

CAUCHY'S FORMULA AND EIGENVAULES (PRINCIPAL STRESSES) (05)

I Main Topics

A Cauchy's formula

B Principal stresses (eigenvectors and eigenvalues)

II Cauchy's formula

A Relates traction vector components to stress tensor components (see Figures 5.1, 5.2, 5.3 for derivation)

B $T_i = \sigma_{ij} n_j$ (5.1)**1 Meaning of terms**a T_i =traction vector component: $\vec{T} = T_1\vec{i} + T_2\vec{j} + T_3\vec{k}$ b σ_{ij} = stress componentc \mathbf{n} =unit normal vector. The components n_j of the unit normal are the direction cosines between \mathbf{n} and the coordinate axes.d $\frac{F_i}{A} = \frac{F_i}{A_1} \frac{A_1}{A} + \frac{F_i}{A_2} \frac{A_2}{A} + \frac{F_i}{A_3} \frac{A_3}{A}$

2 This represents the physics directly

3 The traction component that acts *in* the i-direction reflects the contribution of the stresses that act *in* that direction.**4 Note that the j's "cancel out"****5 Note that the subscripts on the T and the n differ**6 σ is symmetric ($\sigma_{ij} = \sigma_{ji}$), so ...C $T_i = \sigma_{ij} n_j$ Standard form of Cauchy's formula**1 The subscript j's still "cancel out"**

