

## STRAIN (06)

### I Main topics

- A General deformation**
- B Tensor notations for strain**
- C Relationship between stress and strain**

### II General deformation (changes of position of points in a body)

#### **A Rigid body translation**

- 1 All points displaced by an equal vector (equal magnitude and direction); no displacement of points relative to one another
- 2  $[X'] = [u] + [X]$  matrix addition

#### **B Rigid body rotation**

- 1 All points rotated by an equal amount about a common axis; no angular displacement of points relative to one another
- 2  $[X'] = [a][X]$  matrix multiplication; rows in  $[a]$  are dir. cosines!

#### **C Change in shape (distortional strain)**

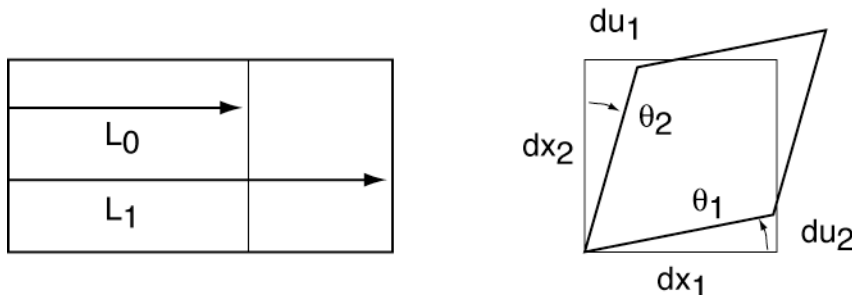
- 1 Change in linear dimension (normal strain or elongation)

$$\varepsilon = \frac{\Delta L}{L_o} = \frac{L_1 - L_o}{L_o} \quad \text{dimensionless!} \quad (6.1)$$

- 2 Change in angles (shear strain)

$$\gamma = \theta_1 + \theta_2 = 2\varepsilon_{xy} = 2\varepsilon_{yx} \quad \text{dimensionless!} \quad (6.2)$$

For small angle changes,  $\theta_1 = \tan^{-1}(du_2/dx_1) \approx du_2/dx_1$



A positive shear strain corresponds to a **decrease** in the right angle

#### **D Change in volume (dilation)**

$$\Delta = \frac{\Delta V}{V_o} = \frac{V_1 - V_o}{V_o} \quad \text{dimensionless!} \quad (6.3)$$

### III Tensor notations for infinitesimal strain

Displacement gradient matrix = Strain matrix + rotation matrix  
 $J_{ij} = \epsilon_{ij} + \omega_{ij}$  (6.4)

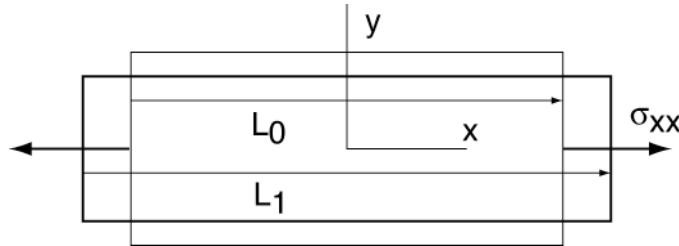
$$J_{ij} = \begin{pmatrix} \frac{\partial u_i}{\partial x_j} \\ \frac{\partial u_j}{\partial x_i} \end{pmatrix} \quad \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

In 2-D

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \\ \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) & \frac{1}{2} \left( \frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2} \right) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \left( \frac{\partial u_1}{\partial x_1} - \frac{\partial u_1}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right) \\ \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) & \frac{1}{2} \left( \frac{\partial u_2}{\partial x_2} - \frac{\partial u_2}{\partial x_2} \right) \end{bmatrix} \quad (6.5)$$

### IV Relationship between stress and strain

#### A Uniaxial stress



$$\sigma_{xx} = E \epsilon_{xx} \quad (6.6)$$

E = Young's modulus; dimensions of stress

Analog:

$$F = kx$$

k = spring constant; dimensions of stress\*length

$$\epsilon_{yy} = \epsilon_{zz} = -\nu \epsilon_{xx} \quad (6.7)$$

$\nu$  = Poisson's ratio (dimensionless)

**B General conditions** (for an isotropic medium)

$$\varepsilon_{xx} = \frac{1}{E} \left[ \sigma_{xx} - \nu (\sigma_{yy} + \sigma_{zz}) \right] \quad (6.8)$$

$$\varepsilon_{yy} = \frac{1}{E} \left[ \sigma_{yy} - \nu (\sigma_{zz} + \sigma_{xx}) \right] \quad (6.9)$$

$$\varepsilon_{zz} = \frac{1}{E} \left[ \sigma_{zz} - \nu (\sigma_{xx} + \sigma_{yy}) \right] \quad (6.10)$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2G} \sigma_{xy} \quad (6.11)$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2G} \sigma_{yz} \quad (6.12)$$

$$\varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2G} \sigma_{zx} \quad (6.13)$$

For isotropic materials, the principal stresses parallel principal strains!

Relationships Among Elastic Constants for an Isotropic Solid  
(from Mal and Singh, 1991, p. 16)

Pair	$\lambda, G$ ( $\lambda, \mu$ )	$\lambda, \nu$	$G, k$ ( $\mu, k$ )	$\mu, E$	$k, \nu$	$E, \nu$
$\lambda =$	<i>Lame parameter</i>		$k - \frac{2G}{3}$	$\frac{G(E - 2G)}{3G - E}$	$\frac{3k\nu}{1 + \nu}$	$\frac{E\nu}{(1 + \nu)(1 - 2\nu)}$
$G =$ ( $\mu =$ )	<i>Shear modulus</i>	$\frac{\lambda(1 - 2\nu)}{2\nu}$			$\frac{3k(1 - 2\nu)}{2(1 + \nu)}$	$\frac{E}{2(1 + \nu)}$
$k =$	$\lambda + \frac{2G}{3}$	$\frac{\lambda(1 + \nu)}{3\nu}$	<i>Bulk modulus</i>	$\frac{GE}{3(3G - E)}$		$\frac{E}{3(1 - 2\nu)}$
$E =$	$\frac{G(3\lambda + 2G)}{\lambda + G}$	$\frac{\lambda(1 + \nu)(1 - 2\nu)}{\nu}$	$\frac{9kG}{3k + G}$		$3k(1 - 2\nu)$	<i>Young's modulus</i>
$\nu =$	$\frac{\lambda}{2(\lambda + G)}$		$\frac{3k - 2G}{2(3k + G)}$	$\frac{E}{2G} - 1$		<i>Poisson's ratio</i>

References

- Barber, J.R., 1993, *Elasticity*: Kluwer Academic Publishers, Boston, p. 11-19.  
 Chou, P.C., and Pagano, N.J., 1967, *Elasticity*, Dover, New York, p. 170-179.  
 Malvern, L.E., 1969, *Introduction to the mechanics of a continuous medium*: Prentice-Hall, Englewood Cliffs, p. 120-127.  
 Timoshenko, S.P., and Goodier, J.N., 1971: *Theory of elasticity*, McGraw-Hill, New York

