EQUILIBRIUM & COMPATIBILITY (07)

I Main topics

A Equilibrium

B Compatibility

C Plane strain

D Plane stress

E Anti-plane strain

II Equilibrium

The equations of equilibrium describe how stress can vary within a body. They do not have any information on the rheology of the body, so these equations apply to viscous fluids, plastics, and elastic solids. From balancing forces (see diagram on next page)

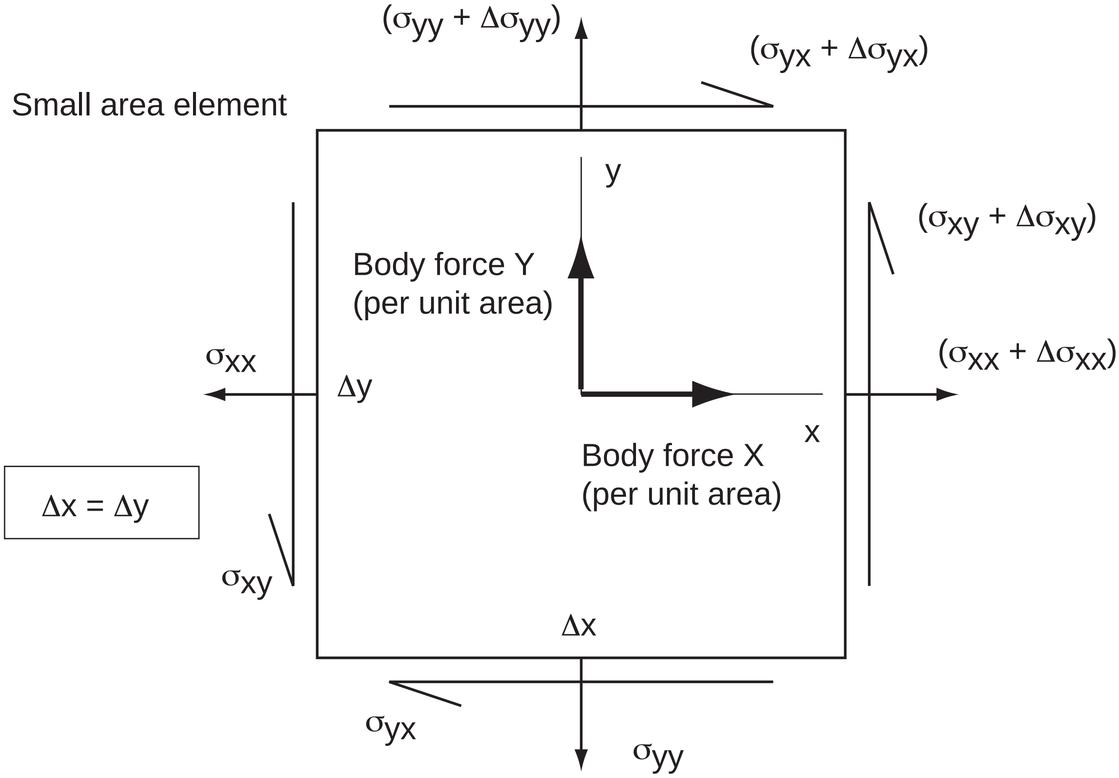
2-D	3-D	
$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + F_1 = 0$	$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + F_1 = 0$	(7.1a)
$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + F_2 = 0$	$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + F_2 = 0$	(7.1b)
$F_3 = 0$	$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + F_3 = 0$	(7.1c)

or in tensor notation

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \tag{7.1d}$$

where F = body force/unit volume. Commonly the only body force is due to gravity, so $F_{vert} = \rho g$. For true plane strain in the x,y plane there can be no body force in the z-direction (Chou & Pagano, p. 70, Barber, p. 70).

EQUATIONS OF EQUILIBRIUM (2-D)



We again turn to force balances: $\sum F_x = 0$ and $\sum F_y = 0$. First we sum forces in the x-direction:

$$(\sigma_{xx} + \Delta\sigma_{xx})(\Delta y) - (\sigma_{xx})(\Delta y) + (\sigma_{yx} + \Delta\sigma_{yx})(\Delta x) - (\sigma_{yx})(\Delta x) + \mathrm{X}\Delta x\Delta y = 0.$$

The terms involving σ_{xx} and σ_{yx} fall out. Dividing through by ΔxΔy gives

$$(\Delta\sigma_{xx})/(\Delta x) + (\Delta\sigma_{yx})/(\Delta y) + \mathrm{X} = 0.$$

Taking the limit as Δx and Δy go to zero: $\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \mathrm{X} = 0.$

Similarly, $\frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{xy}}{\partial x} + \mathrm{Y} = 0.$

These are the equations of equilibrium. Assuming the body forces are constant, decreases in σ_{xx} in the x-direction must be balanced by increases in σ_{yx}in the y-direction, etc.

