DISLOCATIONS (13)

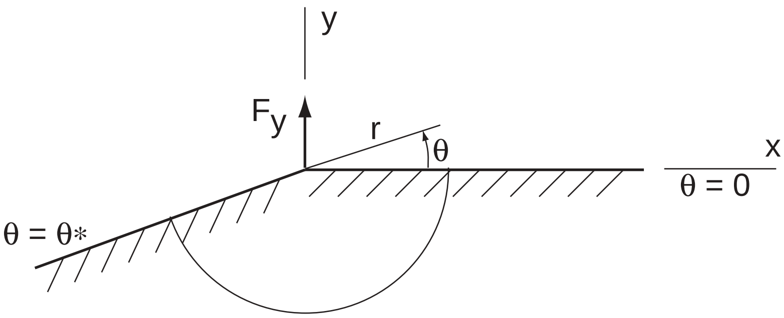
I Main topics

A Point (line) force on a half space (see Barber 12.1-12.3)

B Dislocations (see Barber, 13.2)

A Point force on a half space (Flamant solution)

The solution for the stresses due to surface loads (e.g., houses, volcanoes, glaciers, etc.) is one of the most important in elasticity theory. Consider the surface of a wedge with a 2-D line load at its hinge:



By integrating the tractions along the arc below the surface we can get the total force acting down. This must be enough to exactly balance the force Fy acting up on the surface. In other words

$$\int_{\theta^* r}^{2\pi r} t_y \, ds = -F_y \tag{13.1}$$

The length s of the arc (i.e., its 2-D "area") is $(2\pi-\theta^*)r$. This force balance must hold for an arc of any radius greater than zero. **Thus the tractions (and hence the stresses) must decrease as 1/r for point (line) forces** because the area they are applied over increases as r. The same conclusion holds for a force in the x-direction. A review of Table 8.1 of Barber shows that a stress function that yields stresses that decay as 1/r must be of the following form:

$$\phi = C_1 r\theta \sin\theta + C_2 r\theta \cos\theta + C_3 r(\ln r)\cos\theta + C_4 r(\ln r)\sin\theta \tag{13.2}$$

The associated stresses for all r > 0 are:

$$\sigma_{rr} = r^{-1}(2C_1\cos\theta - 2C_2\sin\theta + C_3\cos\theta + C_4\sin\theta) \tag{13.3}$$

$$\sigma_{r\theta} = r^{-1}\left(C_3 \sin\theta - C_4 \cos\theta\right) \qquad (13.4)$$

$$\sigma_{\theta\theta} = r^{-1}\left(C_3 \sin\theta + C_4 \cos\theta\right) \qquad (13.5)$$

The boundary conditions along the surface

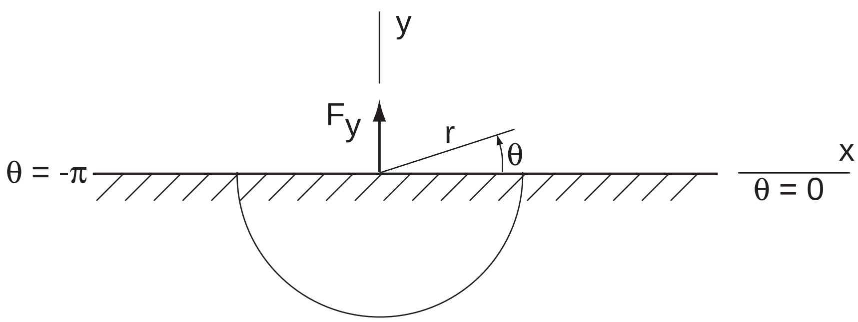
$$\sigma_{\theta r}(\theta=0,\theta^*) = \sigma_{\theta\theta}(\theta=0,\theta^*) = 0 \quad \text{(for all r > 0)} \qquad (13.6)$$

allow us to solve for two constants immediately: $C_3 = C_4 = 0$. This is the case for any wedge angle. So the only non-zero stresses in a polar reference frame are the radial stresses:

$$\sigma_{rr} = r^{-1}\left(2C_1 \cos\theta - 2C_2 \sin\theta\right) \qquad (13.7)$$

This rather amazing result is not "intuitively obvious."

