STRESSES AROUND HOLES (2-D) (11)

I Main Topics: Plane solutions for

A A traction-free circular hole under a uniaxial load

B A traction-free circular hole under biaxial loading

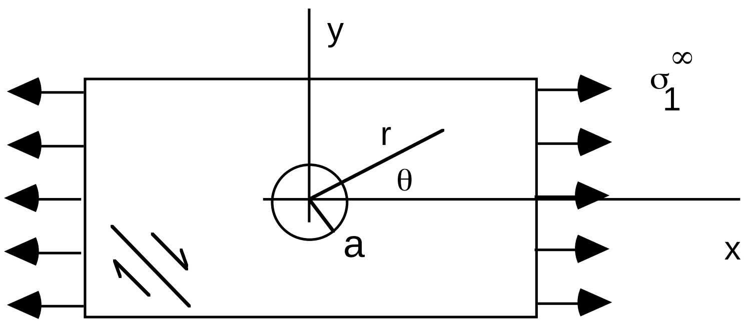
C A pressurized circular hole with no remote load

D A pressurized circular hole with a remote biaxial load

II A stress-free circular hole under a uniaxial load

Preliminary considerations and boundary conditions

We start by considering the problem of a plate under uniaxial tension, where the plate contains a circular hole of radius a.



Boundary conditions

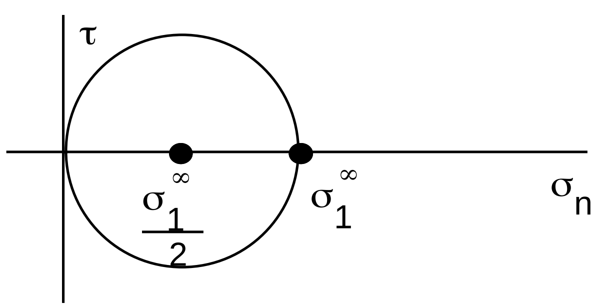
The remote boundary conditions are met far away from the hole. In terms of x and y the tension-positive remote stresses are:

(a) $\sigma_{xx}^\infty = \sigma_1^\infty$ (b) $\sigma_{xy}^\infty = 0$ (c) $\sigma_{yy}^\infty = 0$ (11.1)

In terms of polar coordinates these conditions are

(a) $\sigma_{rr}^\infty = \frac{\sigma_1^\infty}{2}(1+\cos 2\theta)$ (b) $\sigma_{r\theta}^\infty = \frac{-\sigma_1^\infty}{2}\sin 2\theta$ (c) $\sigma_{\theta\theta}^\infty = \frac{\sigma_1^\infty}{2}(1-\cos 2\theta)$ (11.2)

The conditions of equations (11.2) can be visualized with a Mohr circle:



Boundary conditions also exist on the surface of the hole (r=a). These are:

(a) $\sigma_{rr}^{a} = 0$, (b) $\sigma_{r\theta}^{a} = 0$. (11.3)

In other words, no tractions exist on the surface of the hole. Note that $\sigma_{\theta\theta}$ does not act on the surface of the hole and so cannot be a boundary condition.

Governing Equation

$$\nabla^4\phi = \nabla^2\nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0 \qquad (11.4)$$

General solution (from Michell, 1899)

$$\begin{aligned}\phi = \; & A_{01}r^2 + A_{02}r^2\ln r + A_{03}\ln r + A_{04}\theta \\ & +(A_{11}r^3 + A_{12}r\ln r + A_{14}r^{-1})\cos\theta + A_{13}r\theta\sin\theta \\ & +(B_{11}r^3 + B_{12}r\ln r + B_{14}r^{-1})\sin\theta + B_{13}r\theta\cos\theta \\ & +\sum_{n=2}^{\infty}\left(A_{n1}r^{n+2} + A_{n2}r^{-n+2} + A_{n3}r^{n} + A_{n4}r^{-n}\right)\cos n\theta \\ & +\sum_{n=2}^{\infty}\left(B_{n1}r^{n+2} + B_{n2}r^{-n+2} + B_{n3}r^{n} + B_{n4}r^{-n}\right)\sin n\theta\end{aligned} \qquad (11.5)$$

The series contribution is in the form of a Fourier series.

