

## STRESS AND STRAIN (3)

## I Main Topics

- A Stress vector (traction) on a plane
- B Stress at a point
- C Principal stresses
- D Stress transformations
- E Strain

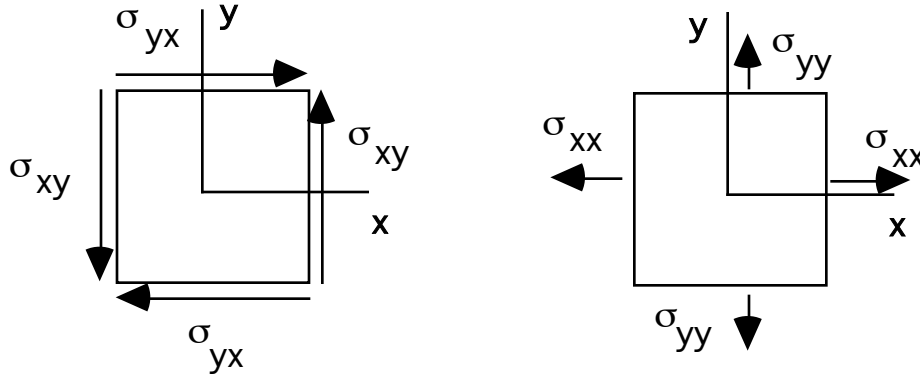
II Stress vectors (traction) on a particular plane:  $\vec{\sigma}$ 

- A  $\vec{\sigma} = \lim_{A \rightarrow 0} \vec{F} / A$ .      Dimensions of force per unit area
- B Traction vectors can be added vectorially.
- C Traction vectors can be resolved into normal and shear components.
  - 1 Normal tractions  $\vec{\sigma}_n$  act perpendicular to a plane
  - 2 Shear tractions  $\vec{\sigma}_s = (\vec{\tau})$  act parallel to a plane
- D Magnitude of the tractions depends on the orientation of the plane
- E The state of stress can (and usually does) vary from point to point.

## III Stress at a point

- A Stresses refer to **balanced** internal "forces". They differ from force vectors, which, if unbalanced, cause accelerations
- B **"On -in convention": The stress component acts on the plane normal to the i-direction and acts in the j-direction**
  - 1 Normal stresses:  $i=j$
  - 2 Shear stresses:  $i \neq j$

$$C \quad \sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad \text{or} \quad \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



D For rotational equilibrium,  $\sigma_{xy} = \sigma_{yx}$ ,  $\sigma_{xz} = \sigma_{zx}$ ,  $\sigma_{yz} = \sigma_{zy}$

#### IV Principal Stresses (these have magnitudes and orientations)

- A Principal stresses act on planes on which no shear stress are resolved
- B The principal stresses are normal stresses.
- C Principal stresses act on perpendicular planes
- D The maximum, intermediate, and minimum principal stresses are usually designated  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$ , respectively. Note that the principal stresses have a single subscript.
- E The principal stresses represent the stress state most simply.

$$F \quad \sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \quad \text{3-D} \quad \text{or} \quad \sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \quad \text{2-D}$$

#### V Transformation of stresses between planes of arbitrary orientation

- We transform stresses from one reference frame to another to represent the forces in a body in their clearest and most useful form
- All the stress components in one reference frame contribute to each stress component in another reference frame (see handouts).

In a two-dimensional case, let there be two reference frames:  $xy$  and  $x'y'$ . The stress component  $\sigma_{x'x'}$  (in terms of the  $xy$  components) is:

$$\sigma'_{x'x'} = a_{x'x}a_{x'x}\sigma_{xx} + a_{x'x}a_{x'y}\sigma_{xy} + a_{x'y}a_{x'x}\sigma_{yx} + a_{x'y}a_{x'y}\sigma_{yy}$$

The terms  $a_{qr'}$  form the set of cosines between the  $q$  and  $r'$  axes.