Now consider the special case of a wedge with an angle of 180°: a half-space with a flat surface.



For a normal point force F_y at x=0, the solution must be symmetric (even) about the y-axis, so C_1 (and C_4) must be zero; only C_2 is not zero. For a shear point force F_x at x=0, the solution must be asymmetric (odd) about the y-axis, so C_2 (and C_3) must be zero; only C_1 is not zero.

To find C_2 (or C_1) we solve equation (13.1) for F_y (or F_x). To get the traction **<u>vectors</u>** t_y and t_x we first use Cauchy's formula and then the vector transformation equation:

$$t_i = n_j \sigma_{ij} \quad \text{Cauchy's formula} \qquad (13.8)$$

$$t_r = n_r \sigma_{rr} + n_\theta \sigma_{r\theta} = n_r \sigma_{rr} + (0)(0) = \cos\theta\left(-2C_2 \sin\theta\right) \qquad (13.9)$$

$$t_\theta = n_r \sigma_{\theta r} + n_\theta \sigma_{\theta\theta} = n_r(0) + (0)(\sigma_{\theta\theta}) = 0 \qquad (13.10)$$

$$t_i = a_{ij} tj \quad \text{vector transformation equation} \qquad (13.11)$$

$$t_x = a_{xr} t_r + a_{x\theta} t_\theta = \cos\theta t_r, \quad t_y = a_{yr} t_r + a_{y\theta} t_\theta = \sin\theta t_r \qquad (13.12)$$

Now we substitute (13.10) into (13.12) and then that into (13.1):

$$\int_0^{\pi r} t_y ds = \int_0^{\pi r} \left([\sin\theta][-2C_2 \sin\theta\cos\theta]\right)(rd\theta) = -\pi C_2 = -F_y \quad \Rightarrow C_2 = F_y / \pi \tag{13.13}$$

$$\int_0^{\pi r} t_x ds = \int_0^{\pi r} \left([\cos\theta][-2C_2 \sin\theta\cos\theta]\right)(rd\theta) = \pi C_1 = -F_x \quad \Rightarrow C_1 = -F_x / \pi \tag{13.14}$$

So for a normal point force with components F_x and F_y:

$$\phi = (-F_x / \pi) r\theta\sin\theta + (F_y / \pi) r\theta\cos\theta \tag{13.15}$$

$$\sigma_{rr} = r^{-1}\left(2[-F_x / \pi]\cos\theta - 2[F_y / \pi]\sin\theta\right) = \frac{-2}{\pi r}\left(F_x \cos\theta + F_y \sin\theta\right) \tag{13.16}$$

$$\sigma_{r\theta} = 0 \tag{13.17}$$

$$\sigma_{\theta\theta} = 0 \tag{13.18}$$

Note that away from the “point” where the load is applied that the induced stresses parallel to the surface match the applied stresses normal to the surface: both equal zero. Immediately beneath the load (at r = 0) the stresses have infinite magnitude, so the stresses “match” there too.

Barber provides the associated displacements (Table 9.1 & p. 142).

$$u_x(y=0) = \frac{-F_x(\kappa+1)\ln|x|}{4\pi\mu} + \frac{F_y(\kappa-1)\mathrm{sgn}(x)}{8\mu} \tag{13.19}$$

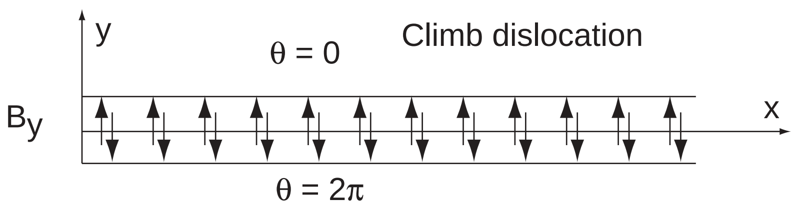
$$u_y(y=0) = \frac{-F_x(\kappa-1)\mathrm{sgn}(x)}{8\mu} - \frac{F_y(\kappa+1)\ln|x|}{4\pi\mu} \tag{13.20}$$

Note that (a) the solutions for stresses and displacements have no length scale (that was the case with the stresses around a hole of radius a), (b) at x = 0 the displacement is undefined, and (c) as $|x| \to \infty$, $u \to \infty$. Point (c) is problematic, and this is one reason that tilts (derivatives of displacements) are used in some geodetic work.

