VECTOR TRANSFORMATIONS (02)

I Main topics

A Why bother with transformations?

B Orthogonal reference frames

C Vector Algebra

D Vector transformations

II Why bother with transformations?

A To allow us to see things in their simplest representation

B To put observations from different reference frames into a common frame

III Orthogonal reference frames (Coordinate systems)

A Cartesian (x,y,z): 3 sets of orthogonal planes

B Cylindrical (r,θ,z): concentric cylinders, radial planes, axis-normal planes

C Spherical (ρ,θ, ϕ); concentric spheres, 2 sets of planes
Example: θ = trend, ϕ = plunge (or co-plunge)

D Elliptical: confocal elliptical cylinders, hyperbolic surfaces, 1 set of planes

E We will use the right-hand rule

IV Vector Algebra

Vectors have magnitude and a single direction; they transform according to certain rules

A Basis vectors: $\boxed{\vec{i}, \vec{j}, \vec{k}}$ These are unit length: $\boxed{|\vec{i}| = |\vec{j}| = |\vec{k}| = 1}$

B Components of a vector:

$$\vec{T} = T_x\vec{i} + T_y\vec{j} + T_z\vec{k} \qquad (2.1)$$

C Length of a vector

$$|\vec{T}| = \sqrt{T_x^2 + T_y^2 + T_z^2} \qquad (2.2)$$

D Dot (scalar) product of two vectors

1 Yields a scalar (i.e., a number with no associated direction)

2 A projection

3 Key equations

$$\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}|\cos\theta_{AB} = \vec{B} \bullet \vec{A} \qquad (2.3)$$

$$\boxed{\vec{i} \bullet \vec{i} = \vec{j} \bullet \vec{j} = \vec{k} \bullet \vec{k} = 1} \qquad (2.4)$$

$$\boxed{\vec{i} \bullet \vec{j} = \vec{j} \bullet \vec{k} = \vec{k} \bullet \vec{i} = 0;\ \vec{j} \bullet \vec{i} = \vec{k} \bullet \vec{j} = \vec{i} \bullet \vec{k} = 0} \qquad \boxed{\vec{i} \perp \vec{j} \perp \vec{k}} \qquad (2.5)$$

$$\boxed{\vec{A} \bullet \vec{B} = \left(A_x\vec{i} + A_y\vec{j} + A_z\vec{k}\right) \bullet \left(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}\right) = A_xB_x + A_yB_y + A_zB_z} \qquad (2.6)$$

$$\boxed{|\vec{A}| = \sqrt{\vec{A} \bullet \vec{A}}} \qquad (2.7)$$

$$\boxed{\vec{A} \bullet \vec{i} = A_x;\ \vec{A} \bullet \vec{j} = A_y;\ \vec{A} \bullet \vec{k} = A_z}$$ Recall the comment about projections! (2.8)

$$\boxed{\vec{A} \bullet \vec{B} = \sum_{i=1}^{3} A_iB_i}$$ **Summation notation** (2.9)

$$\boxed{\vec{A} \bullet \vec{B} = A_iB_i}$$ **Tensor notation** (2.10)

$$\boxed{\vec{A} \bullet \vec{B} = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \begin{bmatrix} B_x & B_y & B_z \end{bmatrix}^T}$$ =A*B' **Matrix notation**

(2.11)

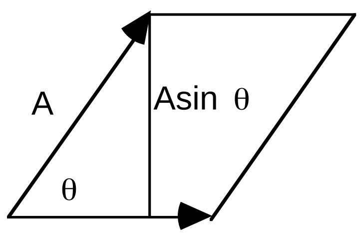
E Cross (vector) product of two vectors

1 Yields a vector

2 Yields an area with an associated direction

3 Key equations

$$\vec{A}\times\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta_{AB}\vec{n} = -\vec{B}\times\vec{A} \qquad \text{where } \boxed{\vec{n}\perp\vec{A},\ \vec{n}\perp\vec{B},\ |\vec{n}| = 1} \tag{2.12}$$



$$\boxed{\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0} \tag{2.13}$$

$$\boxed{\vec{i}\times\vec{j} = \vec{k};\ \vec{j}\times\vec{k} = \vec{i};\ \vec{k}\times\vec{i} = \vec{j}} \qquad \boxed{\vec{j}\times\vec{i} = -\vec{k};\ \vec{k}\times\vec{j} = -\vec{i};\ \vec{i}\times\vec{k} = -\vec{j}} \qquad \boxed{\vec{i}\perp\vec{j}\perp\vec{k}} \tag{2.14}$$