Table 8.1 from Barber shows the stresses associated with each term in the above equation, obtained using the following equations:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \qquad (11.6)$$

$$\sigma_{r\theta} = \frac{-1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}. \qquad (11.7)$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}. \qquad (11.8)$$

An inspection of the table shows that terms involving cos(0θ) yield stresses that do not depend on θ or r. Such terms are needed to describe the boundary conditions both on the hole and far from the hole. We also need functions that yield stresses that vary as cos(2θ) term, and these come only from stress functions that include cos2θ. So the general solution can be trimmed substantially:

$$\phi = A_{01}r^2 + A_{02}r^2\ln r + A_{03}\ln r + A_{04}\theta + \left(A_{21}r^4 + A_{22}r^0 + A_{23}r^2 + A_{24}r^{-2}\right)\cos n\theta \qquad (11.9)$$

The general solution can be simplified more. The coefficients A_{02} and A_{21} must equal zero in order for the stresses to be finite value as r goes to ∞. So (11.9) becomes

$$\phi = A_{01}r^2 + A_{03}\ln r + A_{04}\theta + \left(A_{22}r^{-n+2} + A_{23}r^n + A_{24}r^{-n}\right)\cos n\theta \tag{11.10}$$

The stresses obtained from this stress function are:

$$\sigma_{rr} = 2A_{01} + A_{03}r^{-2} + A_{04}(0) + (-4A_{22}r^{-2} - 2A_{23}r^0 - 6A_{24}r^{-4})\cos 2\theta \tag{11.11}$$

$$\sigma_{r\theta} = 0 + 0 + A_{04}r^{-2} + (-2A_{22}r^{-2} + 2A_{23}r^0 - 6A_{24}r^{-4})\sin 2\theta \tag{11.12}$$

$$\sigma_{\theta\theta} = 2A_{01} - A_{03}r^{-2} + A_{04}(0) - (0A_{22}r^{-2} + 2A_{23}r^0 + 6A_{24}r^{-4})\cos 2\theta \tag{11.13}$$

Particular Solution

Letting r go to ∞, both σ_{rr} and $\sigma_{\theta\theta}$ equal $\sigma_1^\infty/2$ at $2\theta = 90°$. Equations (11.11) and (11.13) require

$$A_{01} = \frac{\sigma_1^\infty}{4}. \tag{11.14}$$

Comparing (11.2c) and (11.13) shows that the term $2A_{23}$ equals $-\sigma_1^\infty/2$, so

$$A_{23} = \frac{-\sigma_1^\infty}{4}. \tag{11.15}$$

So the stresses are:

$$\sigma_{rr} = \left[\frac{\sigma_1^\infty}{2} + A_{03}r^{-2}\right] + (-4A_{22}r^{-2} + \frac{\sigma_1^\infty}{2} - 6A_{24}r^{-4})\cos 2\theta \tag{11.16}$$

$$\sigma_{r\theta} = \left[A_{04}r^{-2}\right] + (-2A_{22}r^{-2} - \frac{\sigma_1^\infty}{2} - 6A_{24}r^{-4})\sin 2\theta \tag{11.17}$$

$$\sigma_{\theta\theta} = \frac{\sigma_1^\infty}{2} - A_{03}r^{-2} + (-\frac{\sigma_1^\infty}{2} + 6A_{24}r^{-4})\cos 2\theta \tag{11.18}$$

At r = a, $\sigma_{rr} = 0$ and $\sigma_{r\theta} = 0$ (for all values of 2θ), so the bracketed terms in (11.16 and 11.17) must sum to zero. So

$$A_{03} = \frac{-\sigma_1^\infty a^2}{2}. \tag{11.19}$$

$$A_{04} = 0. \tag{11.20}$$

So the equations for the stresses reduce to:

$$\sigma_{rr} = \left[\frac{\sigma_1^\infty}{2} - \frac{\sigma_1^\infty}{2}a^2r^{-2}\right] + (-4A_{22}r^{-2} + \frac{\sigma_1^\infty}{2} - 6A_{24}r^{-4})\cos 2\theta \tag{11.21}$$

$$\sigma_{r\theta} = (-2A_{22}r^{-2} - \frac{\sigma_1^\infty}{2} - 6A_{24}r^{-4})\sin 2\theta \tag{11.22}$$

$$\sigma_{\theta\theta} = \frac{\sigma_1^\infty}{2} + \frac{\sigma_1^\infty}{2}a^2r^{-2} + (-\frac{\sigma_1^\infty}{2} + 6A_{24}r^{-4})\cos 2\theta \tag{11.23}$$

All that is left now is to find A_{22} and A_{24}.