B Dislocations

If we return to the wedge problem, note that we could let the angle $\theta^* = 0^\circ$. This would be analogous to making a cut in a material and taking the cut to an infinite distance. We can imagine applying a series of equal but opposite forces along this cut. This infinite cut with forces or tractions acting along it is termed a dislocation. We will now consider a special dislocation that we will use (a) as a fundamental solution for some key fracture problems, and (b) as a basis for our boundary element numerical modeling technique.

Consider the cut below cut that extends along the x-axis from x= 0 to ∞.



Now suppose we apply pairs of equal and opposite normal tractions along the opposing faces of the cut such that there is a constant relative displacement of the cut faces by an amount B_y. This is called a dislocation [or more specifically a climb dislocation (Barber, 1992)].

Preliminary comments

Before proceeding, think about the magnitude of the tractions that would be required to produce a dislocation with a constant gap: would they increase, decrease, or stay constant as $x \rightarrow 0$?

Because a series of point loads are applied to the faces of the dislocation (including at the origin), we expect that the stresses in the solution might have the same form as in the half-space problem (i.e., the stresses must decay as r^{-1}, where r is the distance from the origin). We also want to preserve the discontinuity in displacement across the surface θ = 0,2π so we do not want consider stress functions that readily admit solutions where θ can exceed 2π.

General solution

Terms in the Michell solution that yields stresses of the form $\sigma \sim r^{-1}$ yield (again) the following stress function:

$$\phi = C_1 r(\theta \sin\theta) + C_2 r(\theta\cos\theta) + C_3 r \ln r(\cos\theta) + C_4 r \ln r(\sin\theta) \quad (13.21)$$

The associated stress components are

$$\sigma_{rr} = r^{-1}(2C_1\cos\theta - 2C_2\sin\theta + C_3\cos\theta + C_4\sin\theta) \quad (13.22)$$

$$\sigma_{r\theta} = r^{-1}(C_3\sin\theta - C_4\cos\theta) \quad (13.23)$$

$$\sigma_{\theta\theta} = r^{-1}(C_3\cos\theta + C_4\sin\theta) \quad (13.24)$$

The stresses should be symmetric about the x-axis, as was the case with the point force in the x-direction in the wedge (Flamant) problem, so

$$\sigma_{rr}(r,\theta) = \sigma_{rr}(r,-\theta), \quad (13.25)$$

$$\sigma_{r\theta}(r,\theta) = -\sigma_{r\theta}(r,-\theta), \text{ and} \quad (13.26)$$

$$\sigma_{\theta\theta}(r,\theta) = \sigma_{\theta\theta}(r,-\theta). \quad (13.27)$$

As a result, the coefficients for the <u>sine</u> terms in the <u>normal</u> stress in (13.22) and (13.24) must be zero, and the coefficients of the <u>cosine</u> term in (13.23) must be zero. So C_2 and C_4 =0. <u>Unlike</u> the case with the point force, here the <u>net</u> force at the origin is zero. From the wedge (Flamant) problem we recall that the expression for C_1 is:

$$C_1 = \frac{-F_x}{2\pi}. \quad (13.28)$$

If F_x equals zero, then C_1 = 0 also, so the only nonzero coefficient is C_3!

$$\sigma_{rr} = r^{-1}(C_3\cos\theta) \quad (13.29)$$

$$\sigma_{r\theta} = r^{-1}(C_3\sin\theta) \quad (13.30)$$

$$\sigma_{\theta\theta} = r^{-1}(C_3\cos\theta) \quad (13.31)$$

These surprisingly simple results have some surprising consequences. First, the radial (13.29) and hoop (13.31) stresses are equal everywhere, including along the walls of the dislocation. So the normal tractions <u>perpendicular</u> to the walls <u>match</u> the normal stresses <u>parallel</u> to the walls. Second, the stresses near the end of the dislocation are *singular* (they approach infinite levels as $r\rightarrow 0$) and vary as 1/r. Third, the tractions along the wall along the wall at $y=0^+$ have the same magnitude but opposite sign as those along the wall at $y=0^-$. So in contrast to the line load solution, the net force associated with a dislocation <u>is</u> zero.

