ELASTICITY PROBLEMS IN POLAR COORDINATES (10)

I Main topics

- A Motivation
- B Cartesian Approach
- C Transformation of coordinates
- D Equilibrium equations in polar coordinates
- E Biharmonic equation in polar coordinates
- F Stresses in polar coordinates

II Motivation
- A Many key problems in geomechanics (e.g., stress around a borehole, stress around a tunnel, stress around a magma chamber) involve cylindrical geometries. Our reference frame should fit the features we examine.
- B Introduces the concept of stress concentration due to factors inside a body (as opposed to concentrated boundary loads).

III Approach
- A Transform elastic equations from xy form to polar form
- B Alternative: vector and tensor approaches (see C&P, Ch. 11)

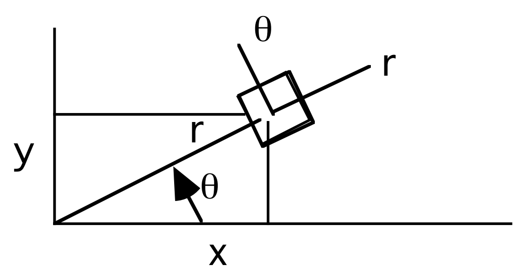
IV Transformation of coordinates (See Fig. 34.1)
- A Rectangular to polar:

$$\theta = \tan^{-1}(y/x) \qquad r = (x^2 + y^2)^{1/2} \tag{10.1}$$

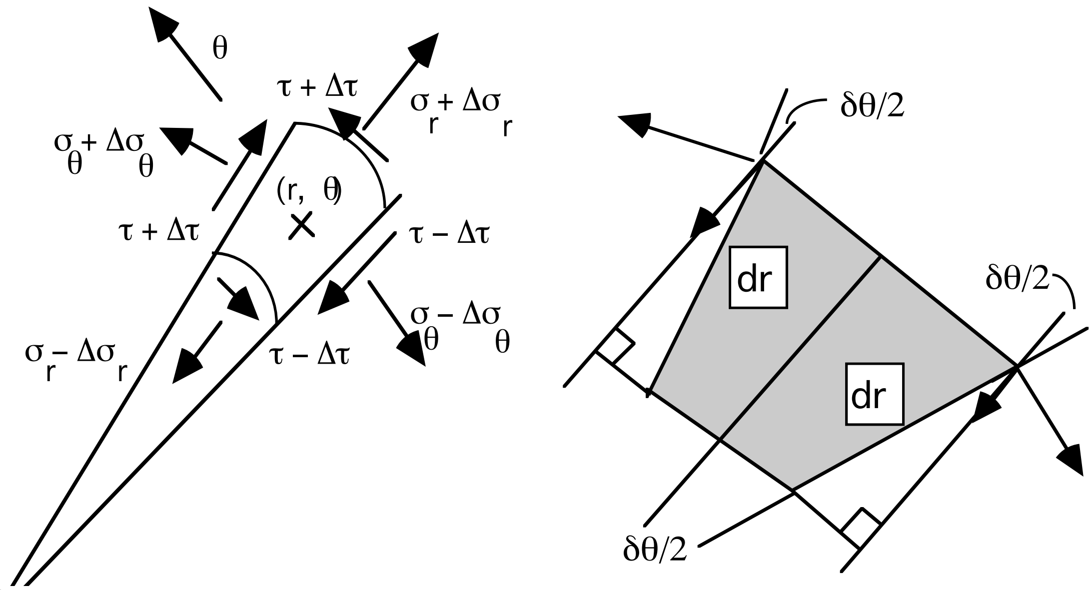
- B Polar to rectangular:

$$x = r\cos\theta \qquad y = r\sin\theta \tag{10.2}$$

- C Stress convention: Still use on-in convention $(\theta_{rr}, \theta_{r\theta}, \theta_{\theta r}, \theta_{\theta\theta})$