$$\begin{aligned}\vec{A}\times\vec{B} &= \left(A_x\vec{i} + A_y\vec{j} + A_z\vec{k}\right)\times\left(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}\right)\\ &= \left(A_yB_z - A_zB_y\right)\vec{i} - \left(A_xB_z - A_zB_x\right)\vec{j} + \left(A_xB_y - A_yB_x\right)\vec{k}\end{aligned} \tag{2.15}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix} \tag{2.16}$$

Determinant form

E Scalar triple product of three vectors

This equals the volume of a parallelepiped with edges defined by vectors **A**, **B**, and **C**. **AxB** is the area of the base; dotting **AxB** into **C** multiplies the basal area by the height. The scalar triple product is a scalar.

$$\begin{aligned}\left(\vec{A}\times\vec{B}\right)\bullet\vec{C} &= \left(\left(A_x\vec{i} + A_y\vec{j} + A_y\vec{k}\right)\times\left(B_x\vec{i} + B_y\vec{j} + B_y\vec{k}\right)\right)\bullet\left(C_x\vec{i} + C_y\vec{j} + C_y\vec{k}\right)\\ &= \left(A_yB_z - A_zB_y\right)C_x - \left(A_xB_z - A_zB_x\right)C_y + \left(A_xB_y - A_yB_x\right)C_y\end{aligned} \tag{2.17}$$

$$\left(\vec{A} \times \vec{B}\right) \bullet \vec{C} = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (2.18)$$

V Vector transformations

A Summation Notation

$$t_{i'} = \sum_{j=1}^{3} a_{i'j} T_j \qquad \text{Example:} \quad t_{1'} = \cos\theta_{1'1} T_1 + \cos\theta_{1'2} T_2 + \cos\theta_{1'3} T_3 \quad (2.19)$$

All components of T contribute to t!

B Tensor notation

$$t_{i'} = a_{i'j} T_j \quad (2.20)$$

C Matrix notation

$$\begin{bmatrix} t_{1'} & t_{2'} & t_{3'} \end{bmatrix} = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix} \begin{bmatrix} a_{1'1} & a_{2'1} & a_{3'1} \\ a_{1'2} & a_{2'2} & a_{3'2} \\ a_{1'3} & a_{2'3} & a_{3'3} \end{bmatrix} \quad (2.21)$$

If T and t are (1x3) row vectors, then t = T*a

$$\begin{bmatrix} t_{1'} \\ t_{2'} \\ t_{3'} \end{bmatrix} = \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \quad (2.22)$$

If T and t are (3x1) column vectors, then t = a'*T, where a' is the transpose of a.

References

Akivis, M.A., and Goldberg, V.V., 1972, An introduction to linear algebra and tensors: Dover, New York, 167 p.

Barber, J.R., 1993, Elasticity: Kluwer Academic Publishers, Boston, p. 7-9

Chou, P.C., and Pagano, N.J., 1967, Elasticity, Dover, New York, p. 227-229.

Mal, A.K., and Singh, S.J., 1991, Deformation of elastic solids: Prentice-Hall, Englewood Cliffs, 341 p.

Direction Cosines from Geologic Angle Measurements (Spherical coordinates)

Positive z-axis up
y = north; x = east

xy plane is horizontal plane

RIGHT-HANDED COORDINATES

Remember: Trends are azimuths and are measured in a horizontal plane. Plunges are inclinations and are measured in a vertical plane.

