

## STRESS FUNCTIONS IN RECTANGULAR COORDINATES (09)

### I Main topics

- A Airy stress functions and the biharmonic equation**
- B Example**
- C Finding stress functions**
- D Stress functions where body forces exist (Appendix)**

### II Airy stress functions and the biharmonic equation

- A Airy stress functions ( $\phi$ ) are potential functions for solving 2-D problems**
- B We use derivatives of potential functions to get useful quantities, not the functions themselves (e.g.,  $U=mgh$ ;  $dU/dh = mg = F_g$ )**
- C The stress function  $\phi$  must be consistent with the boundary conditions for a given problem and satisfy the governing equation(s)**

For no body forces (i.e.,  $F_i = 0$ ), the stresses are:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}, \quad (9.1)$$

Our governing equation for plane strain is

$$\nabla^2 \left\{ \sigma_{xx} + \sigma_{yy} \right\} = -\frac{1}{1-\nu} \left\{ \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right\}. \quad (9.2)$$

If the body forces are zero, the right side of (9.2) is zero, and

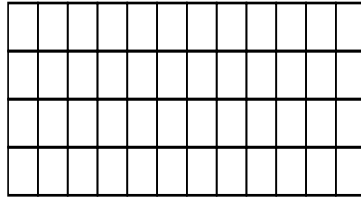
$$\nabla^2 \left\{ \sigma_{xx} + \sigma_{yy} \right\} = 0. \quad (9.3)$$

This is the Laplace equation  $\nabla^2 \{f\} = 0$ , with the term in braces being the first stress invariant. So the first stress invariant satisfies the Laplace equation.

Dividing both sides of (9.3) by two yields:

$$\frac{1}{2} \nabla^2 \left\{ \sigma_{xx} + \sigma_{yy} \right\} = \nabla^2 \left\{ \frac{\sigma_{xx} + \sigma_{yy}}{2} \right\} = 0. \quad (9.4)$$

So the mean normal stress here satisfies the Laplace equation. This tells us something about how the mean normal stresses at one point is related to the mean stress at neighboring points in an elastic body. For a function that solves the Laplace equation, the function value at a point on a rectangular grid equals average of the function at the four nearest points.



The mean normal stress also scales with the change in area of an infinitesimal element (i.e., the dilation)

Stresses determined by an Airy stress function do satisfy the equilibrium equations (7.1)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x = 0 \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \quad (9.5)$$

Substituting the equations of (9.1) into (9.4) yields

$$\frac{\partial^3 \phi}{\partial y^2 \partial x} + \frac{-\partial^3 \phi}{\partial y^2 \partial x} + F_x = 0 \quad \frac{\partial^3 \phi}{\partial y^2 \partial x} + \frac{-\partial^3 \phi}{\partial y^2 \partial x} + F_y = 0 \quad (9.6)$$

The stresses defined by the Airy stress function must also satisfy the compatibility equation.

$$(1 - \nu) \left\{ \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} \right\} - \nu \left\{ \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} \right\} = 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad (9.7)$$

This can be re-arranged to give the following:

$$\left\{ \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} \right\} - \nu \left\{ \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} \right\} = 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad (9.8)$$

or

$$\left\{ \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} \right\} - \nu \left\{ \frac{\partial^2 (\sigma_{xx} + \sigma_{yy})}{\partial x^2} + \frac{\partial^2 (\sigma_{xx} + \sigma_{yy})}{\partial y^2} \right\} = 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad (9.9)$$

From (9.3), the second term in braces is zero:

$$\left\{ \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} \right\} = 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad (9.10)$$

Expressing the stresses in terms of derivatives of  $\phi$  (see eq. 9.1) we find

$$\left\{ \frac{\partial^2 \left( \frac{\partial^2 \phi}{\partial y^2} \right)}{\partial y^2} + \frac{\partial^2 \left( \frac{\partial^2 \phi}{\partial x^2} \right)}{\partial x^2} \right\} = 2 \frac{\partial^2 \left( \frac{-\partial^2 \phi}{\partial x \partial y} \right)}{\partial x \partial y} \quad (9.11)$$

Upon re-arranging we obtain:

$$\frac{\partial^4 \phi}{\partial y^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial x^4} = \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} \right) \equiv \nabla^4 \phi = 0 \quad (9.12)$$

This is known as the **biharmonic equation**. Any stress function  $\phi$  that satisfies this equation is a solution to a 2-D stress problem.

