

(14.A3) $\sigma_{rr} = r^{-1}(C_3 \cos\theta) = r^{-1}(C_3 \frac{x}{r}) = C_3 \frac{x}{r^2}$

(14.A4) $\sigma_{r\theta} = r^{-1}(C_3 \sin\theta) = r^{-1}(C_3 \frac{y}{r}) = C_3 \frac{y}{r^2}$

(14.A5) $\sigma_{\theta\theta} = r^{-1}(C_3 \cos\theta) = r^{-1}(C_3 \frac{x}{r}) = C_3 \frac{x}{r^2}$

Recalling that

(14.A6) $\sigma_{ij} = n_{ip} n_{jq} \sigma_{pq}$

and noting that

(14.A7a) $n_{xr} = \cos\theta,$

(14.A7b) $n_{x\theta} = -\sin\theta,$

(14.A7c) $n_{yr} = \sin\theta,$

(14.A7d) $n_{y\theta} = \cos\theta$

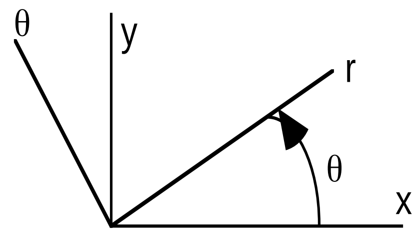
we obtain

(14.A8) $\sigma_{xx} = n_{xr} n_{xr} \sigma_{rr} + n_{xr} n_{x\theta} \sigma_{r\theta} + n_{x\theta} n_{xr} \sigma_{\theta r} + n_{x\theta} n_{x\theta} \sigma_{\theta\theta}$

(14.A9) $\sigma_{xy} = n_{xr} n_{yr} \sigma_{rr} + n_{xr} n_{y\theta} \sigma_{r\theta} + n_{x\theta} n_{yr} \sigma_{\theta r} + n_{x\theta} n_{y\theta} \sigma_{\theta\theta}$

(14.A10) $\sigma_{yx} = \sigma_{xy}$

(14.A11) $\sigma_{yy} = n_{yr} n_{yr} \sigma_{rr} + n_{yr} n_{y\theta} \sigma_{r\theta} + n_{y\theta} n_{yr} \sigma_{\theta r} + n_{y\theta} n_{y\theta} \sigma_{\theta\theta}$



Substituting in for the direction cosines

(14.A12) $\sigma_{xx} = \cos^2\theta\sigma_{rr} + \cos\theta(-\sin\theta)\sigma_{r\theta} + (-\sin\theta)\cos\theta\sigma_{\theta r} + \sin^2\theta\sigma_{\theta\theta}$

(14.A13) $\sigma_{xy} = \cos\theta\sin\theta\sigma_{rr} + \cos\theta\cos\theta\sigma_{r\theta} + (-\sin\theta)\sin\theta\sigma_{\theta r} + (-\sin\theta)\cos\theta\sigma_{\theta\theta}$

(14.A14) $\sigma_{yx} = \sigma_{xy}$

(14.A15) $\sigma_{yy} = \sin^2\theta\sigma_{rr} + \sin\theta\cos\theta\sigma_{r\theta} + \cos\theta\sin\theta\sigma_{\theta r} + \cos^2\theta\sigma_{\theta\theta}$

Now we substitute the expressions for the stresses in polar form, the expression for C_3, and substitute x/r = cosθ and y/r = sinθ to obtain

(14.A16) $\sigma_{xx} = C_3 x \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}$ or $\sigma_{xx} = \frac{-2\mu B_y}{\pi(\kappa+1)} x \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}$

(14.A17) $\sigma_{xy} = C_3 y \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}$ or $\sigma_{xy} = \frac{-2\mu B_y}{\pi(\kappa+1)} y \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}$

(14.A18) $\sigma_{yx} = \sigma_{xy}$

(14.A19) $\sigma_{yy} = C_3 x \frac{x^2 + 3y^2}{\left(x^2 + y^2\right)^2}$ or $\sigma_{yy} = \frac{-2\mu B_y}{\pi(\kappa+1)} x \frac{x^2 + 3y^2}{\left(x^2 + y^2\right)^2}$

These are the stresses for an opening mode dislocation that extends along the x-axis from x = 0 to x = ∞. The term 'x' in the above expressions thus refers to the x-distance from the end of the dislocation to the point (x,y).