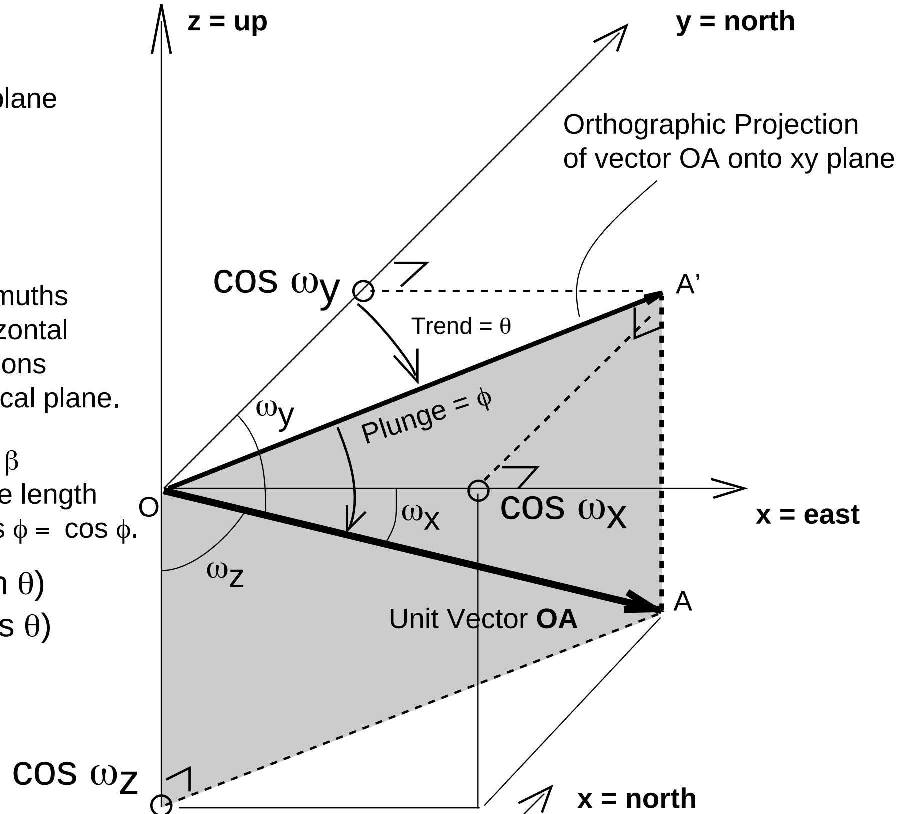
The direction cosines α and β are determined from OA', the length of OA' being $|OA'| = |OA| \cos\phi = \cos\phi$.

$\alpha = \cos\omega_x = (\cos\phi)(\sin\theta)$

$\beta = \cos\omega_y = (\cos\phi)(\cos\theta)$

$\gamma = \cos\omega_z = -(\sin\phi)$

$\alpha^2 + \beta^2 + \gamma^2 = 1$



Positive z-axis down
x = north; y = east

xy plane is horizontal plane

RIGHT-HANDED COORDINATES

Remember: Trends are azimuths and are measured in a horizontal plane. Plunges are inclinations and are measured in a vertical plane.

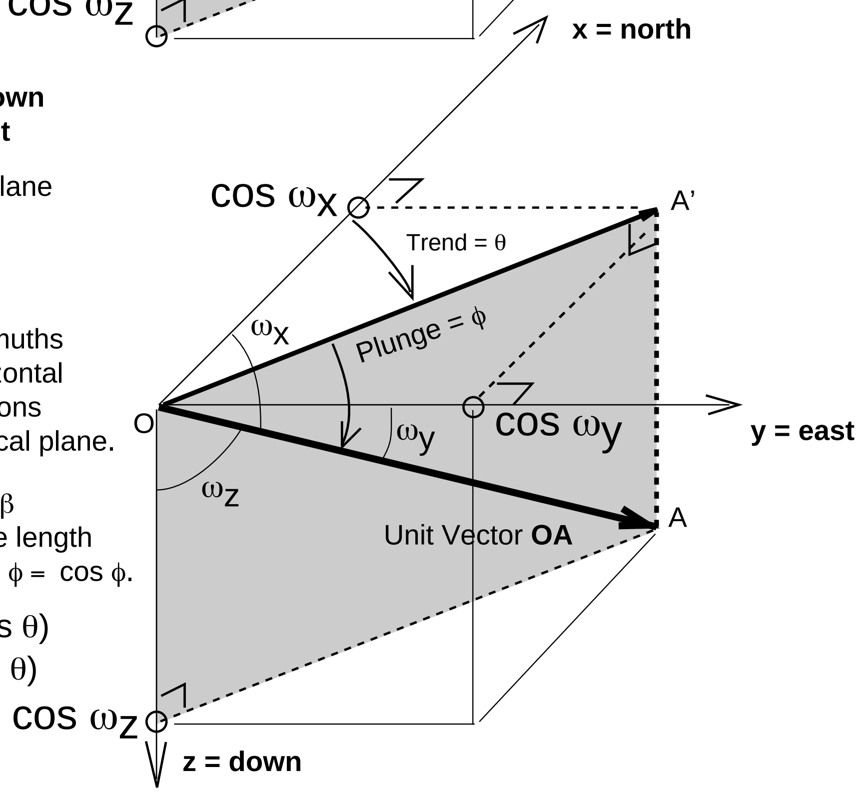
The direction cosines α and β are determined from OA', the length of OA' being $|OA'| = |OA| \cos\phi = \cos\phi$.