2 The subscripts on the T and the n still differ

3 Easier(?) to remember than "B"

D Full expansion

$$T_i = \sigma_{ji} n_j$$

$$T_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

$$T_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$

$$T_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$

$$T_i = \sigma_{ij} n_j$$

$$= \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$

$$= \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$$

$$= \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$$

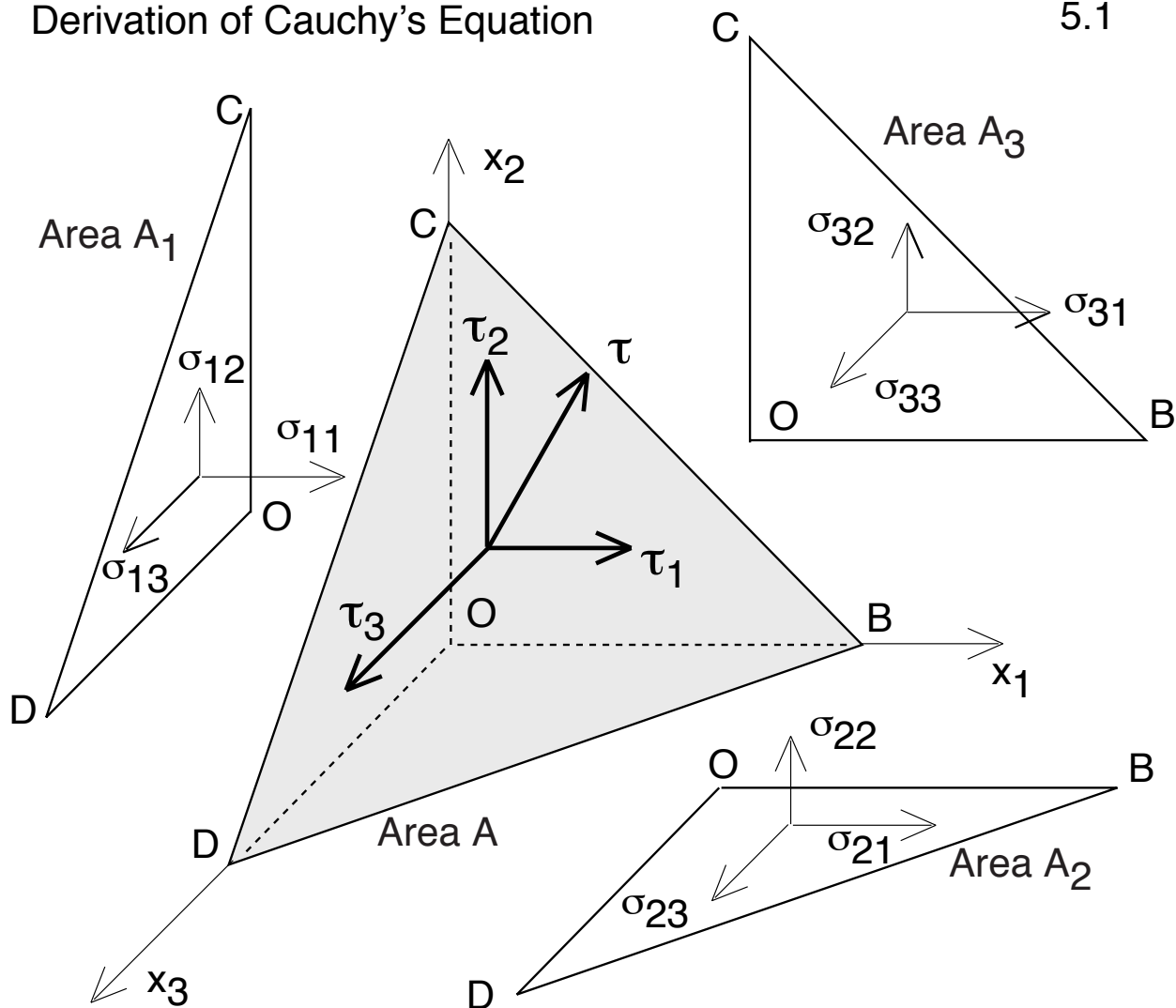
E Matrix form

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Derivation of Cauchy's Equation

5.1



$$\Sigma \mathbf{F}_1 = 0, \text{ so } (\Sigma \mathbf{F}_1)/A = 0.$$

$$\tau_1(A/A) = (\sigma_{11})(A_1/A) + (\sigma_{21})(A_2/A) + (\sigma_{31})(A_3/A).$$

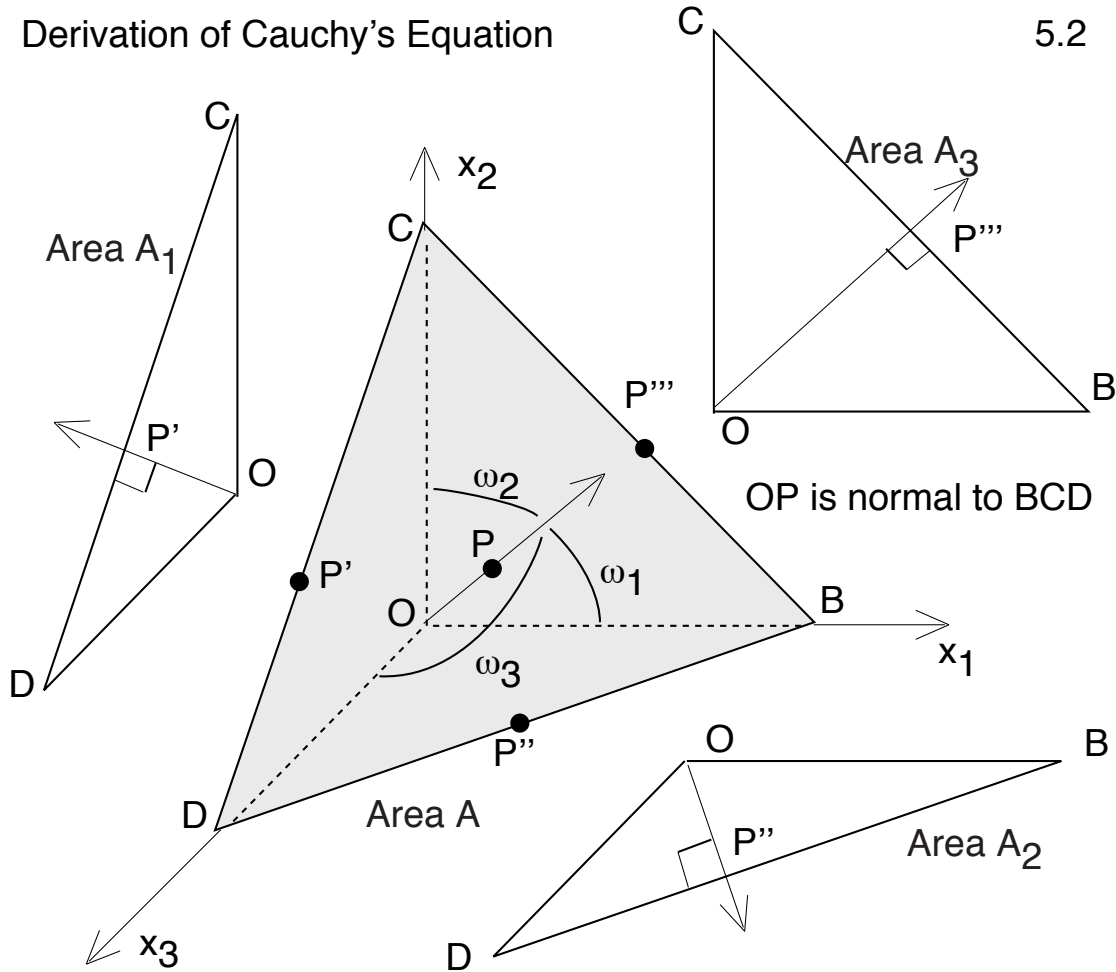
Similarly, $\Sigma \mathbf{F}_2 = 0$ and $\Sigma \mathbf{F}_3 = 0$, so