At r = a, $\sigma_{r\theta}$ = 0 for all values of 2θ, so the terms in parentheses in (11.22) that multiply sin2θ must sum to zero.

$$-2A_{22}a^{-2} - \sigma_1^\infty/2 - 6A_{24}a^{-4} = 0. \quad (11.24)$$

Similarly, at r = a, σ_{rr} = 0 for all values of 2θ, so the terms in parentheses in (11.21) that multiply cos2θ must sum to zero.

$$-4A_{22}a^{-2} + \sigma_1^\infty/2 - 6A_{24}a^{-4} = 0. \quad (11.25)$$

Subtracting (11.25) from (11.24) gives $2A_{22}a^{-2} - \sigma_1^\infty = 0$, hence

$$A_{22} = \frac{\sigma_1^\infty a^2}{2}. \quad (11.26)$$

Inserting (11.26) into (11.25) yields

$$-4\frac{\sigma_1^\infty a^2}{2}a^{-2} + \frac{\sigma_1^\infty}{2} - 6A_{24}a^{-4} = 0. \quad (11.27)$$

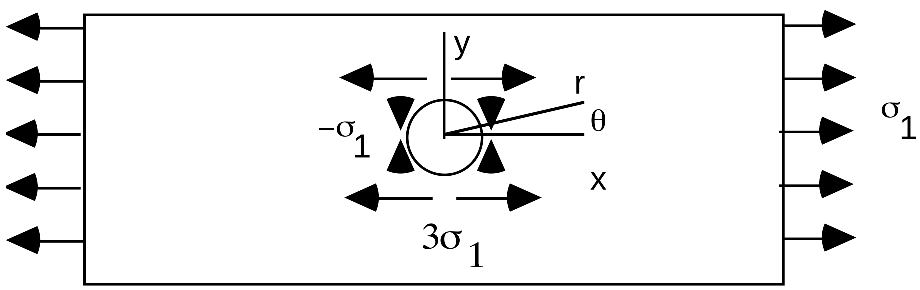
This simplifies to

$$-6A_{24}a^{-4} = 3\frac{\sigma_1^\infty}{2}. \quad (11.28)$$

Finally,

$$A_{24} = -\frac{\sigma_1^\infty a^4}{4}. \quad (11.29)$$

So here is our solution



$$\sigma_{rr} = \frac{\sigma_1^\infty}{2}\left[\left(1-\frac{a^2}{r^2}\right) + \left(1-4\frac{a^2}{r^2}+3\frac{a^4}{r^4}\right)\cos 2\theta\right]. \quad (11.30)$$

$$\sigma_{r\theta} = \frac{-\sigma_1^\infty}{2}\left(1+2\frac{a^2}{r^2}-3\frac{a^4}{r^4}\right)\sin 2\theta. \quad (11.31)$$

$$\sigma_{\theta\theta} = \frac{\sigma_1^\infty}{2}\left[\left(1+\frac{a^2}{r^2}\right) - \left(1+3\frac{a^4}{r4}\right)\cos 2\theta\right] \quad (11.32)$$

Comments on solution

At this point we should examine the solution to see what insight it provides.

A Even though the walls of the hole are traction-free, they are not stress-free, because at r = a, $\sigma_{\theta\theta}$ = 0 only where cos2θ = 1/2.