$\alpha = \cos\omega_x = (\cos\phi)(\cos\theta)$

$\beta = \cos\omega_y = (\cos\phi)(\sin\theta)$

$\gamma = \cos\omega_z = +(\sin\phi)$

$\alpha^2 + \beta^2 + \gamma^2 = 1$



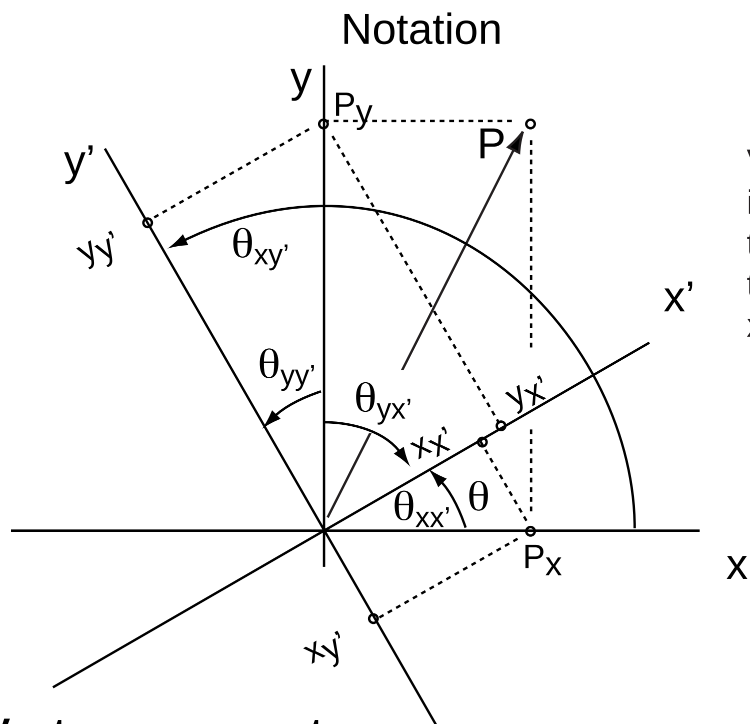


Fig. 2.1

Vector P can be considered in terms of its components in the x,y reference frame, or in terms of components in the x',y' reference frame.

Vector components

The x component of P (i.e., $x = P_x$) has components in the x' and y' directions:
$x_{x'}$ = x' component of x $\quad$ $x_{y'}$ = y' component of x

The y component of P (i.e., $y = P_y$) has components in the x' and y' directions:
$y_{x'}$ = x' component of y $\quad$ $y_{y'}$ = y' component of y

Angles

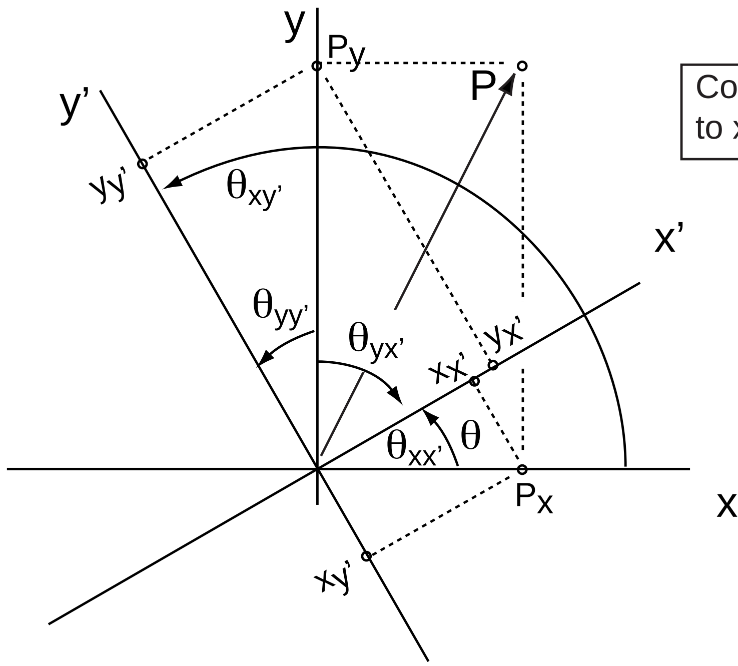
$\theta_{xx'}$ is the angle from the x axis to the x' axis; $\quad \theta_{xx'} = -\theta_{x'x}$
$\theta_{xy'}$ is the angle from the x axis to the y' axis; $\quad \theta_{xy'} = -\theta_{y'x}$
$\theta_{yx'}$ is the angle from the y axis to the x' axis; $\quad \theta_{yx'} = -\theta_{x'y}$
$\theta_{yy'}$ is the angle from the y axis to the y' axis; $\quad \theta_{yy'} = -\theta_{y'y}$
Switching the order of the angle subscripts changes the sign of the angle