$$\tau_2(A/A) = (\sigma_{12})(A_1/A) + (\sigma_{22})(A_2/A) + (\sigma_{32})(A_3/A).$$

$$\tau_3(A/A) = (\sigma_{13})(A_1/A) + (\sigma_{23})(A_2/A) + (\sigma_{33})(A_3/A).$$

Derivation of Cauchy's Equation

5.2



Note that ΔDCB of area A projects onto the x_1 - x_2 plane as ΔOCB , onto the x_2 - x_3 plane as ΔOCD , and onto the x_3 - x_1 plane as ΔODB .

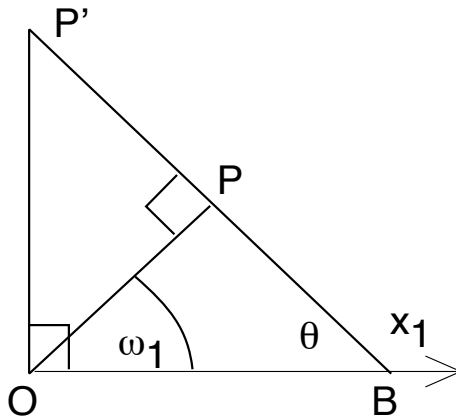
BOP' is perpendicular to CD , and because CD is a line in BCD , BOP' is perpendicular to BCD . Similarly, COP'' is perpendicular to BD , so COP'' is perpendicular to BCD . The intersection of BOP' and COP'' is perpendicular to BCD , and that intersection is OP .

$\omega_1, \omega_2, \omega_3$, are angles between OP and x_1, x_2 , and x_3 , respectively.

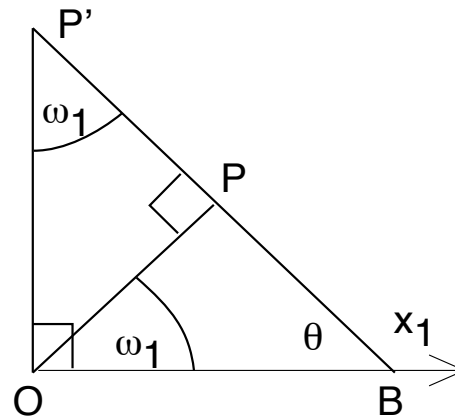
$$\frac{A_1}{A} = \frac{1/2 (\text{base } OCD)(\text{height } OCD)}{1/2 (\text{base } DCB)(\text{height } CBD)} = \frac{(CD)(OP')}{(CD)(BP')} = \frac{OP'}{BP'}$$

Derivation of Cauchy's Equation

5.3



Triangles BOP and BP'O are similar right triangles; they both have angle OBP (i.e., θ) in common.



Therefore, angle BP'O = ω_1 .

$$\frac{A_1}{A} = \frac{OP'}{BP'} = \cos \omega_1 = n_1$$

Similarly, $\frac{A_2}{A} = \frac{OP''}{CP''} = \cos \omega_2 = n_2$ and $\frac{A_3}{A} = \frac{OP''' }{DP'''} = \cos \omega_3 = n_3$ so

$$\tau_1(A/A) = (\sigma_{11})(A_1/A) + (\sigma_{21})(A_2/A) + (\sigma_{31})(A_3/A) \quad \text{becomes}$$

$$\tau_1 = (\sigma_{11})(n_1) + (\sigma_{21})(n_2) + (\sigma_{31})(n_3). \quad \text{Similarly,}$$

$$\tau_2(A/A) = (\sigma_{12})(A_1/A) + (\sigma_{22})(A_2/A) + (\sigma_{32})(A_3/A) \quad \text{becomes}$$

$$\tau_2 = (\sigma_{12})(n_1) + (\sigma_{22})(n_2) + (\sigma_{32})(n_3), \quad \text{and}$$

$$\tau_3(A/A) = (\sigma_{13})(A_1/A) + (\sigma_{23})(A_2/A) + (\sigma_{33})(A_3/A) \quad \text{becomes}$$

$$\tau_3 = (\sigma_{13})(n_1) + (\sigma_{23})(n_2) + (\sigma_{33})(n_3).$$

So $\tau_i = \sigma_{ji} n_j$, but $\sigma_{ij} = \sigma_{ji}$, so $\tau_i = \sigma_{ij} n_j$