### III Example

Let  $\phi = Cx^2y$ . First check to see if  $\nabla^4 \phi = 0$ .

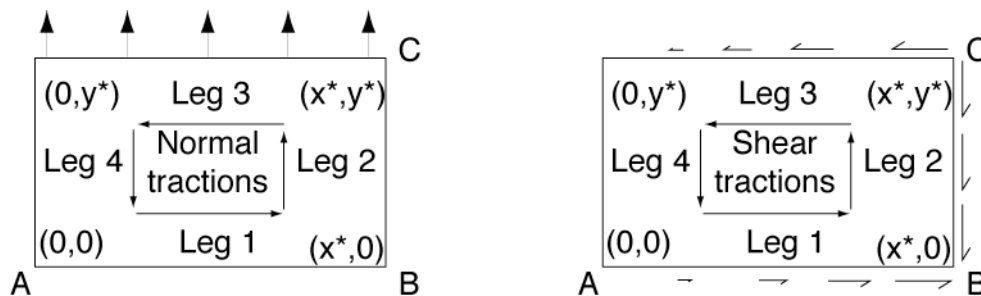
$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \frac{\partial^2 (2Cy)}{\partial x^2} + \frac{\partial^2 (2Cy)}{\partial y^2} + 0 = 0 + 0 + 0 = 0$$

So this is a solution. The stresses are as follows:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 (Cx^2y)}{\partial y^2} = \frac{\partial (Cx^2)}{\partial y} = 0$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 (Cx^2y)}{\partial x^2} = \frac{\partial (2Cxy)}{\partial x} = 2Cy$$

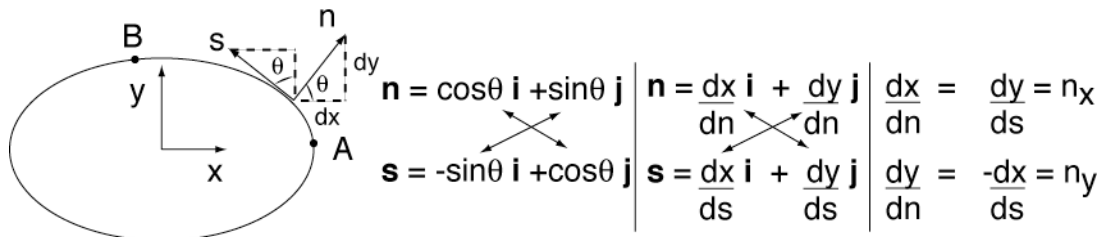
$$\sigma_{xy} = \frac{-\partial^2 \phi}{\partial x \partial y} = \frac{-\partial^2 (Cx^2y)}{\partial x \partial y} = \frac{-\partial (2Cxy)}{\partial y} = -2Cx \quad \sigma_{xy} \propto -x$$



## V Finding stress functions

- A Pick a stress function, see which boundary conditions it satisfies, and that problem is solved
- B Guess which stress function might solve the problem of interest (semi-inverse method). Not the most satisfying method, but it works
- C Solve for the stress function analytically from the body geometry and boundary conditions by contour integration (for some simple problems)

If second derivatives of  $\phi$  yield the stresses, then two integrations of the stresses should give  $\phi$ . We start with Cauchy's formula



$$t_i = \sigma_{ij} n_j = \sigma_{ji} n_j, \text{ or} \quad (9.13)$$

$$t_x = \sigma_{xx} n_x + \sigma_{xy} n_y \quad \text{and} \quad t_y = \sigma_{yx} n_x + \sigma_{yy} n_y \quad (9.14)$$

Substituting for the stresses in terms of  $\phi$  (9.1), and for  $n_x$  and  $n_y$  gives:

$$t_x = \frac{\partial^2 \phi}{\partial y^2} \frac{dy}{ds} + \frac{-\partial^2 \phi}{\partial x \partial y} \frac{-dx}{ds} = \frac{\partial^2 \phi}{\partial y^2} \frac{dy}{ds} + \frac{\partial^2 \phi}{\partial x \partial y} \frac{dx}{ds} = \frac{d}{ds} \left( \frac{\partial \phi}{\partial y} \right) \quad (9.15)$$

$$t_y = \frac{-\partial^2 \phi}{\partial x \partial y} \frac{dy}{ds} + \frac{\partial^2 \phi}{\partial x^2} \frac{-dx}{ds} = -\frac{d}{ds} \left( \frac{\partial \phi}{\partial x} \right) \quad (\text{The chain rule applies}) \quad (9.16)$$