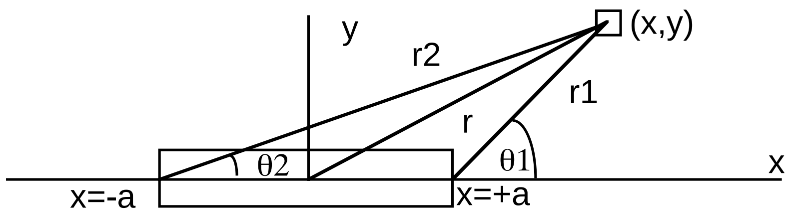
In order to get the stress fields for a dislocation discontinuity we superpose the field from a dislocation of strength B_y with its end at x = -a and one of strength $-B_y$ with its end at x = +a.

(14.A20) $$\sigma_{xx} = \frac{-2\mu D_y}{\pi(\kappa+1)}\left[(x+a)\frac{(x+a)^2 - y^2}{\left((x+a)^2 + y^2\right)^2} - (x-a)\frac{(x-a)^2 - y^2}{\left((x-a)^2 + y^2\right)^2}\right]$$

(14.A21) $$\sigma_{xy} = \frac{-2\mu D_y}{\pi(\kappa+1)}\left[y\frac{(x+a)^2 - y^2}{\left((x+a)^2 + y^2\right)^2} - y\frac{(x-a)^2 - y^2}{\left((x-a)^2 + y^2\right)^2}\right]$$

(14.A22) $\sigma_{yx} = \sigma_{xy}$

(14.A23) $$\sigma_{yy} = \frac{-2\mu D_y}{\pi(\kappa+1)}\left[(x+a)\frac{(x+a)^2 + 3y^2}{\left((x+a)^2 + y^2\right)^2} - (x-a)\frac{(x-a)^2 + 3y^2}{\left((x-a)^2 + y^2\right)^2}\right]$$



$-\pi < \theta \leq \pi$

Note that for y=0, $\sigma_{xx} = \sigma_{yy}$, and σ_{xy} = 0.

Omitting the tedious algebra, the plane strain stresses can also be written as (Crouch and Starfield, 1983):

(14.A24) $\sigma_{xx} = 2\mu B_x\left[+2 f_{,xy} + yf_{,xyy}\right] + 2\mu B_y\left[f_{,yy} + yf_{,yyy}\right]$

(14.A25) $\sigma_{xy} = 2\mu B_x\left[f_{,yy} + yf_{,yyy}\right] + 2\mu B_y\left[-yf_{,xyy}\right]$

(14.A24) $\sigma_{yy} = 2\mu B_x\left[-yf_{,xyy}\right] + 2\mu B_y\left[f_{,yy} - yf_{,yyy}\right]$

The plain strain displacements are (Crouch and Starfield, 1983):

(14.A27) $u_x = B_x\left[2(1-\nu) f_{,y} - yf_{,xx}\right] + B_y\left[-(1-2\nu) f_{,x} - yf_{,xy}\right]$

(14.A28) $u_y = B_x\left[(1-2\nu) f_{,x} - yf_{,xy}\right] + B_y\left[2(1-\nu) f_{,y} - yf_{,yy}\right]$

where

(14.A29)
$$f(x,y) = \frac{-1}{4\pi(1-\nu)}\left[y\left(\tan^{-1}\frac{y}{x-a} - \tan^{-1}\frac{y}{x+a}\right)\right] - \frac{1}{4\pi(1-\nu)}\left[-(x-a)\ln[(x-a)^2+y^2] + (x+a)\ln[(x+a)^2+y^2]\right]$$

(14.A30)
$$f_{,xyy} = -f_{,xxx} = \frac{+1}{4\pi(1-\nu)}\left[\frac{(x-a)^2 - y^2}{\left\{(x-a)^2+y^2\right\}^2} - \frac{(x+a)^2 - y^2}{\left\{(x+a)^2+y^2\right\}^2}\right]$$

(14.A31)
$$f_{,yyy} = -f_{,xxy} = \frac{+2y}{4\pi(1-\nu)}\left[\frac{x-a}{\left\{(x-a)^2+y^2\right\}^2} - \frac{x+a}{\left\{(x+a)^2+y^2\right\}^2}\right]$$

Note that the terms in the expressions here correspond to distances, angles, or trigonometric functions of angles relative to the ends of a displacement discontinuity.