The associated displacements (see Table 9.1 of Barber) are:

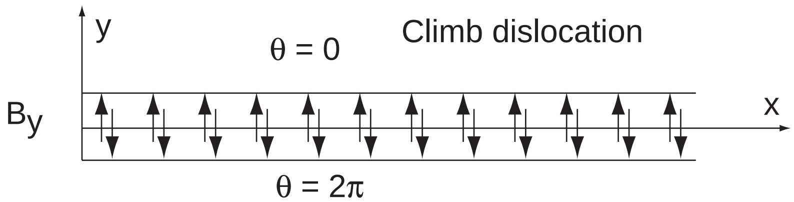
$$u_r = \frac{C_3}{4\mu}\left[(\kappa+1)(\theta\sin\theta) - \cos\theta + (\kappa-1)\ln r(\cos\theta)\right] \qquad (13.32)$$

$$u_\theta = \frac{C_3}{4\mu}\left[(\kappa+1)(\theta\cos\theta) - \sin\theta + (\kappa-1)\ln r(\sin\theta)\right] \qquad (13.33)$$

where

$\kappa = 3 - 4\nu$ (=2 for ν=0.25) for plane strain, and

$\kappa = (3-\nu)/(1+\nu)$ (=2.2 for ν=0.25) for plane stress.



Note that the displacements u_θ are <u>discontinuous</u> across the x-axis (in contrast to the point force solution) because of the θcosθ term. So a pre-existing line drawn across the x-axis would be broken. In this sense a dislocation is like a fracture. The direction of displacement on one face of the dislocation is opposite that of the opposing face. The strength of the discontinuity is given by the relative displacement across the dislocation:

$$B_y = u_y(\theta=0) - u_y(\theta=2\pi) = u_y^{+} - u_y^{-}. \qquad (13.34)$$

Whereas $u_\theta(\theta=0) = u_y$, substituting (13.32) into (13.34) with $\theta = 0$ yields

$$B_y = \frac{C_3}{4\mu}\left[(\kappa+1)(0)\cos 0) - (\kappa+1)(2\pi)\cos 2\pi\right] = \frac{-\pi(\kappa+1)C_3}{2\mu} \qquad (13.35)$$

So we can see that the constant C_3 in (13.29)-(13.31) is determined by the gap B_y - this makes sense.

$$C_3 = B_y \frac{-2\mu}{\pi(\kappa+1)} \qquad (13.36)$$

Notice that in this solution the positive side of the dislocation is not displaced in the y-direction. Commonly a factor of 1/2 B_y is added to the solution so that the symmetry about the y-axis is preserved. This rigid body translation will not affect the stresses or the relative displacements within the body with the dislocation.

We now revisit (13.33) to examine the tangential, or shear, displacement discontinuity across an opening dislocation.

$$u_r = \frac{C_3}{4\mu}\left[(\kappa+1)(\theta\sin\theta) - \cos\theta + (\kappa-1)\ln r(\cos\theta)\right] \tag{13.37}$$

Along the x-axis $\theta = 0$, $u_\theta = u_y$, and r = x. So

$$\Delta u_x = u_x^+ - u_x^- = u_r(\theta = 0) - u_r(\theta = 2\pi). \tag{13.38}$$

Making the appropriate substitutions from (13.33) into (13.38)

$$\Delta u_x = \frac{C_3}{4\mu}\left[-1 + (\kappa-1)\ln x\right] - \frac{C_3}{4\mu}\left[-1 + (\kappa-1)\ln x\right] = 0.$$

So the opening of a climb dislocation produces no **relative** shear displacement across the dislocation. Note, however, that this does **not** mean there is no shear displacement along the dislocation faces. In fact, there **is** a shear (tangential) displacement along both faces, but because the amounts are equal and in the same direction, there is no relative shear displacement from one dislocation face to another. **The lack of a relative displacement in a given direction across a dislocation (or a fracture) thus does not mean there is no absolute displacement in that direction**.

By analogous procedures the stresses and displacements for sliding mode dislocation can be found. In this case the relative tangential displacement across the dislocation is constant (and non-zero) but the relative opening displacement across the dislocation is zero