V Equilibrium equations in polar coordinates



$$\sum F_r = 0 \quad (10.3)$$

We will assume body forces are negligible.
General form of the terms is $\sigma_{ij} = \{\sigma_{ij\,ref} + (\sigma_{ij}\text{ gradient})(\text{distance})\}$.

$$\left(\sigma_{rr} + \frac{\partial\sigma_{rr}}{\partial r}\frac{dr}{2}\right)\left(r + \frac{dr}{2}\right)d\theta - \left(\sigma_{rr} + \frac{\partial\sigma_{rr}}{\partial r}\frac{dr}{2}\right)\left(r + \frac{dr}{2}\right)d\theta \;+$$
$$\cos\frac{d\theta}{2}\left(\sigma_{r\theta} + \frac{\partial\sigma_{r\theta}}{\partial\theta}\frac{d\theta}{2}\right)dr - \cos\frac{d\theta}{2}\left(\sigma_{r\theta} - \frac{\partial\sigma_{r\theta}}{\partial\theta}\frac{d\theta}{2}\right)dr \;+$$
$$-\sin\frac{d\theta}{2}\left(\sigma_{\theta\theta} + \frac{\partial\sigma_{\theta\theta}}{\partial\theta}\frac{d\theta}{2}\right)dr - \sin\frac{d\theta}{2}\left(\sigma_{\theta\theta} - \frac{\partial\sigma_{\theta\theta}}{\partial\theta}\frac{d\theta}{2}\right)dr = 0. \quad (10.4)$$

This equation reduces to

$$\left(2\frac{\partial\sigma_{rr}}{\partial r}\frac{dr}{2}r d\theta\right) + \left(2\sigma_{rr}\frac{dr}{2}d\theta\right) + \left(2\frac{\partial\sigma_{rr}}{\partial\theta}\frac{dr}{2}\frac{dr}{2}d\theta\right)$$
$$+\left(2\frac{\partial\sigma_{rr}}{\partial\theta}\frac{d\theta}{2}dr\cos\frac{d\theta}{2}\right) - \left(2\sigma_{\theta\theta}dr\sin\frac{d\theta}{2}\right) = 0 \quad . \quad (10.5)$$

Dividing through by dr, and noting that for small angles $\sin(d\theta/2) \approx d\theta/2$ and $\cos(d\theta/2) \approx 1$, equation (10.5) reduces to

$$\left(\frac{\partial\sigma_{rr}}{\partial r}r d\theta\right) + \left(\sigma_{rr}d\theta\right) + \left(\frac{\partial\sigma_{rr}}{\partial r}\frac{dr^2}{2}d\theta\right) + \left(\frac{\partial\sigma_{rr}}{\partial\theta}d\theta\right) - \left(2\sigma_{\theta\theta}\frac{d\theta}{2}\right) = 0. \quad (10.5)$$

The third term drops out because dr^2 is tiny relative to the other terms.
Dividing this (10.5) through by r dθ yields

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r} + \frac{1}{r}\frac{\partial \sigma_{rr}}{\partial \theta} - \frac{\sigma_{\theta\theta}}{r} = 0. \tag{10.6}$$

This expression is commonly rearranged as (see eq. 5.41 in C&P):

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{rr}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0. \tag{10.7}$$

By summing the forces in the θ-direction one can obtain

$$\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0. \tag{10.8}$$

Equations (10.7) and (10.8) are the equilibrium equations in polar coordinates.

VI Biharmonic equation in polar coordinates

Our starting point is the biharmonic equation

$$\nabla^4 \phi = \nabla^2 \nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0, \tag{10.9}$$

and the stresses in terms of the stress function ϕ:

$$\text{(a)}\ \sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \qquad \text{(b)}\ \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \qquad \text{(c)}\ \sigma_{xy} = \frac{-\partial^2 \phi}{\partial x \partial y}. \tag{10.10}$$

In order to transform these equations to polar form, we need to know how to express derivatives with respect to x and y in terms of r and θ. We get these relationships from the chain rule:

$$\text{(a)}\ \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial \theta}\frac{\partial \theta}{\partial x} \qquad \text{(b)}\ \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial \theta}\frac{\partial \theta}{\partial y} \tag{10.11}$$

We therefore need to know how r and θ relate to x and y.

