DISPLACEMENTS IN POLAR COORDINATES (12)

I Main topics

A Sample problem for determining displacements in polar coordinates (from section 9.2 of Barber)

B Displacement components from the Michell solution (Table 9.1 of Barber)

II Sample problem: A plate under plane stress with a circular hole that perturbs a pure shear stress field

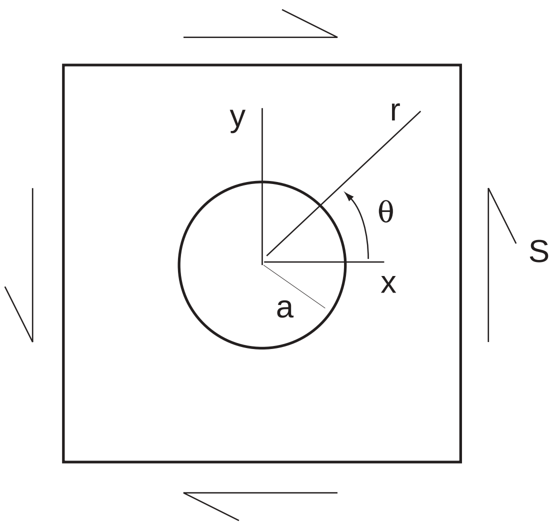
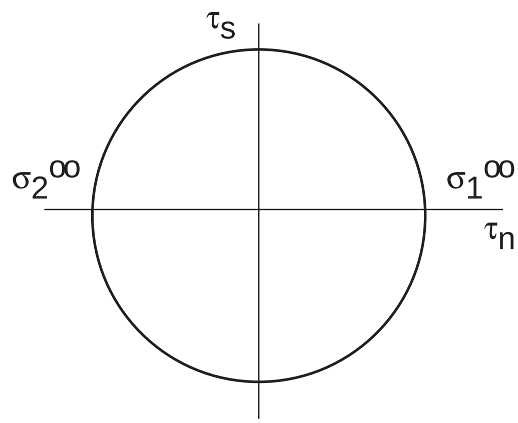


Plate under Pure Shear Stress



Mohr Diagram for Pure Shear Stress

A Under pure shear, $\sigma_1^\infty = -\sigma_2^\infty = \tau_{max}^\infty$

B In plane stress, $\sigma_{zz} = 0$.

C Stresses (from Barber, eq. 8.40-8.42).

$$\sigma_{rr} = S\left(1 - 4\left(\frac{a}{r}\right)^2 + 3\left(\frac{a}{r}\right)^4\right)\sin 2\theta \quad (12.1)$$

$$\sigma_{r\theta} = S\left(1 + 2\left(\frac{a}{r}\right)^2 - 3\left(\frac{a}{r}\right)^4\right)\cos 2\theta \quad (12.2)$$

$$\sigma_{\theta\theta} = S - \left(1 - 3\left(\frac{a}{r}\right)^4\right)\sin 2\theta \quad (12.3)$$

D Strains (from Hooke's Law)

$$e_{rr} = \frac{\sigma_{rr}}{E} - \nu\frac{\sigma_{\theta\theta}}{E} = \frac{S}{E}\left(1 + \nu - 4\left(\frac{a}{r}\right)^2 + 3(1+\nu)\left(\frac{a}{r}\right)^4\right)\sin 2\theta \quad (12.4)$$

$$e_{r\theta} = (1+\nu)\frac{\sigma_{r\theta}}{E} = \frac{S}{E}\left(1+\nu+2(1+\nu)\left(\frac{a}{r}\right)^2 - 3(1+\nu)\left(\frac{a}{r}\right)^4\right)\cos 2\theta \quad (12.5)$$

$$e_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \nu\frac{\sigma_{rr}}{E} = \frac{S}{E}\left(-1-\nu+4\nu\left(\frac{a}{r}\right)^2 - 3(1+\nu)\left(\frac{a}{r}\right)^4\right)\sin 2\theta \quad (12.6)$$

These strains are integrated to obtain displacements.

E Displacement-strain relationships in polar coordinates

These are obtained by using chain rule on strain-displacement relationships in Cartesian coordinates (see Barber, Chapter 8)

$$e_{xx} = \frac{\partial u_x}{\partial x} \quad (12.7)$$

$$e_{rr} = \frac{\partial u_r}{\partial r} \quad (12.8)$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \quad (12.9)$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \quad (12.10)$$

$$e_{yy} = \frac{\partial u_y}{\partial y} \quad (12.11)$$

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \quad (12.12)$$

We now integrate the simplest of the strains in the polar reference frame, e_{rr}.

$$\int e_{rr}\,dr = \int \frac{\partial u_r}{\partial r}dr = u_r^{(1)} + f(\theta) \quad (12.13)$$

The function of theta must be included - take the partial derivative of the right side of (12.13) to see why.

Integrating

$$e_{rr} = \frac{\sigma_{rr}}{E} - \nu\frac{\sigma_{rr}}{E} = \frac{S}{E}\left(1+\nu-4\left(\frac{a}{r}\right)^2+3(1+\nu)\left(\frac{a}{r}\right)^4\right)\sin 2\theta$$

with respect to r gives

$$u_r = \frac{S}{E}\left((1+\nu)r+4\frac{a^2}{r^1}-(1+\nu)\frac{a^4}{r^3}\right)\sin 2\theta + f(\theta) \qquad (12.14)$$

We now proceed to u_θ. We obtain $\frac{\partial u_\theta}{\partial \theta}$ by multiplying both sides of (12.12) by r and then subtracting u_r from both sides.

$$re_{\theta\theta} - u_r = \frac{\partial u_\theta}{\partial \theta} \qquad (12.15)$$