⋮

III. Principal stresses from tensor and matrix perspectives

Consider a plane with a normal vector \mathbf{n} defined by direction cosines n_1 , n_2 , and n_3 . The components of traction \mathbf{T} on the plane, by Cauchy's formula, are $T_i = \sigma_{ij} n_j$. They also are simply the components of \mathbf{T} : $T_1 = Tn_1$, $T_2 = Tn_2$, and $T_3 = Tn_3$. The components can be equated:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = T \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}. \quad (5.2)$$

The right side of (5.2) can be subtracted from the left side to yield:

$$\begin{bmatrix} \sigma_{11} - T & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - T & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - T \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0. \quad (5.3)$$

Equation (5.3) can be rewritten

$$[\sigma - IT] [\mathbf{n}] = 0, \quad \text{where } I \text{ is the identity matrix} \quad (5.4)$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \text{For any square matrix } [A], [A][I] = [A]. \quad (5.5)$$

According to theorems of linear algebra, equation (5.3) can be solved only if the determinant $|\sigma - IT|$ equals zero:

$$\begin{vmatrix} \sigma_{11} - T & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - T & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - T \end{vmatrix} = 0 \quad (5.6)$$

In many cases the components of σ are known but T is must be solved for.

Problems of the form of equation (5.4) are common in many branches of mathematics, engineering, and physics, and they have a special name:

eigenvalue problems. The values of T (i.e., $|T|$, the principal values) that solve the equation are called eigenvalues, and the vectors \mathbf{n} (the principal directions) that give the directions of \mathbf{T} are called eigenvectors. Because these problems are so common, many mathematics packages, including Matlab, have special routines to solve for eigenvalues and eigenvectors.

Solving (5.6) by hand requires finding the roots of a cubic equation (not easy), so we consider the easier 2-D case, which yields a quadratic equation.

$$\begin{vmatrix} \sigma_{11} - T & \sigma_{12} \\ \sigma_{21} & \sigma_{22} - T \end{vmatrix} = 0 \quad \text{Note: } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (5.7)$$

$$(\sigma_{11} - T)(\sigma_{22} - T) - (\sigma_{12})(\sigma_{21}) = 0 \quad (5.8)$$

$$T^2 - T(\sigma_{11} + \sigma_{22}) + (\sigma_{11})(\sigma_{22}) - (\sigma_{12})(\sigma_{21}) = 0 \quad (5.9)$$

$$T^2 - T(\sigma_{11} + \sigma_{22}) + [(\sigma_{11})(\sigma_{22}) - (\sigma_{12})^2] = 0 \quad (5.10a) \quad \text{or} \quad T^2 - T(I_1) + [I_2] = 0 \quad (5.10b)$$

The term T in equations (5.10) is solved using the quadratic formula:

$$T = \frac{(\sigma_{11} + \sigma_{22}) \pm \sqrt{(\sigma_{11} + \sigma_{22})^2 - 4[(\sigma_{11})(\sigma_{22}) - (\sigma_{12})^2]}}{2} = \frac{I_1 \pm \sqrt{I_1^2 - 4I_2}}{2} \quad (5.11)$$

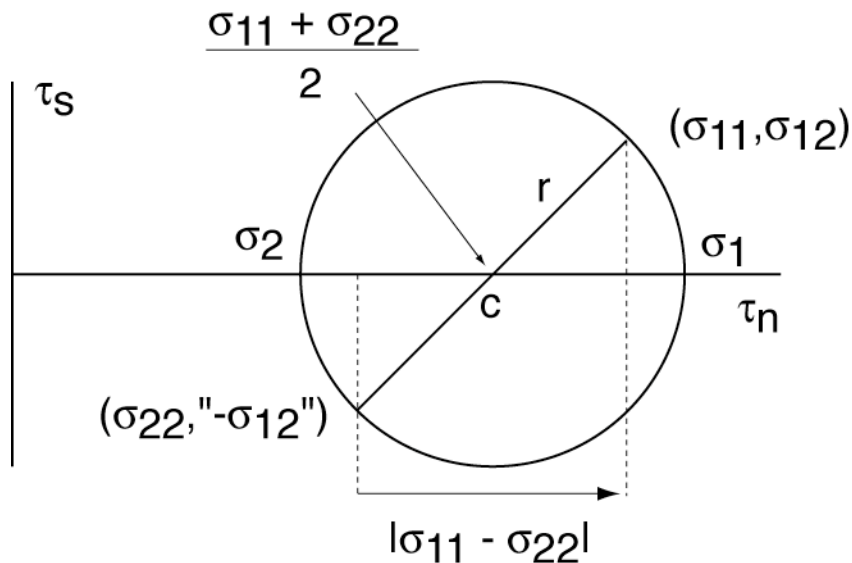
$$T = \frac{(\sigma_{11} + \sigma_{22}) \pm \sqrt{(\sigma_{11}^2 + 2\sigma_{11}\sigma_{22} + \sigma_{22}^2) - 4[(\sigma_{11})(\sigma_{22}) - (\sigma_{12})^2]}}{2} \quad (5.12)$$