(a) $\theta = \tan^{-1}(y/x)$ (b) $r = (x^2 + y^2)^{1/2}$ (10.12)

(a) $x = r\cos\theta$ (b) $y = r\sin\theta$ (10.13)

Now to the derivatives. Starting with eq. (10.12b)

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}(2x) = \frac{x}{\left(x^2 + y^2\right)^{1/2}} = \frac{x}{r} = \cos\theta \tag{10.14}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}(2y) = \frac{y}{\left(x^2 + y^2\right)^{1/2}} = \frac{y}{r} = \sin\theta \tag{10.15}$$

Now take derivatives of eq. (10.12a). Recalling that $d(\tan^{-1}u) = du/(1+u^2)$

$$\frac{\partial\theta}{\partial x}=\frac{\partial\left(\tan^{-1}(y/x)\right)}{\partial x}=\frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)^2}\frac{\partial(y/x)}{\partial x}=\frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)^2}\frac{-y}{x^2}=\frac{-y}{r^2}=\frac{-\sin\theta}{r}$$

(10.16)

Similarly
(16)

$$\frac{\partial\theta}{\partial y}=\frac{\partial\left(\tan^{-1}(y/x)\right)}{\partial y}=\frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)^2}\frac{\partial(y/x)}{\partial y}=\frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)^2}\frac{1}{x}\frac{x}{x}=\frac{x}{r^2}=\frac{\cos\theta}{r}$$

(10.17)

Now we can find the derivatives of ϕ in terms of r and θ by substituting eqs. (10.16) and (10.17) into chain rule equations (10.11):

(a) $\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial r}\cos\theta-\frac{\partial\phi}{\partial\theta}\frac{\sin\theta}{r}$, (b) $\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial r}\sin\theta+\frac{\partial\phi}{\partial\theta}\frac{\cos\theta}{r}$. (10.18)

Second derivatives are found by operating on first derivatives.

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial\left(\frac{\partial\phi}{\partial x}\right)}{\partial x}=\frac{\partial\left(\frac{\partial\phi}{\partial x}\right)}{\partial r}\cos\theta-\frac{\partial\left(\frac{\partial\phi}{\partial x}\right)}{\partial\phi}\frac{\sin\theta}{r}. \tag{10.19}$$

Substituting eq. (10.18a) into eq. (10.19) yields

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial\left(\frac{\partial\phi}{\partial x}\right)}{\partial x}=\frac{\partial\left(\frac{\partial\phi}{\partial r}\cos\theta-\frac{\partial\phi}{\partial\theta}\frac{\sin\theta}{r}\right)}{\partial r}\cos\theta-\frac{\partial\left(\frac{\partial\phi}{\partial r}\cos\theta-\frac{\partial\phi}{\partial\theta}\frac{\sin\theta}{r}\right)}{\partial\theta}\frac{\sin\theta}{r}.$$

(10.20)

This can be expanded as

$$\begin{aligned}\frac{\partial^2\phi}{\partial x^2}=&\frac{\partial^2\phi}{\partial r^2}\cos^2\theta-\frac{\partial^2\phi}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r}+\frac{\partial\phi}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}\\&-\frac{\partial^2\phi}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r}+\frac{\partial\phi}{\partial r}\frac{\sin^2\theta}{r}+\frac{\partial^2\phi}{\partial\theta^2}\frac{\sin^2\theta}{r^2}+\frac{\partial\phi}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}\end{aligned} \tag{10.21}$$

The expression for $\frac{\partial^2\phi}{\partial y^2}$ can be found by the same procedure:

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial r^2}\sin^2\theta + \frac{\partial^2\phi}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r} - \frac{\partial\phi}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}$$
$$+\frac{\partial^2\phi}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r} + \frac{\partial\phi}{\partial r}\frac{\cos^2\theta}{r} + \frac{\partial^2\phi}{\partial\theta^2}\frac{\cos^2\theta}{r^2} - \frac{\partial\phi}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2} \qquad (10.22)$$

Equation (10.21) is the expression for σ_{yy} and eq. (10.22) is the expression for σ_{xx}. Notice the term-by-term "symmetry" between these two equations.