The moments must balance as well. If moments are taken about the center of the box, only the shear stresses contribute to the moment (the normal stresses and body forces act through the center of the box and hence don’t contribute).

By inspection of the diagram above, one can see that $\Delta\sigma_{yx} = \Delta\sigma_{xy}$, so the shear stresses σ_{yx} and σ_{xy} since they must be equal at one point must be equal at all points.

III Compatibility

A The strains of all the elements of a body must be compatible so that all elements "fit together" during deformation without opening holes.

B The compatibility equation brings information on the elastic response of a body; the equations of equilibrium do not.

C Compatibility conditions can be expressed in terms of strain (better for 2-D problems) or displacement (better for 3-D problems).

The equations for strain in two dimensions are:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad i=1,2;\ j=1,2 \tag{7.2a}$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \tag{7.2b}$$

$$\varepsilon_{12} = \left(\frac{1}{2}\right)\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \varepsilon_{21} \tag{7.2c}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \tag{7.2d}$$

These <u>three</u> equations <u>are not independent</u>; they rely on only <u>two</u> variables, the displacements u_1 and u_2. We can combine the equations by taking partial derivatives (we don't have many other options!):

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^{\,2}} = \frac{\partial^2\left(\frac{\partial u_1}{\partial x_1}\right)}{\partial x_2^{\,2}} = \frac{\partial^3 u_1}{\partial x_2^{\,2}\partial x_1} \tag{7.3}$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_1^{\,2}} = \frac{\partial^2\left(\frac{\partial u_2}{\partial x_2}\right)}{\partial x_1^{\,2}} = \frac{\partial^3 u_2}{\partial x_1^{\,2}\partial x_2} \tag{7.4}$$

$$\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)}{\partial x_1 \partial x_2} = \frac{1}{2}\left(\frac{\partial^3 u_1}{\partial x_1 \partial x_2^{\,2}} + \frac{\partial^3 u_2}{\partial x_1^{\,2}\partial x_2}\right) \tag{7.5}$$

Half the sum of (7.3) and (7.4) equals (7.5), so

$$\frac{1}{2}\left(\frac{\partial^2 \varepsilon_{11}}{\partial x_2^{\,2}} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^{\,2}}\right) = \frac{\partial^2 \varepsilon_{12}}{\partial x_2 \partial x_1} \quad \text{condition of compatibility} \tag{7.6}$$

This gives us three variables (3 strains) and three equations (7.1a, 7.1b, and 7.6). These equations are sufficient to determine how the stress and deformation fields vary within an elastic body with no cavities (need to check whether displacements are single-valued in a body with cavities).

IV Plane strain

A Displacements permitted only in two directions, and they do not vary as a function of the third direction; displacements in the third direction are zero

B Restrained, frictionless boundaries

C Infinite 2-D bodies by symmetry ("thick plate")

Suppose displacements are not allowed in the z (or x_3) directions. Then

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = 0 + 0 = 0 = \varepsilon_{31} \qquad (7.7)$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = 0 + 0 = 0 = \varepsilon_{32} \qquad \text{5 strain terms are zero} \qquad (7.8)$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0 \;\; = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right],\ so\ \ \sigma_{33} = \nu\left(\sigma_{11} + \sigma_{22}\right) \qquad (7.9)$$

The four strains that can be non-zero are:

$$\varepsilon_{11} = \frac{1}{E}\left[\left(1-\nu^2\right)\sigma_{11} - \nu(1+\nu)\sigma_{22}\right] \qquad (7.10)$$

$$\varepsilon_{22} = \frac{1}{E}\left[\left(1-\nu^2\right)\sigma_{22} - \nu(1+\nu)\sigma_{11}\right] \qquad \text{(note similarity to 7.10)} \qquad (7.11)$$

$$\varepsilon_{12} = \frac{1}{E}\left[(1+\nu)\sigma_{12}\right] = \varepsilon_{21} \qquad (7.12)$$