B The mean normal stress at θ = ±45° dies off as $\frac{a^2}{r^2}$

C On the boundary of the hole (r=a) the hoop stress $\sigma_{\theta\theta}$ is

$$\sigma_{\theta\theta}\big|_{r=a} = \sigma_1^\infty(1-2\cos2\theta). \tag{11.33}$$

The hoop stress at the perimeter of the hole at θ = ± π/2 has the same sign as σ_1^∞ but a higher magnitude, so <u>the hole concentrates stress</u>:

$$(\sigma_{\theta\theta})_{\max} = 3\sigma_1^\infty. \qquad \text{(at } \theta = \pm\pi/2) \tag{11.34}$$

The hoop stress at θ = 0 and θ = π is the negative of σ_1^∞:

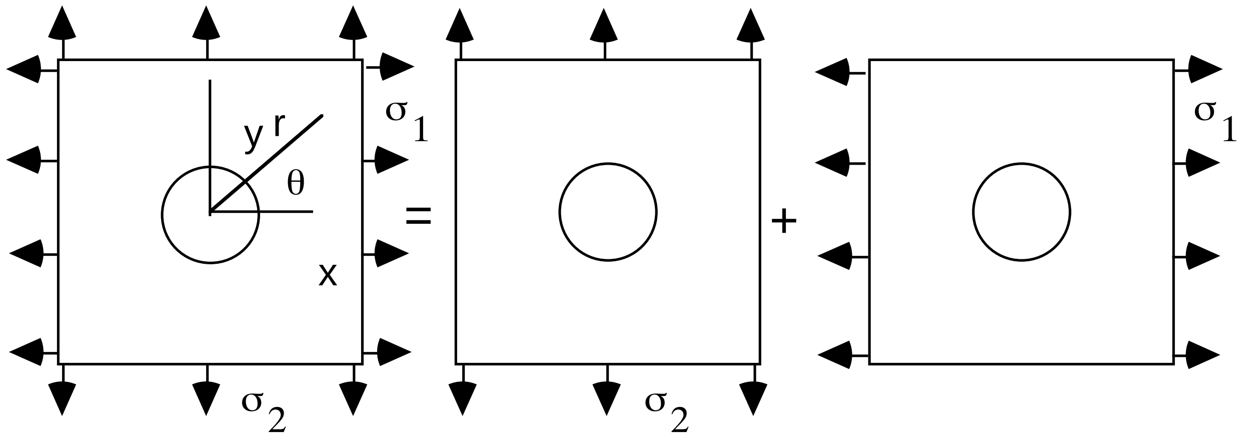
$$(\sigma_{\theta\theta})_{\min} = -\sigma_1^\infty. \qquad \text{(at } \theta = 0 \text{ and } \theta = \pi) \tag{11.35}$$

* **Localized tensile stresses can exist even though the ambient stress field is compressive (and vice-versa).**

* **The magnitude of the stresses around the perimeter of the hole are <u>independent</u> of the radius of the hole. So tiny holes can concentrate stresses just the same as large ones.**

III Solution for stresses about a hole under biaxial loading

The solution for the stresses about a hole under biaxial loading can be obtained by superposing the solutions for uniaxial loads at right angles



We use the solutions of (11.30)-(11.32) to find the stress state for the uniaxial load of σ_2 along the y-direction, substituting σ_2 for σ_1 and $(\theta - \pi/2)$ for θ. The resulting equations differ in form from (11.30)-(11.32) by a sign change in the trigonometric terms.

$$\sigma_{rr} = \frac{\sigma_2^\infty}{2}\left[\left(1-\frac{a^2}{r^2}\right) - \left(1-4\frac{a^2}{r^2}+3\frac{a^4}{r^4}\right)\cos 2\theta\right]. \tag{11.36}$$

$$\sigma_{r\theta} = \frac{\sigma_2^\infty}{2}\left(1+2\frac{a^2}{r^2}-3\frac{a^4}{r^4}\right)\sin 2\theta. \tag{11.37}$$

$$\sigma_{\theta\theta} = \frac{\sigma_2^\infty}{2}\left[\left(1+\frac{a^2}{r^2}\right) + \left(1+3\frac{a^4}{r^4}\right)\cos 2\theta\right]. \tag{11.38}$$

Superposing (11.30)-(11.32) and (11.36)-(11.38)

$$\sigma_{rr} = \left(\frac{\sigma_1^\infty + \sigma_2^\infty}{2}\right)\left(1-\frac{a^2}{r^2}\right) + \left[\left(\frac{\sigma_1^\infty - \sigma_2^\infty}{2}\right)\left(1-4\frac{a^2}{r^2}+3\frac{a^4}{r^4}\right)\cos 2\theta\right]. \tag{11.39}$$