Substituting (12.6) for $e_{\theta\theta}$ in (12.12) and substituting (12.14) for u_r in (12.12) gives

$$\frac{\partial u_\theta}{\partial \theta} = r\left[\frac{S}{E}\left(-1-\nu+4\nu\left(\frac{a}{r}\right)^2-3(1+\nu)\left(\frac{a}{r}\right)^4\right)\sin 2\theta\right] - \frac{S}{E}\left((1+\nu)r+4\frac{a^2}{r^1}-(1+\nu)\frac{a^4}{r^3}\right)\sin 2\theta + f(\theta)$$

(12.16)

This is now integrated with respect to θ to give

$$u_\theta = \frac{S}{E}\left((1+\nu)r+3(1-\nu)\frac{a^2}{r}+(1+\nu)a^4r^3\right)\cos 2\theta - F(\theta)+g(r)$$

(12.17)

where $F(\theta) = \int f(\theta)d\theta$.

Now for the last steps. The displacements obtained so far using the normal strains also have to be consistent with the shear strains - this constraint allows the functions F(θ) and g(r) to be found. Substituting (12.14) for u_r, (12.17) for u_θ, and (12.5) for $e_{r\theta}$ into (12.10) and performing the differentiations yields conditions requiring that
the functions of r and θ, that according to Barber, correspond to rigid body rotations and translations. These rigid body displacements will not affect the stresses. So the functions can be set to zero, yielding

$$u_r = \frac{S}{E}\left((1+\nu)r+4\frac{a^2}{r^1}-(1+\nu)\frac{a^4}{r^3}\right)\sin 2\theta \qquad (12.18)$$

$$u_\theta = \frac{S}{E}\left((1+\nu)r+3(1-\nu)\frac{a^2}{r}+(1+\nu)a^4r^3\right)\cos 2\theta \qquad (12.19)$$

The Michell Solution - displacement components*
(Modified from Barber, 1992, p. 104)

ϕ	$2\mu u_r$	$2\mu u_\theta$
r^2	$(\kappa-1)\, r$	0
$r^2 \ln r$	$(\kappa-1)\, r \ln r - r$	$(\kappa+1)\, r\theta$ **
$\ln r$	$-1/r$	0
θ	0	$-1/r$
$r^3 \cos\theta$	$(\kappa-2) r^2 \cos\theta$	$(\kappa+2) r^2 \sin\theta$
$r\theta \sin\theta$	$(1/2)[(\kappa-1)\theta \sin\theta - \cos\theta + (\kappa+1)(\ln r)\cos\theta]$**	$(1/2)[(\kappa-1)\theta \cos\theta - \sin\theta - (\kappa+1)(\ln r)\sin\theta]$**
$r(\ln r)\cos\theta$	$(1/2)[(\kappa+1)\theta \sin\theta - \cos\theta + (\kappa-1)(\ln r)\cos\theta]$**	$(1/2)[(\kappa+1)\theta \cos\theta - \sin\theta - (\kappa-1)(\ln r)\sin\theta]$**
$(\cos\theta)/r$	$(\cos\theta)/r^2$	$(\sin\theta)/r^2$
$r^3 \sin\theta$	$(\kappa-2) r^2 \sin\theta$	$-(\kappa+2) r^2 \cos\theta$
$r\theta \cos\theta$	$(1/2)[(\kappa-1)\theta \cos\theta + \sin\theta - (\kappa+1)(\ln r)\sin\theta]$**	$(1/2)[-(\kappa-1)\theta \sin\theta - \cos\theta - (\kappa+1)(\ln r)\cos\theta]$**
$r(\ln r)\sin\theta$	$(1/2)[-(\kappa+1)\theta \cos\theta - \sin\theta + (\kappa-1)(\ln r)\sin\theta]$**	$(1/2)[(\kappa+1)\theta \sin\theta + \cos\theta + (\kappa-1)(\ln r)\cos\theta]$**
$(\sin\theta)/r$	$(\sin\theta)/r^2$	$(-\cos\theta)/r^2$
$r^{n+2} \cos n\theta$	$(\kappa-n-1) r^{n+1} \cos n\theta$	$(\kappa+n+1) r^{n+1} \sin n\theta$
$r^{-n+2} \cos n\theta$	$(\kappa+n-1) r^{-n+1} \cos n\theta$	$-(\kappa-n+1) r^{-n+1} \sin n\theta$
$r^n \cos n\theta$	$-n r^{n-1} \cos n\theta$	$n r^{n-1} \sin n\theta$
$r^{-n} \cos n\theta$	$n r^{-n-1} \cos n\theta$	$n r^{-n-1} \sin n\theta$
$r^{n+2} \sin n\theta$	$(\kappa-n-1) r^{n+1} \sin n\theta$	$-(\kappa+n+1) r^{n+1} \cos n\theta$
$r^{-n+2} \sin n\theta$	$(\kappa+n-1) r^{-n+1} \sin n\theta$	$(\kappa-n+1) r^{-n+1} \cos n\theta$
$r^n \sin n\theta$	$-n r^{n-1} \sin n\theta$	$-n r^{n-1} \cos n\theta$
$r^{-n} \sin n\theta$	$n r^{-n-1} \sin n\theta$	$-n r^{-n-1} \cos n\theta$

For plane strain: $\kappa = 3-4\nu$
For plane stress: $\kappa = (3-\nu)/(1+\nu)$

* Values have not been checked independently

** Note that the absolute value of θ should not exceed 2π! Otherwise, the displacements (e.g., θ at $= 0$ and $\theta = 2\pi$) will be multi-valued.

References

Barber, J.R., 1993, Elasticity: Kluwer Academic Publishers, Boston, p. 83-104