Direction cosines

$a_{xx'} = \cos\theta_{xx'} = \cos(-\theta_{xx'}) = \cos\theta_{x'x} = a_{xx'} = \cos\theta$
$a_{xy'} = \cos\theta_{xy'} = \cos(-\theta_{xy'}) = \cos\theta_{y'x} = a_{yx'} = -\sin\theta$
$a_{yx'} = \cos\theta_{yx'} = \cos(-\theta_{yx'}) = \cos\theta_{x'y} = a_{xy'} = \sin\theta$
$a_{yy'} = \cos\theta_{yy'} = \cos(-\theta_{yy'}) = \cos\theta_{y'y} = a_{yy'} = \cos\theta$
Switching the order of the cosine subscripts does not change the sign of the cosine. This makes using direction cosines somewhat immune to sign errors.

Transformation of 2-D Vectors (I)

Fig. 2.2



Conversion from x,y components to x',y' components

Note that $\theta_{x'x} = \theta$

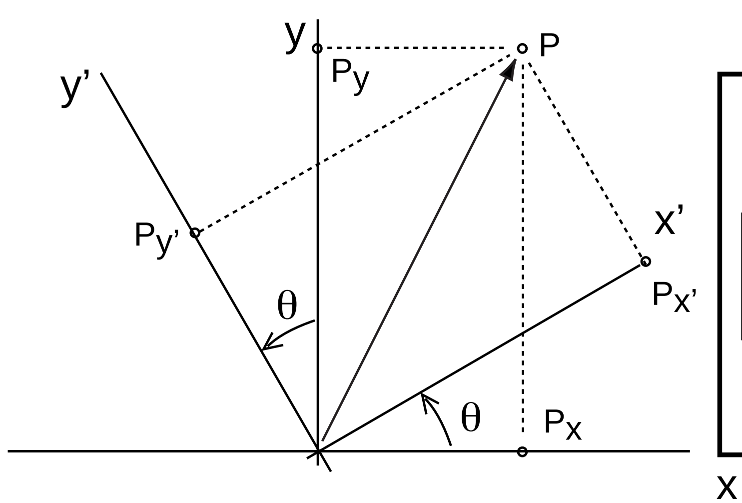
P_x (i.e., x) has components in the x' and y' directions:
$x_{x'} = (a_{x'x})\,(x) = (\cos\theta)(x)$ $\quad x_{y'} = (a_{y'x})\,(x) = (-\sin\theta)(x)$

P_y (i.e., y) has components in the x' and y' directions:
$y_{x'} = (a_{x'y})\,(y) = (\sin\theta)(y)$ $\quad y_{y'} = (a_{y'y})\,(x) = (\cos\theta)(y)$

The x' and y' components of P are then:

$x' = x_{x'} + y_{x'}$ $\quad x' = a_{x'x}\,x + a_{x'y}\,y$ $\quad x' = \cos(\theta_{x'x})\,x + \cos(\theta_{x'y})\,y$

$y' = x_{y'} + y_{y'}$ $\quad y' = a_{y'x}\,x + a_{y'y}\,y$ $\quad y' = \cos(\theta_{y'x})\,x + \cos(\theta_{y'y})\,y$