$$T = \frac{(\sigma_{11} + \sigma_{22}) \pm \sqrt{(\sigma_{11}^2 - 2\sigma_{11}\sigma_{22} + \sigma_{22}^2) + 4[\sigma_{12}^2]}}{2} \quad (5.13)$$

$$T = \frac{(\sigma_{11} + \sigma_{22}) \pm \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4[\sigma_{12}^2]}}{2} = \frac{I_1 \pm \sqrt{I_1^2 - 4I_2}}{2} \quad (5.14)$$

$$T = \left[\frac{\sigma_{11} + \sigma_{22}}{2} \right] \pm \left[\sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2} \right)^2 + \sigma_{12}^2} \right] = [c] \pm [r] = \left[\frac{I_1}{2} \right] \pm \left[\sqrt{\left(\frac{I_1}{2} \right)^2 - I_2} \right] = \sigma_1, \sigma_2 \quad (5.15)$$

An inspection of the diagram below shows that the first term in brackets in equation (5.15) is the mean normal stress (i.e., the center of the Mohr circle) and the second term in brackets is the maximum possible shear stress (i.e., the radius of the Mohr circle). So the principal stresses lie at the end of a horizontal diameter through the Mohr circle. The terms c , r , I_1 , and I_2 are called invariants and are independent of the frame of reference.



Example

Suppose the stress state at a point is given by

$$\sigma_{ij} = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}, \quad \text{where dimensions are in MPa.}$$

Solving for the principal values using eq. (14) yields

$$T = \frac{(10+2)}{2} \pm \sqrt{\left(\frac{10-2}{2}\right)^2 + 3^2} = 6 \pm \sqrt{25} = 11 \text{ and } 1$$

Now we substitute these back into (5.3)

$$\begin{bmatrix} 10-11 & 3 \\ 3 & 2-11 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } T = \sigma_1 = 11 \text{ MPa.}$$

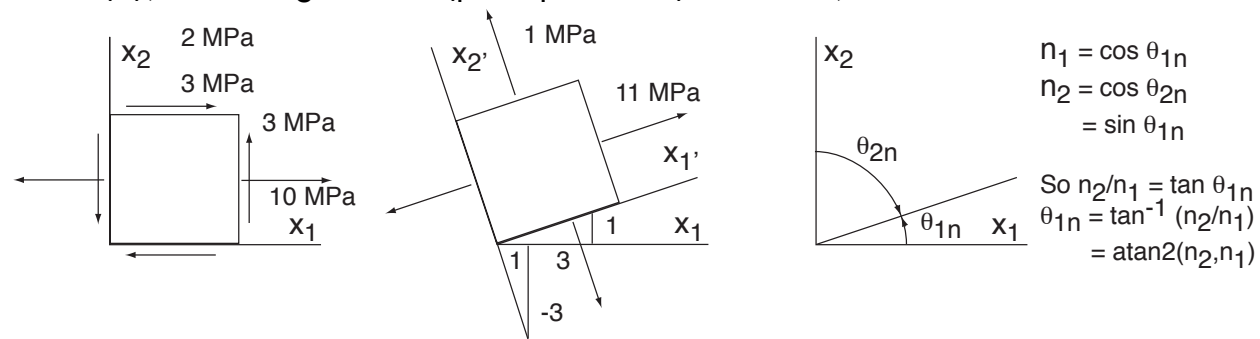
$$\begin{bmatrix} 10-1 & 3 \\ 3 & 2-1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } T = \sigma_2 = 1 \text{ MPa.}$$

These relations yield

$$(a) \quad -n_1 + 3n_2 = 0 \quad (\sigma_1 = 11 \text{ MPa}) \quad (b) \quad 3n_1 + n_2 = 0 \quad (\sigma_2 = 1 \text{ MPa}).$$

From (a), for an eigenvalue (principal value) of 11 MPa, $n_1 = 3n_2$.