By contour integrating (9.15) and (9.16) around the body, we obtain

$$\int_C d \left( \frac{\partial \phi}{\partial x} \right) = \frac{\partial \phi}{\partial x} = -\int_C t_y ds + C_1 \quad \text{and} \quad \int_C d \left( \frac{\partial \phi}{\partial y} \right) = \frac{\partial \phi}{\partial y} = \int_C t_x ds + C_2 \quad (9.17)$$

Integrating the tractions over arc AB gives the net force over the arc; this is what  $\partial \phi / \partial x$  and  $\partial \phi / \partial y$  mean. We now apply the chain rule again:

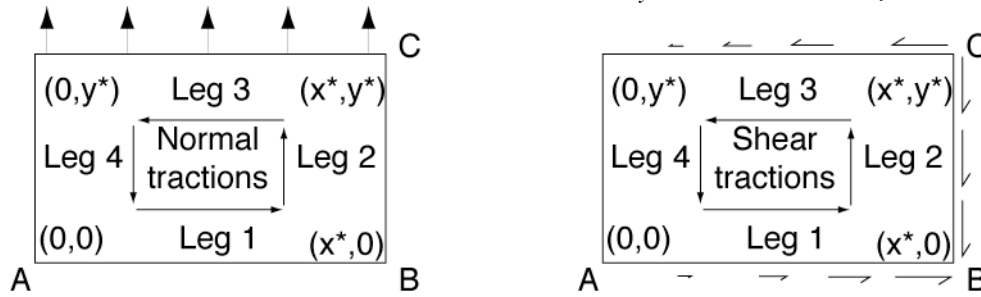
$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x} \frac{dx}{ds} + \frac{\partial \phi}{\partial y} \frac{dy}{ds} \quad \text{and} \quad \frac{d\phi}{dn} = \frac{\partial \phi}{\partial x} \frac{dx}{dn} + \frac{\partial \phi}{\partial y} \frac{dy}{dn} = \frac{\partial \phi}{\partial x} \frac{dy}{ds} - \frac{\partial \phi}{\partial y} \frac{dx}{ds} \quad (9.18)$$

By integrating  $\partial \phi / \partial s$  around the contour of the body we get  $\phi$ .

$$\phi = \int_C \frac{\partial \phi}{\partial s} ds + C_3 \quad (9.19)$$

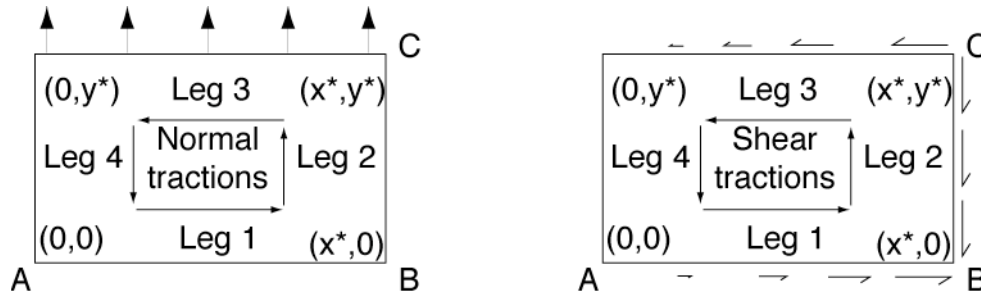
$C_3$  is arbitrary but  $C_1$  ( $= \partial \phi / \partial x|_A$ ) and  $C_2$  ( $= \partial \phi / \partial y|_A$ ) are not.

**Example:**  $\phi = Cx^2y$  (a)  $\sigma_{xx}=0$  (b)  $\sigma_{xy}=-2Cx$  (c)  $\sigma_{yx}=-2Cx$  (d)  $\sigma_{yy}=2Cy$