These equations can be added to give the expression for $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$.

Recalling that $\sin^2\theta + \cos^2\theta = 1$, we get

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \nabla^2\phi. \qquad (10.23)$$

The biharmonic equation is obtained by allowing the harmonic equation (10.23) to operate on itself.

$$\nabla^4\phi = \nabla^2\nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0. \qquad (10.24)$$

VIIStresses in polar coordinates

We are now left with the problem of how to determine the stresses in polar coordinates from the stress function ϕ. We know that the mean normal stress (and hence twice the mean stress) is an invariant term - it does not depend on the choice of the system of coordinates. As a result

$$2\sigma_{mean} = \sigma_{xx} + \sigma_{yy} = \sigma_{rr} + \sigma_{\theta\theta} = \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = \nabla^2\phi. \qquad (10.25)$$

By comparing equations (10.23 and 10.25) we know that if one of the terms on the right side of eq. (10.24) equals $\sigma_{\theta\theta}$, then the other terms must sum to equal σ_{rr}. If we can show this for any particular position (r,θ), then the result will hold for all positions. We therefore choose a simple case. Let the r-direction be along the x-axis and the θ-direction be parallel to the y-axis, so $\sigma_{rr} = \sigma_{xx}$ and $\sigma_{\theta\theta} = \sigma_{yy}$. The x-axis corresponds to $\theta = 0°$ and the y-direction to $\theta = 90°$. So

$$\frac{\partial^2\phi}{\partial x^2} = \sigma_{yy} = \sigma_{\theta\theta} = \begin{array}{c} \dfrac{\partial^2\phi}{\partial r^2}\cos^2\theta - \dfrac{\partial^2\phi}{\partial\theta\partial r}\dfrac{\sin\theta\cos\theta}{r} + \dfrac{\partial\phi}{\partial\theta}\dfrac{\sin\theta\cos\theta}{r^2} \\ -\dfrac{\partial^2\phi}{\partial\theta\partial r}\dfrac{\sin\theta\cos\theta}{r} + \dfrac{\partial\phi}{\partial r}\dfrac{\sin^2\theta}{r} + \dfrac{\partial^2\phi}{\partial\theta^2}\dfrac{\sin^2\theta}{r^2} + \dfrac{\partial\phi}{\partial\theta}\dfrac{\sin\theta\cos\theta}{r^2} \end{array}. \tag{10.26}$$

For our case of $\theta = 0°$, all terms with $\sin\theta$ are zero and this simplifies to

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \tag{10.27}$$

Similarly

$$\frac{\partial^2\phi}{\partial y^2} = \sigma_{xx} = \sigma_{rr} = \begin{array}{c} \dfrac{\partial^2\phi}{\partial r^2}\sin^2\theta + \dfrac{\partial^2\phi}{\partial\theta\partial r}\dfrac{\sin\theta\cos\theta}{r} - \dfrac{\partial\phi}{\partial\theta}\dfrac{\sin\theta\cos\theta}{r^2} \\ +\dfrac{\partial^2\phi}{\partial\theta\partial r}\dfrac{\sin\theta\cos\theta}{r} + \dfrac{\partial\phi}{\partial r}\dfrac{\cos^2\theta}{r} + \dfrac{\partial^2\phi}{\partial\theta^2}\dfrac{\cos^2\theta}{r^2} - \dfrac{\partial\phi}{\partial\theta}\dfrac{\sin\theta\cos\theta}{r^2} \end{array}. \tag{10.28}$$

Again $\theta = 0°$, so all terms with $\sin\theta$ are zero, and this simplifies to

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}. \tag{10.29}$$

Note that $\sigma_{\theta\theta} + \sigma_{rr}$ does indeed equal $\nabla^2\phi = \dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$.