D Compatibility equation for plane strain

Substituting (7.7)-(7.12) into the compatibility condition (7.6) yields:

$$\frac{1}{2}\left(\frac{\partial^2\left[\left(1-\nu^2\right)\sigma_{11} - \nu(1+\nu)\sigma_{22}\right]}{\partial x_2{}^2} + \frac{\partial^2\left[\left(1-\nu^2\right)\sigma_{22} - \nu(1+\nu)\sigma_{11}\right]}{\partial x_1{}^2}\right) = \frac{\partial^2\left[(1+\nu)\sigma_{12}\right]}{\partial x_2 \partial x_1} \qquad (7.13)$$

$$(1-\nu^2)\left\{\frac{\partial^2\sigma_{11}}{\partial x_2{}^2} + \frac{\partial^2\sigma_{22}}{\partial x_1{}^2}\right\} - \nu(1+\nu)\left\{\frac{\partial^2\sigma_{11}}{\partial x_1{}^2} + \frac{\partial^2\sigma_{22}}{\partial x_2{}^2}\right\} = 2(1+\nu)\frac{\partial^2\sigma_{12}}{\partial x_1 \partial x_2} \qquad (7.14)$$

Dividing both sides through by $(1+\nu)$

$$(1-\nu)\left\{\frac{\partial^2\sigma_{xx}}{\partial y^2}+\frac{\partial^2\sigma_{yy}}{\partial x^2}\right\}-\nu\left\{\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}\right\}=2\frac{\partial^2\sigma_{xy}}{\partial x\partial y} \quad (7.15)$$

The right side of (7.15) can be derived from the equilibrium equations

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+X=0, \quad (7.16a) \qquad \frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{xy}}{\partial x}+Y=0. \quad (7.16b)$$

Take the derivatives of (7.16a) with respect to x and (7.16b) relative to y to get terms of $\partial^2\sigma_{xy}/\partial x\partial y$.

$$\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yx}}{\partial x\partial y}+\frac{\partial X}{\partial x}=0, \quad (7.17a) \qquad \frac{\partial^2\sigma_{yy}}{\partial y^2}+\frac{\partial^2\sigma_{yx}}{\partial x\partial y}+\frac{\partial Y}{\partial y}=0. \quad (7.17b)$$

Adding these together and isolating the $\partial^2\sigma_{xy}/\partial y^2$ term:

$$\frac{2\partial^2\sigma_{xy}}{\partial x\partial y}=-\left\{\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}+\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right\}. \quad (7.18)$$

Equating the right side of (7.18) with the left side of (7.15):

$$(1-\nu)\left\{\frac{\partial^2\sigma_{xx}}{\partial y^2}+\frac{\partial^2\sigma_{yy}}{\partial x^2}\right\}-\nu\left\{\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}\right\}=-\left\{\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}+\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right\}. \quad (7.19)$$

This can be simplified

$$(1-\nu)\left\{\frac{\partial^2\sigma_{xx}}{\partial y^2}+\frac{\partial^2\sigma_{yy}}{\partial x^2}\right\}+(1-\nu)\left\{\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}\right\}=-\left\{\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right\}. \quad (7.20)$$

Now let $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ (∇^2 = Laplacian operator).

$$\nabla^2\left\{\sigma_{xx}+\sigma_{yy}\right\}=-\frac{1}{1-\nu}\left\{\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right\}. \quad (7.21)$$

This gives the governing equation for 2-D plane strain. Note that if the body forces are constant, then the stress variation is independent of the elastic properties (E and ν) of the material; steel behaves as plastic.

V Plane stress (Generalized plane stress)

A Stresses in one direction are zero (thin plate approximation)

B A thin plate, plane stress solution is approximate; stresses are really those averaged over the thickness of the thin plate. The mid-plane of a thin plate under "plane stress" actually feels plane strain!

C Plane stress solutions can be converted to plane strain solutions by substituting $\boxed{E = \frac{E'}{1-\nu'^2};\ \nu = \frac{\nu'}{1-\nu'}}$ **in the plane stress solutions.**

VI Anti-plane strain

A Displacements permitted only in one direction (e.g., the z-direction), do not vary with z, but can vary as a function of x and y.

B Because the displacement fields are one-dimensional, anti-plane strain is simpler to address than plane strain or plane stress

C Strains (five terms equal zero, including all the normal strains)

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0 \tag{7.22}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0 + 0 = 0 = \varepsilon_{21} \tag{7.23}$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left(0 + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\frac{\partial u_3}{\partial x_1} = \varepsilon_{31} \tag{7.24}$$

$$\varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = 0 + 0 = 0 = \varepsilon_{12} \tag{7.25}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0 \tag{7.26}$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(0 + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\frac{\partial u_3}{\partial x_2} = \varepsilon_{32} \tag{7.27}$$

$$\varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + 0\right) = \frac{1}{2}\frac{\partial u_3}{\partial x_1} = \varepsilon_{13} \tag{7.28}$$

$$\varepsilon_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + 0\right) = \frac{1}{2}\frac{\partial u_3}{\partial x_2} = \varepsilon_{23} \tag{7.29}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0 \tag{7.30}$$

