FORMULATION OF ELASTICITY BOUNDARY VALUE PROBLEMS (08)

I Main topics (CRITICAL FOR NUMERICAL MODELING!)

A Governing equations

B Boundary conditions

C Strain energy density

D Principle of superposition

E Uniqueness

F Saint-Venant's Principle

II Governing equations (How a body responds to a load)

A Equations that relate stress, strain, and/or displacements at points within a body

B Example

$$\nabla^2\left\{\sigma_{xx}+\sigma_{yy}\right\}=-\frac{1}{1-\nu}\left\{\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right\}. \tag{8.1}$$

III Boundary conditions (the "loading conditions")

A Tractions on boundaries (e.g., crack problem)

B Displacements on boundaries (e.g., dislocation problem)

C Mixed boundary conditions (e.g., stamp problem). Tractions are specified for part of boundary and displacements for the rest. Can't specify both at the same point!

D If the boundary forces are not in equilibrium, the body will accelerate.

IV Strain energy density U_0 in an initially unstrained body

Let U_0 = dU/dV = Strain energy/unit volume = strain energy density

For an elastic body, the strain energy equals the work done by external forces; body forces do not contribute (see Chou and Pagano, section 7.3).

Work = Δstrain energy = [Integrated average force][displacement]

= [1/2 (final stress)(incremental area)][(du_i/ dx_j) (dx_j)]

= [1/2 (final stress)(strain)][incremental volume]

$$U_0 = \frac{1}{2}[\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3] \qquad (8.2)$$

The strain energy density can be cast in terms of strain using Hooke's Law:

$$U_0 = \frac{1}{2}[\lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 + G(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)], \qquad (8.3)$$

where $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ and $G = \frac{E}{2(1+\nu)}$

The strain energy at a point depends on the sum of squared strain terms; **U_0 can not be negative!** In terms of the stresses, U_0 is:

$$U_0 = \frac{1}{2}\left[\frac{1}{E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{2\nu}{E}\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right] \qquad (8.4)$$

The derivative of U_0 (see 8.4) with respect to σ_1 component equals ε_1,

$$\frac{\partial U_0}{\partial \sigma_1} = \left[\frac{1}{E}\left(\sigma_1\right) - \frac{\nu}{E}\left(\sigma_2 + \sigma_3\right)\right] = \varepsilon_1 \qquad (8.5)$$

and vice versa (Timoshenko and Godier, 1971, p. 246):

$$\frac{\partial U_0}{\partial \sigma_1} = \varepsilon_1, \quad \frac{\partial U_0}{\partial \varepsilon_1} = \sigma_1; \quad \frac{\partial U_0}{\partial \sigma_{xx}} = \varepsilon_{xx}, \quad \frac{\partial U_0}{\partial \varepsilon_{xx}} = \sigma_{xx}, \text{ etc.}$$

The general tensor form for the strain energy density is

$$U_0 = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \qquad (8.6)$$

This reduces to equation 8.2 if the stresses are principal stresses.

V Principle of superposition in linear elasticity

- **A Solutions for stresses, strains, and displacements arising from external loads and body forces acting separately can be superimposed to give the solutions for external loads and body forces acting together**
- **B Critical for BEM modeling**
- **C Applicable provided displacement gradients are small (Infinitesimal theory applies)**

VI Uniqueness

- **A Analytical two-dimensional solutions are unique**
- **B Numerical solutions are approximate; they are not unique**

The proof of “A” follows from superposition. Consider the stresses and strains that arise in a body a result of identical external tractions on two identical bodies. Suppose two different solutions exist, and we wish to find the difference:

	Solution 1	Solution 2	Difference
Surface tractions (i=1,2,3)	T_i^1	T_i^2	$T_i^0 = T_i^1 - T_i^2 = 0$
Boundary Conditions (i=1,2,3)	$T_i^1 = \sum_{j=1}^{j=3} \sigma_{ij}^1 n_j$	$T_i^2 = \sum_{j=1}^{j=3} \sigma_{ij}^2 n_j$	$T_i^0 = \sum_{j=1}^{j=3} \sigma_{ij}^0 n_j$
Body forces (i=1,2,3)	F_i^1	F_i^2	$F_i^0 = F_i^1 - F_i^2 = 0$
Equilibrium Equations (j=1,2,3)	$\sum_{i=1}^{i=3} \frac{\partial \sigma_{ij}^1}{\partial x_i} + F_i^1 = 0$	$\sum_{i=1}^{i=3} \frac{\partial \sigma_{ij}^2}{\partial x_i} + F_i^2 = 0$	$\sum_{i=1}^{i=3} \frac{\partial \sigma_{ij}^0}{\partial x_i} = 0$
Compatibility of strain	ε_{ij}^1	ε_{ij}^2	$\varepsilon_{ij}^0 = \varepsilon_{ij}^1 - \varepsilon_{ij}^2$

Conditions satisfying the third column are: no surface forces (i.e., $T^0 = 0$) and no body forces (i.e., $F^0 = 0$). Hence, no work on the body is done and its strain energy must be zero. This can occur only if the strain energy density is zero at every point in the body. In light of the solution for the strain energy, this means the stresses and the strains in the body are zero (i.e., a body with no surface or body forces is unstressed and unstrained). Thus there can be no difference between solution 1 and solution 2.

VII Saint-Venant's Principle

"The stresses due to two statically equivalent loadings applied over a small area are significantly different only in the vicinity of the area on which the loadings are applied; and at distances which are large in comparison with the linear dimensions of the area on which the loadings are applied, the effects due to these two loading area the same." (From Chou & Pagano, p. 84.)

A Impossible to uniquely invert stress/strain/displacement measurements to infer details of internal deformation by distant measurements

B Provides a basis for boundary element modeling

References

Chou, P.C., and Pagano, N.J., 1967, Elasticity, Dover, New York, p. 65-88.