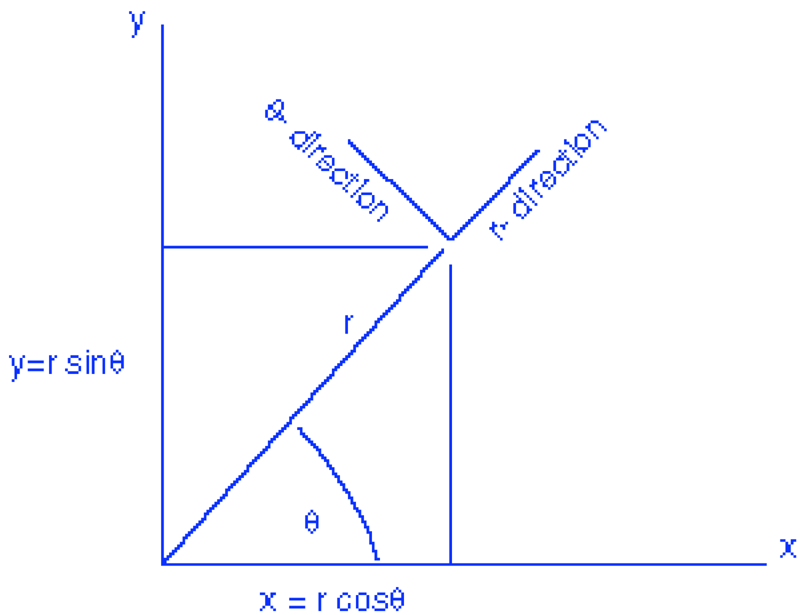
The shear stress $\theta_{r\theta}$ can be determined from σ_{xy} and is given by

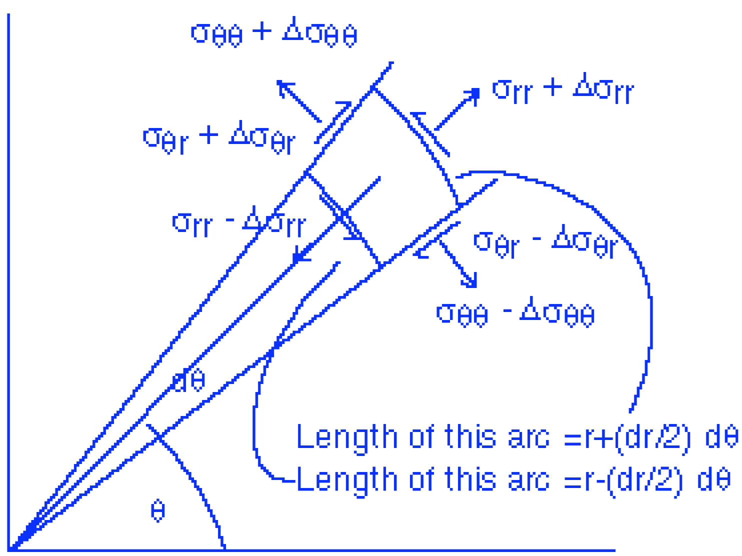
(a) $\sigma_{r\theta} = \dfrac{-\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right)$ or $\sigma_{r\theta} = \dfrac{-1}{r}\dfrac{\partial^2\phi}{\partial r\partial\theta} + \dfrac{1}{r^2}\dfrac{\partial\phi}{\partial\theta}$. (10.30)