D Stresses (five terms equal zero, including all the normal stresses)

$$\sigma_{11} = \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + \frac{E}{(1+\nu)}\varepsilon_{11} = 0+0=0 \quad (7.31)$$

$$\sigma_{12} = 2G\varepsilon_{12} = 0 \quad (7.32)$$

$$\varepsilon_{13} = 2G\varepsilon_{13} = G\frac{\partial u_3}{\partial x_1} \quad (7.33)$$

$$\varepsilon_{21} = 2G\varepsilon_{21} = 0 \quad (7.34)$$

$$\sigma_{22} = \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + \frac{E}{(1+\nu)}\varepsilon_{22} = 0+0=0 \quad (7.35)$$

$$\varepsilon_{23} = 2G\varepsilon_{23} = G\frac{\partial u_3}{\partial x_2} \quad (7.36)$$

$$\varepsilon_{31} = 2G\varepsilon_{31} = G\frac{\partial u_3}{\partial x_1} \quad (7.37)$$

$$\varepsilon_{32} = 2G\varepsilon_{32} = G\frac{\partial u_3}{\partial x_2} \quad (7.38)$$

$$\sigma_{33} = \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + \frac{E}{(1+\nu)}\varepsilon_{33} = 0+0=0 \quad (7.39)$$

E Equilibrium (<u>in the absence of body forces</u>)

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}+F_1=0 \Rightarrow 0+0+0=0 \ \ \textit{Identically solved} \quad (7.40)$$

$$\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3}+F_2=0 \Rightarrow 0+0+0=0 \ \ \textit{Identically solved} \quad (7.41)$$

$$\frac{\partial\sigma_{31}}{\partial x_1}+\frac{\partial\sigma_{32}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}+F_3=0 \Rightarrow \frac{\partial\sigma_{13}}{\partial x_1}+\frac{\partial\sigma_{23}}{\partial x_2}=0 \Rightarrow G\left(\frac{\partial^2 u_3}{\partial x_1^{\,2}}+\frac{\partial^2 u_3}{\partial x_2^{\,2}}=0\right) \Rightarrow \nabla^2 u_3 = 0 \quad (7.42)$$

F Compatibility (<u>in the absence of body forces</u>)

The displacements must yield compatible strains

$$\varepsilon_{13} = \frac{1}{2}\frac{\partial u_3}{\partial x_1} \Rightarrow \frac{\partial\varepsilon_{13}}{\partial x_2} = \frac{1}{2}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} \quad (7.43)$$

$$\varepsilon_{23} = \frac{1}{2}\frac{\partial u_3}{\partial x_2} \Rightarrow \frac{\partial\varepsilon_{23}}{\partial x_1} = \frac{1}{2}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} \quad (7.44)$$

$$\frac{\partial\varepsilon_{13}}{\partial x_2} = \frac{\partial\varepsilon_{23}}{\partial x_1} \Rightarrow \frac{\partial^2\varepsilon_{13}}{\partial x_1 \partial x_2} = \frac{\partial^2\varepsilon_{23}}{\partial x_1^{\,2}} \quad (7.45)$$

In light of the equilibrium condition (7.42), written for strains,

$$\frac{\partial \varepsilon_{13}}{\partial x_1} = \frac{-\partial \varepsilon_{23}}{\partial x_2} \Rightarrow \frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_2} = \frac{-\partial^2 \varepsilon_{23}}{\partial {x_2}^2} \qquad (7.46)$$

Subtracting (7.46) from (7.45) yields

$$\nabla^2 \varepsilon_{23} = 0 \qquad (7.47)$$

Substituting for the shear stress using (7.36) yields

$$\nabla^2 \sigma_{23} = 0 \qquad (7.48)$$

One can show by analogous procedures that

$$\nabla^2 \varepsilon_{13} = 0 \qquad (7.49)$$

and

$$\nabla^2 \sigma_{13} = 0 \qquad (7.50)$$

So for anti-plane strain, the stresses, strains, and displacements in an x,y,z reference frame all obey the Laplace equation.

F <u>Plane strain and anti-plane strain solutions are completely independent and can be superposed.</u>

<u>References</u>

Barber, J.R., 1993, Elasticity: Kluwer Academic Publishers, Boston, p. 21-37

Timoshenko, S.P., and Goodier, J.N., 1971: Theory of elasticity, McGraw-Hill, New York, p. 26-33.