TENSOR TRANSFORMATIONS (03)

I Main topics

A Review of vector (first-order tensor) transformations

B Second-order tensor transformations

II Review of vector (first-order tensor) transformations

Consider a vector **T.** It has a magnitude and one associated direction (one subscript). If T transforms according to the "rules" of (3.1), then it is a first-order tensor.

Consider **T** in three coordinate systems: X, X', X''.

$$T_{i}'' = a_{i''j'} T_{j'} \quad \text{and} \quad T_{j}' = a_{j'k} T_k \tag{3.1}$$

These can be expressed in matrix form too

$$\begin{bmatrix} T_{1''} \\ T_{2''} \\ T_{3''} \end{bmatrix} = \begin{bmatrix} a_{1''1} & a_{1''2} & a_{1''3} \\ a_{2''1} & a_{2''2} & a_{2''3} \\ a_{3''1} & a_{3''2} & a_{3''3} \end{bmatrix} \begin{bmatrix} T_{1'} \\ T_{2'} \\ T_{3'} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} T_{1'} \\ T_{2'} \\ T_{3'} \end{bmatrix} = \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \tag{3.2}$$

Substituting the expression for T_j' on the right into the one of the left:

$$T_{i}'' = a_{i''j'} a_{j'k} T_k \tag{3.3}$$

$$\begin{bmatrix} T_{1''} \\ T_{2''} \\ T_{3''} \end{bmatrix} = \begin{bmatrix} a_{1''1'} & a_{1''2'} & a_{1''3'} \\ a_{2''1'} & a_{2''2}' & a_{2''3}' \\ a_{3''1}' & a_{3''2'} & a_{3''3'} \end{bmatrix} \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \tag{3.4}$$

If the X'' and X reference frames are the same, then $T_i'' = T_k$

Let us replace each i” term in (3.3) by k:

$$T_k = a_{kj'} a_{j'k} T_k \tag{3.5}$$

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} a_{11'} & a_{12'} & a_{13'} \\ a_{21'} & a_{22}' & a_{23}' \\ a_{31}' & a_{32'} & a_{33'} \end{bmatrix} \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \tag{3.6}$$

This means that $a_{kj'} a_{j'k}$ must equal the identity matrix I:

$$\begin{bmatrix} a_{11'} & a_{12'} & a_{13'} \\ a_{21'} & a_{22'} & a_{23'} \\ a_{31'} & a_{32'} & a_{33'} \end{bmatrix} \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.7}$$

We can get this same result another way. We know that if we have three perpendicular unit vectors **n1**, **n2**, and **n3**, with x, y, and z components, then
n1.n1 =1, **n1.n2** = 0, and **n1.n3** = 0;
n2.n1 =0, **n2.n2** = 1, and **n2.n3** = 0;
n3.n1 =0, **n3.n2** = 0, and **n3.n3** = 1.

These equations are expressed with matrices as:

$$\begin{bmatrix} \begin{bmatrix} n1_x & n1_y & n1_z \end{bmatrix} \\ \begin{bmatrix} n2_x & n2_y & n2_z \end{bmatrix} \\ \begin{bmatrix} n3_x & n3_y & n3_z \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} n1_x \\ n1_y \\ n1_z \end{bmatrix} & \begin{bmatrix} n2_x \\ n2_y \\ n2_z \end{bmatrix} & \begin{bmatrix} n3_x \\ n3_y \\ n3_z \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; or \; n * n^T = I \tag{3.8}$$

So the sum of the squares of the direction cosines for a particular direction equals 1 (if the direction cosines for a given axis are dotted into themselves, then the answer is one; the directions are parallel). If the direction cosines for perpendicular directions are dotted into each other, the result is zero.

Replacing x with 1’, y with 2’, and z with 3’

$$\begin{bmatrix} a_{11'} & a_{12'} & a_{13'} \\ a_{21'} & a_{22'} & a_{23'} \\ a_{31'} & a_{32'} & a_{33'} \end{bmatrix} \begin{bmatrix} a_{11'} & a_{21'} & a_{31'} \\ a_{12'} & a_{22'} & a_{32'} \\ a_{13'} & a_{23'} & a_{33'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.9}$$

Because $\cos\theta_{ij} = \cos\theta_{ji}$, we can switch the order of the subscripts on the terms in the second matrix:

$$\begin{bmatrix} a_{11'} & a_{12'} & a_{13'} \\ a_{21'} & a_{22'} & a_{23'} \\ a_{31'} & a_{32'} & a_{33'} \end{bmatrix} \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.10}$$

This matches (3.7).

III Second-order tensor transformations

Consider two vectors u_i and v_k. Then if their components are transformed into different reference frames:

$$u_i' = a_{i'j} u_j \quad \text{and} \quad v_{\ell}' = a_{\ell' k} v_k \qquad (3.11)$$

Suppose we multiply each component of u_i' by each component of $v_{\ell'}$ to give a 3x3 matrix:

$$u_{j'} v_{\ell'} = \left(a_{j'i} u_i\right)\left(a_{\ell' k} v_k\right) = a_{j'i} a_{\ell' k} u_i v_k \qquad (3.12)$$

Now let the 3x3 matrix product uv = A

$$A_{j'\ell'} = a_{j'i} a_{\ell' k} A_{ik} \qquad (3.13)$$

$$\left[A_{j'\ell'}\right] = \left[a_{j'i}\right]\left[A_{ik}\right]\left[a_{\ell' k}\right] \qquad (3.14)$$

Quantities which transform this way are called second-order tensors. The components have magnitudes and two associated directions (2 subscripts). Stress components transform like this, so stress is considered to be a second-order tensor.