From (b), for an eigenvalue (principal value) of 1 MPa, $n_2 = -3n_1$.



For $\sigma_1 = 11 \text{ MPa}^*$

$$\theta_{x_1, normal} = \theta_{x_1, x_1'} = \tan^{-1} \left(\frac{x_2 / |\mathbf{n}|}{x_1 / |\mathbf{n}|} \right) = \tan^{-1} \left(\frac{n_2}{n_1} \right) = \tan^{-1} \left(\frac{n_2}{3n_2} \right) = \tan^{-1} \left(\frac{1}{3} \right) = 18.5^\circ$$

For $\sigma_2 = 1 \text{ MPa}^*$

$$\theta_{x_1, normal} = \theta_{x_1, x_2'} = \tan^{-1} \left(\frac{x_2 / |\mathbf{n}|}{x_1 / |\mathbf{n}|} \right) = \tan^{-1} \left(\frac{n_2}{n_1} \right) = \tan^{-1} \left(\frac{-3n_1}{n_1} \right) = \tan^{-1}(-3) = -71.5^\circ$$

The two eigenvectors are perpendicular, as they are supposed to be.

* In the first expression for θ , the normal direction is the x_1' direction, and n_1 and n_2 are the direction cosines for a unit vector along x_1' . In the second expression for θ , the normal direction is the x_2' direction, and n_1 and n_2 are the direction cosines for a unit vector along x_2' .

V Matrix treatments of stress transformation

In matrix form, $\sigma_{ij}' = a_{i'k} a_{j'l} \sigma_{kl}$ becomes (Mal & Singh, 1991, p. 37)

$$\sigma' = [a] [\sigma] [a^T], \quad (5.16)$$

where

$$a = \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} \quad (5.17)$$

$$a^T = \begin{bmatrix} a_{1'1} & a_{2'1} & a_{3'1} \\ a_{1'2} & a_{2'2} & a_{3'2} \\ a_{1'3} & a_{2'3} & a_{3'3} \end{bmatrix} \quad (5.18)$$

The proper order of matrix multiplication is **essential** in order to reproduce the expansions of lecture 17: $[a] [\sigma] [a^T] \neq [a^T] [\sigma] [a]$!

In MATLAB, equation (5.16) would be written:

$$\text{sigmaprime} = a * \text{sigma} * a'$$

The term a' signifies $[a^T]$. Matlab also has a function “eig” to find eigenvectors (given in terms of the direction cosines) and eigenvalues.

$$[V,D] = \text{eig}(\text{sigma})$$

Example

$$\gg \text{sigmaxy} = [10 \ 3; 3 \ 2]$$

$$\text{sigmaxy} =$$

$$\begin{array}{cc} 10 & 3 \\ 3 & 2 \end{array}$$

$$\gg \mathbf{a} = [3/\sqrt{10} \ 1/\sqrt{10}; -1/\sqrt{10} \ 3/\sqrt{10}]$$

$$\mathbf{a} =$$

$$\begin{array}{cc} 0.9487 & 0.3162 \\ -0.3162 & 0.9487 \end{array}$$

The first row of matrix "a" is the negative of the first column of matrix V below.

The second row of matrix "a" is the negative of the second column of matrix V below.

$$\gg \text{sigmaprime} = \mathbf{a} * \text{sigmaxy} * \mathbf{a}'$$

$$\text{sigmaprime} =$$

$$\begin{array}{cc} 11.0000 & -0.0000 \\ -0.0000 & 1.0000 \end{array}$$

$$\gg [\mathbf{V}, \mathbf{D}] = \text{eig}(\text{sigmaxy})$$

$$\mathbf{V} =$$

$$\begin{array}{cc} -0.9487 & 0.3162 \\ -0.3162 & -0.9487 \end{array}$$

Column 1 in V relates to column 1 in D

Column 2 in V relates to column 2 in D

$$\mathbf{D} =$$

$$\begin{array}{cc} 11 & 0 \\ 0 & 1 \end{array}$$

The direction cosines (eigenvectors) in the first column of V correspond to the eigenvalue in the first column of D.

The direction cosines (eigenvectors) in the second column of V correspond to the eigenvalue in the second column of D.