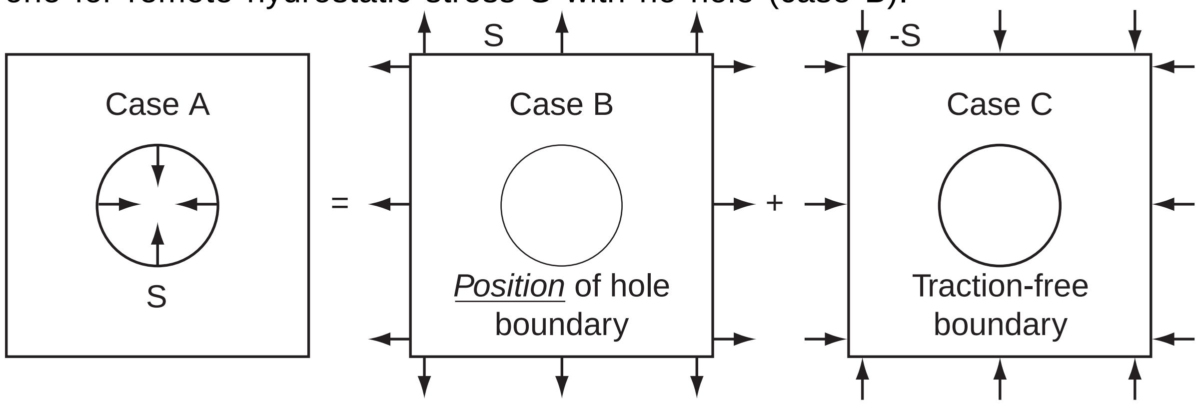
$$\sigma_{r\theta} = -\left(\frac{\sigma_1^\infty - \sigma_2^\infty}{2}\right)\left(1+2\frac{a^2}{r^2}-3\frac{a^4}{r^4}\right)\sin 2\theta. \tag{11.40}$$

$$\sigma_{\theta\theta} = \left(\frac{\sigma_1^\infty + \sigma_2^\infty}{2}\right)\left(1+\frac{a^2}{r^2}\right) - \left[\left(\frac{\sigma_1^\infty - \sigma_2^\infty}{2}\right)\left(1+3\frac{a^4}{r^4}\right)\cos 2\theta\right].. \tag{11.41}$$

Remember that σ_1^∞ and σ_2^∞ refer to <u>remote</u> principal stresses.

IV Solution for a pressurized hole with no remote load

We obtain the solution for a pressurized hole here by superposition. Let σ_1^∞ and σ_2^∞ in the biaxial solution both equal -S (case C). Superpose this solution with one for remote hydrostatic stress S with no hole (case B).



We check the boundary conditions to make sure that the desired conditions are satisfied. In case B the position of the hole boundary is shown, but no hole exists. The following conditions apply at the boundaries r=a and r=∞:

Case B (hydrostatic pressure, no hole)

$\sigma_{rr}^{r=a} = S$ $\sigma_{r\theta}^{r=a} = 0$ $\sigma_{rr}^{r=\infty} = S$ $\sigma_{r\theta}^{r=\infty} = 0$ ($\sigma_{rr} = \sigma_{\theta\theta} = S$ *everywhere*).

Case C (hydrostatic remote tension, traction-free hole)

$\sigma_{rr}^{r=a} = 0$ $\sigma_{r\theta}^{r=a} = 0$ $\sigma_{rr}^{r=\infty} = -S$ $\sigma_{r\theta}^{r=\infty} = 0$

Case A (By superposition of A and B)

$\sigma_{rr}^{r=a} = S$ $\sigma_{r\theta}^{r=a} = 0$ $\sigma_{rr}^{r=\infty} = 0$ $\sigma_{r\theta}^{r=\infty} = 0$

The stress state between our boundaries is found by setting the remote normal stresses in (11.39)-(11.41) to -S and superposing a hydrostatic stress S:

$$\sigma_{rr} = -\left\{\left(\frac{S+S}{2}\right)\left(1-\frac{a^2}{r^2}\right)+\left[\left(\frac{S-S}{2}\right)\left(1-4\frac{a^2}{r^2}+3\frac{a^4}{r^4}\right)\cos 2\theta\right]\right\}+S \quad or \quad \sigma_{rr} = +S\frac{a^2}{r^2} \qquad (11.42)$$