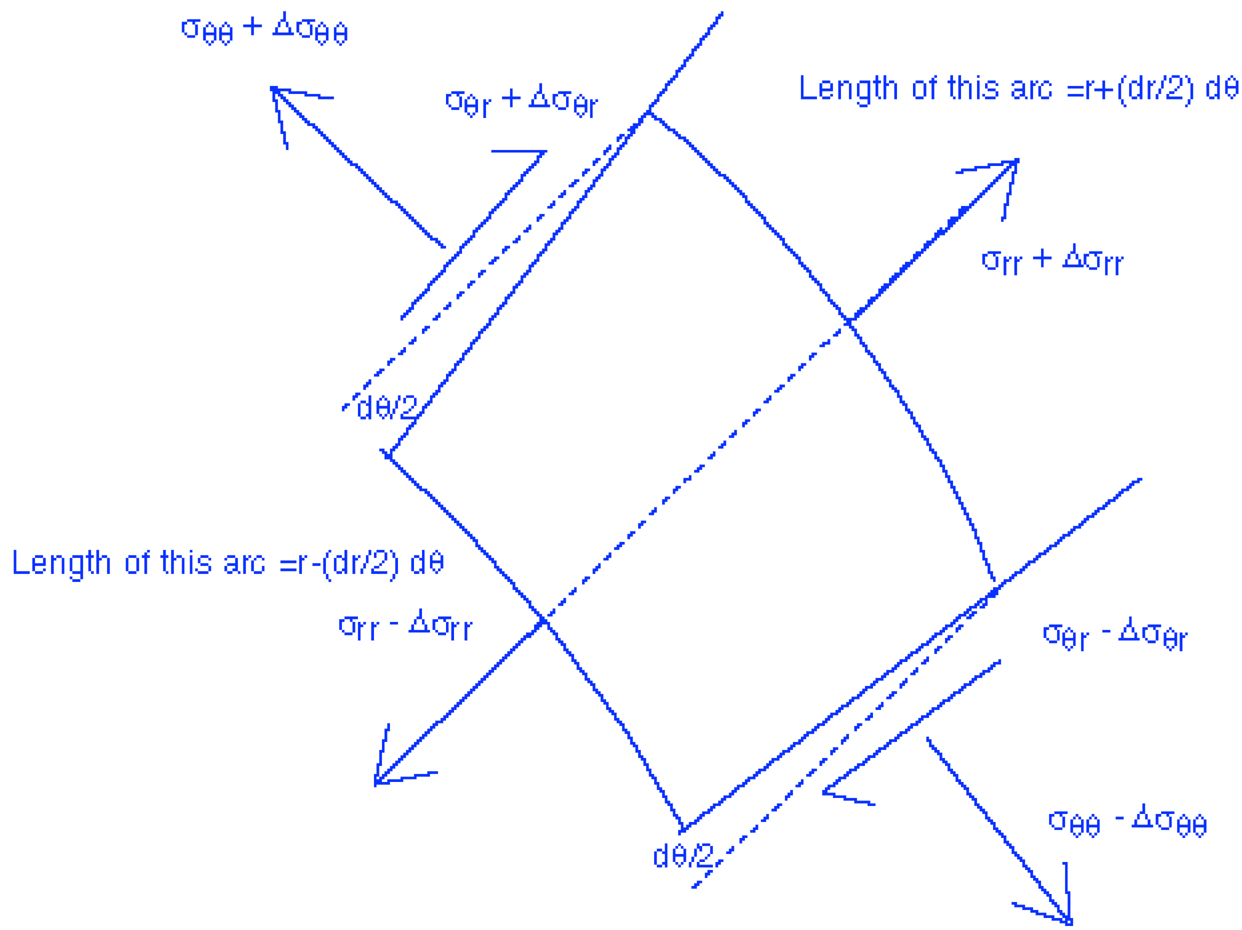
Stresses in Polar Coordinates

$r = (x^2 + y^2)^{1/2}$

$\theta = \tan^{-1}(y/x)$







$\sigma_{\theta\theta} + \Delta\sigma_{\theta\theta}$
$\sigma_{\theta r} + \Delta\sigma_{\theta r}$
Length of this arc =r+(dr/2) dθ
$\sigma_{rr} + \Delta\sigma_{rr}$
dθ/2
Length of this arc =r-(dr/2) dθ
$\sigma_{rr} - \Delta\sigma_{rr}$
$\sigma_{\theta r} - \Delta\sigma_{\theta r}$
$\sigma_{\theta\theta} - \Delta\sigma_{\theta\theta}$
dθ/2