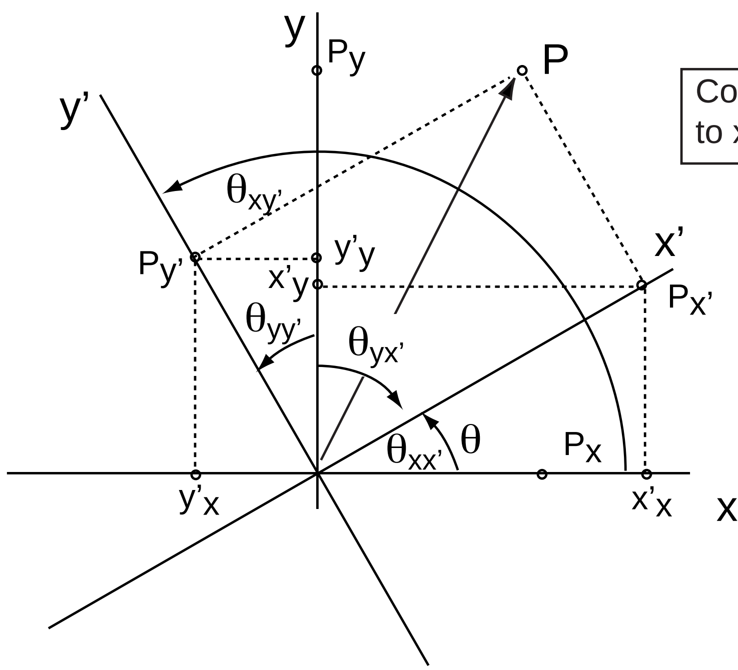
In matrix notation these two equations are written:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{x'x} & a_{x'y} \\ a_{y'x} & a_{y'y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation matrix
with direction cosines

Transformation of 2-D Vectors (II)

Fig. 2.3



Conversion from x',y' components to x,y components

Note that $\theta_{x'x} = \theta$

$P_{x'}$ (i.e., x') has components in the x and y directions:

$x'_x = (a_{xx'})\,(x') = (\cos\theta_{xx'})(x')$ $\quad$ $x'_y = (a_{yx'})\,(x') = (\cos\theta_{yx'})(x')$

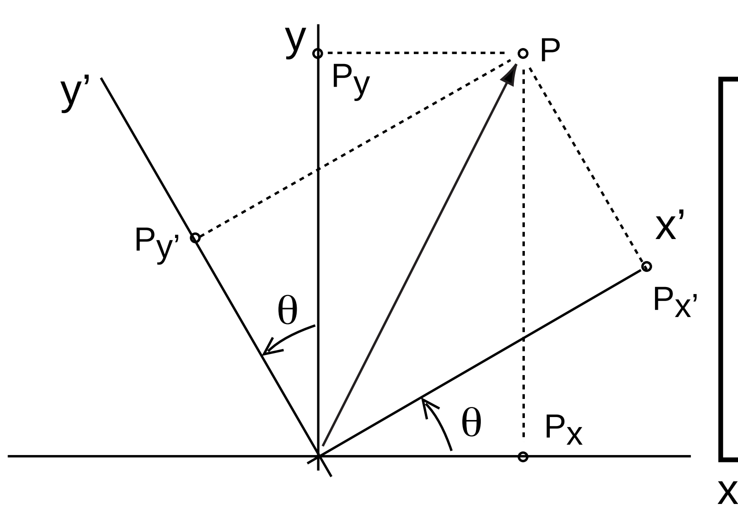
$P_{y'}$ (i.e., y') has components in the x and y directions:

$y'_x = (a_{xy'})\,(y') = (\cos\theta_{xy'})(y')$ $\quad$ $y'_y = (a_{yy'})\,(x) = (\cos\theta_{xy'})(y')$

The x and y components of P are then:

$x = x'_x + y'_x$ $\quad$ $x = a_{xx'}\,x' + a_{xy'}\,y'$ $\quad$ $x = \cos(\theta_{xx'})\,x' + \cos(\theta_{xy'})\,y'$

$y = x'_y + y'_y$ $\quad$ $y = a_{yx'}\,x' + a_{yy'}\,y'$ $\quad$ $y = \cos(\theta_{yx'})\,x' + \cos(\theta_{yy'})\,y'$



In matrix notation these two equations are written:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{xx'} & a_{xy'} \\ a_{yx'} & a_{yy'} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Rotation matrix with direction cosines