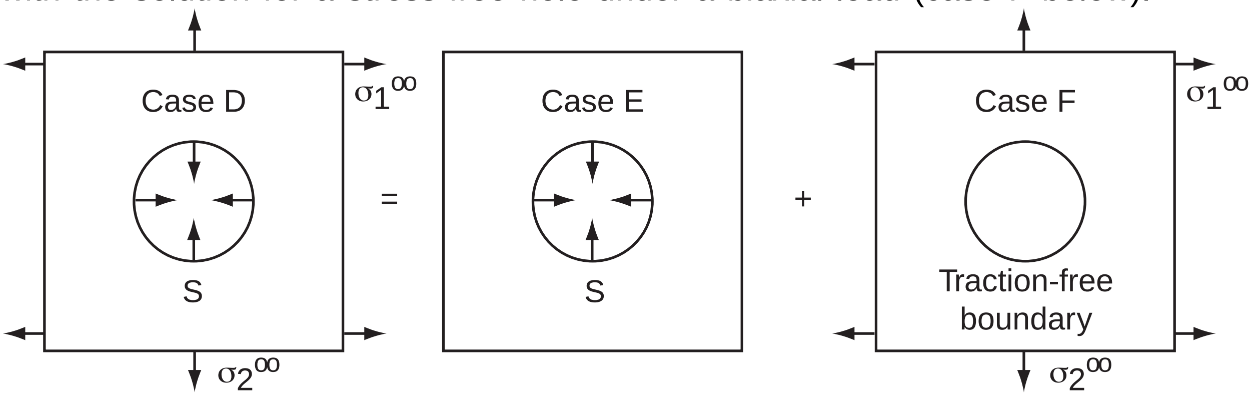
$$\sigma_{r\theta} = -\left\{-\left(\frac{S-S}{2}\right)\left(1+2\frac{a^2}{r^2}-3\frac{a^4}{r^4}\right)\sin 2\theta\right\} \quad or \quad \sigma_{r\theta} = 0. \qquad (11.43)$$

$$\sigma_{\theta\theta} = -\left\{\left(\frac{S+S}{2}\right)\left(1+\frac{a^2}{r^2}\right)-\left[\left(\frac{S-S}{2}\right)\left(1+3\frac{a^4}{r^4}\right)\cos 2\theta\right]\right\}+S \quad or \quad \sigma_{\theta\theta} = -S\frac{a^2}{r^2} \qquad (11.44)$$

The stresses decay as $(a/r)^2$; so St. Venant's principal holds. Also note that a pressure in the hole causes a circumferential tension of equal magnitude at the hole wall. Finally, the mean normal stress equals zero everywhere, and the shear stress on radial planes is zero everywhere.

V Solution for a stressed hole with a remote load

The solution for a pressurized hole with a remote load is obtained here also by superposition. We will superpose the solution for a stressed hole (case E below) with the solution for a stress-free hole under a biaxial load (case F below).



We again check the boundary conditions to make sure that the desired conditions are satisfied at the boundaries r= a and r=∞:

Case E (Pressurized hole, no remote load)

$\sigma_{rr}^{r=a} = S \quad \sigma_{r\theta}^{r=a} = 0 \quad \sigma_{rr}^{r=\infty,\theta=0} = 0 \quad \sigma_{\theta\theta}^{r=\infty,\theta=0} = 0 \quad \sigma_{r\theta}^{r=\infty} = 0.$

Case F (biaxial remote load, stress-free hole)

$\sigma_{rr}^{r=a} = 0 \quad \sigma_{r\theta}^{r=a} = 0 \quad \sigma_{rr}^{r=\infty,\theta=0} = \sigma_1^\infty \quad \sigma_{\theta\theta}^{r=\infty,\theta=0} = \sigma_2^\infty \quad \sigma_{r\theta}^{r=\infty} = 0$

Case D (By superposition of E and F)

$\sigma_{rr}^{r=a} = S \quad \sigma_{r\theta}^{r=a} = 0 \quad \sigma_{rr}^{r=\infty,\theta=0} = \sigma_1^\infty \quad \sigma_{\theta\theta}^{r=\infty,\theta=0} = \sigma_2^\infty \quad \sigma_{r\theta}^{r=\infty} = 0$