	Leg 1	Leg 2	Leg 3	Leg 4
(e) $n_x$	0	1	0	-1
(f) $n_y$	-1	0	1	0
$t_x$ (ae+bf)	$2Cx$	0	$-2Cx$	0
$t_y$ (ce+df)	0	$-2Cx$	$2Cy$	0
<p>(g) <math>\partial\phi/\partial x = -\int_C t_y ds = 2Cxy</math></p> <p>Leg 1  <math>\partial\phi/\partial x = -\int_0^x (0)dx = 0</math> <span style="float:right"><math>I_{1a}(x^*,0) = 0</math></span></p> <p>Leg 2  <math>\partial\phi/\partial x = I_{1a} - \int_0^y (-2Cx)dy = 2Cxy</math> <span style="float:right"><math>I_{2a}(x^*,y^*) = 2Cx^* y^*</math></span></p> <p>Leg 3  <math>\partial\phi/\partial x = I_{2a} - \int_{x^*}^x (2Cy^*)(-dx) = 2Cx^* y^* + 2Cy^*(x - x^*) = 2Cxy^*</math> <span style="float:right"><math>I_{3a}(0,y^*) = 0</math></span></p> <p>Leg 4  <math>\partial\phi/\partial x = I_{3a} - \int_{y^*}^y (0)(-dy) = 0 - 0 = 0</math> <span style="float:right"><math>\sum F_y = 0</math></span></p>				
<p>(h) <math>\partial\phi/\partial y = \int_C t_x ds = Cx^2</math></p> <p>Leg 1  <math>\partial\phi/\partial y = \int_0^x (2Cx)dx = Cx^2</math> <span style="float:right"><math>I_{1b}(x^*,0) = Cx^{*2}</math></span></p> <p>Leg 2  <math>\partial\phi/\partial y = I_{1b} + \int_0^y (0)dy = Cx^{*2} + 0 = Cx^{*2}</math> <span style="float:right"><math>I_{2b}(x^*,y^*) = Cx^{*2}</math></span></p> <p>Leg 3  <math>\partial\phi/\partial y = I_{2b} + \int_{x^*}^x (-2Cx)(-dx) = Cx^{*2} + C(x^2 - x^{*2}) = Cx^2</math> <span style="float:right"><math>I_{3b}(0,y^*) = 0</math></span></p> <p>Leg 4  <math>\partial\phi/\partial y = I_{3b} - \int_{y^*}^y (0)(-dy) = 0 + 0 = 0</math> <span style="float:right"><math>\sum F_x = 0</math></span></p>				

Example:  $\phi = Cx^2y$



	Leg 1	Leg 2	Leg 3	Leg 4
(g) $\partial\phi/\partial x$	0	$2Cxy$	$2Cxy^*$	0
(h) $\partial\phi/\partial y$	$Cx^2$	$Cx^{*2}$	$Cx^2$	0
(i) $dx/ds$	1	0	-1	0
(j) $dy/ds$	0	1	0	-1
$d\phi/ds$ (gi+hj)	0	$Cx^{*2}$	$-2Cxy^*$	0

(k) $\phi = \int_C \frac{d\phi}{ds} ds = Cx^2y$	
Leg 1 $\phi = \int_0^x (0) dx = 0$	$I_{1c}(x^*, 0) = 0$
Leg 2 $\phi = I_{1c} + \int_0^y (Cx^{*2}) dy = 0 + Cx^{*2}y = Cx^{*2}y$	$I_{2c}(x^*, y^*) = Cx^{*2}y$
Leg 3 $\phi = I_{2c} + \int_{x^*}^x (-2Cxy^*)(-dx) = Cx^{*2}y^* + Cy^*(x^2 - x^{*2}) = Cx^2y^*$	$I_{3c}(0, y^*) = 0$
Leg 4 $\phi = I_{3c} + \int_{y^*}^y (0)(-dy) = 0 + 0 = 0$	

Note that along all legs,  $\phi = Cx^2y$  provides a valid solution.

#### References

Barber, J.R., 1993, Elasticity: Kluwer Academic Publishers, Boston, p. 39-43.

Malvern, L.E., 1969, Introduction to the mechanics of a continuous medium: Prentice-Hall, Englewood Cliffs, p. 510-511.

Timoshenko, S.P., and Goodier, J.N., 1971, Theory of elasticity: McGraw-Hill, New York, p. 31-33.

**V Finding stress functions where body forces exist**

- A Find the body force distribution (for gravitational body forces this can be  $pgh$ , where  $h$  is the distance above some datum).
- B Subtract the body force stresses from the total stress field, including along the boundary of the body. This yields a modified problem with modified boundary conditions.
- C Solve for the stress function for the modified problem using the procedure above.
- D To obtain the total stress field, superpose the stress field from the modified problem (which has had the body force removed) with the body force.