The stress state between our boundaries is given by superposing the solutions of eqs. (11.39)-(11.41) and eqs. (11.42)-(11.44).

$$\sigma_{rr} = \left(\frac{\sigma_1^\infty + \sigma_2^\infty}{2}\right)\left(1 - \frac{a^2}{r^2}\right) + \left[\left(\frac{\sigma_1^\infty - \sigma_2^\infty}{2}\right)\left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta\right] + S\frac{a^2}{r^2}. \quad (11.45)$$

$$\sigma_{r\theta} = -\left(\frac{\sigma_1^\infty - \sigma_2^\infty}{2}\right)\left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\sin 2\theta. \quad (11.46)$$

$$\sigma_{\theta\theta} = \left(\frac{\sigma_1^\infty + \sigma_2^\infty}{2}\right)\left(1 + \frac{a^2}{r^2}\right) - \left[\left(\frac{\sigma_1^\infty - \sigma_2^\infty}{2}\right)\left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta\right] - S\frac{a^2}{r^2}. \quad (11.47)$$

The Michell Solution - stress components*
(Modified from Barber, 1992, p. 93)

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$r^2 \ln r$	$(2 \ln r) + 1$	0	$(2 \ln r) + 3$
$\ln r$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^3 \cos\theta$	$2r \cos\theta$	$2r \sin\theta$	$6r \cos\theta$
$r\theta \sin\theta$	$2 \cos\theta/r$	0	0
$r (\ln r) \cos\theta$	$(\cos\theta)/r$	$(\sin\theta)/r$	$(\cos\theta)/r$
$(\cos\theta)/r$	$(-2 \cos\theta)/r^3$	$(-2 \sin\theta)/r^3$	$(2 \cos\theta)/r^3$
$r^3 \sin\theta$	$2r \sin\theta$	$-2r \cos\theta$	$6r \sin\theta$
$r\theta \cos\theta$	$-2 \sin\theta/r$	0	0
$r (\ln r) \sin\theta$	$(\sin\theta)/r$	$(-\cos\theta)/r$	$(\sin\theta)/r$
$(\sin\theta)/r$	$(-2 \sin\theta)/r^3$	$(2 \cos\theta)/r^3$	$(2 \sin\theta)/r^3$
$r^{n+2} \cos n\theta$	$(1+n)(2-n)r^{n} \cos n\theta$	$n(1+n)r^{n} \sin n\theta$	$(1+n)(2+n)r^{n} \cos n\theta$
$r^{-n+2} \cos n\theta$	$(2+n)(1-n)r^{-n} \cos n\theta$	$n(1-n)r^{-n} \sin n\theta$	$(1-n)(2-n)r^{-n} \cos n\theta$
$r^{n} \cos n\theta$	$n(1-n)r^{n-2} \cos n\theta$	$-n(1-n)r^{n-2} \sin n\theta$	$-n(1-n)r^{n-2} \cos n\theta$
$r^{-n} \cos n\theta$	$-n(1+n)r^{-n-2} \cos n\theta$	$-n(1+n)r^{-n-2} \sin n\theta$	$n(1+n)r^{-n-2} \cos n\theta$
$r^{n+2} \sin n\theta$	$(1+n)(2-n)r^{n} \sin n\theta$	$-n(1+n)r^{n} \cos n\theta$	$(1+n)(2+n)r^{n} \sin n\theta$
$r^{-n+2} \sin n\theta$	$(2+n)(1-n)r^{-n} \sin n\theta$	$-n(1-n)r^{-n} \cos n\theta$	$(1-n)(2-n)r^{-n} \sin n\theta$
$r^{n} \sin n\theta$	$n(1-n)r^{n-2} \sin n\theta$	$n(1-n)r^{n-2} \cos n\theta$	$-n(1-n)r^{n-2} \sin n\theta$
$r^{-n} \sin n\theta$	$-n(1+n)r^{-n-2} \sin n\theta$	$n(1+n)r^{-n-2} \cos n\theta$	$n(1+n)r^{-n-2} \sin n\theta$

* Values have not been checked independently