Similarly, the other three components of  $\sigma'$  also involve all four components of  $\sigma$ :

$$\sigma'_{x'y'} = a_{x'x}a_{y'x}\sigma_{xx} + a_{x'x}a_{y'y}\sigma_{xy} + a_{x'y}a_{y'x}\sigma_{yx} + a_{x'y}a_{y'y}\sigma_{yy}$$

$$\sigma'_{y'x'} = a_{y'x}a_{x'x}\sigma_{xx} + a_{y'x}a_{x'y}\sigma_{xy} + a_{y'y}a_{x'x}\sigma_{yx} + a_{y'y}a_{x'y}\sigma_{yy}$$

$$\sigma'_{y'y'} = a_{y'x}a_{y'x}\sigma_{xx} + a_{y'x}a_{y'y}\sigma_{xy} + a_{y'y}a_{y'x}\sigma_{yx} + a_{y'y}a_{y'y}\sigma_{yy}$$

If the angle from the  $x$  to  $x'$  axis is  $+\theta$ , then the following tables apply

Angle table	to x axis	to y axis
From $x'$ axis ...	$\theta$	$90^\circ - \theta$
From $y'$ axis ...	$90^\circ + \theta$	$\theta$

Cosine table	x axis	y axis
$x'$ axis	$a_{1'1} = \cos(x'x) = \cos\theta$	$a_{1'2} = \cos(x'y) = \sin\theta$
$y'$ axis	$a_{2'1} = \cos(y'x) = -\sin\theta$	$a_{2'2} = \cos(y'y) = \cos\theta$

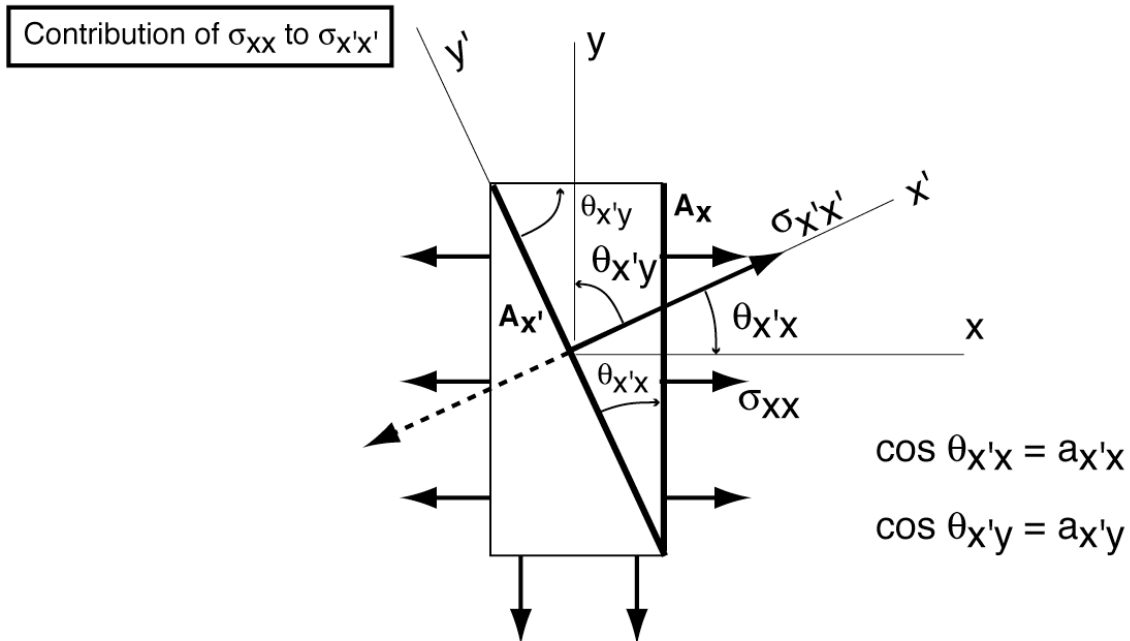
The shorthand tensor notation for the four equations above is

$$\sigma'_{ij} = a_{ik}a_{jl} \sigma_{kl}; \quad i,j,k,l = (x,y) \text{ or } (1,2)$$

It turns out that this exact same notation applies for 3-D!

$$\sigma'_{ij} = a_{ik}a_{jl} \sigma_{kl}; \quad i,j,k,l = (1,2,3)$$

Fig. 3.1



What does  $\sigma_{xx}$  on face  $A_x$  of area  $A_x$  contribute to  $\sigma_{x'x'}$  on face  $A_{x'}$  of area  $A_{x'}$ ?

Start with the definition of stress:  $\sigma_{x'x'}^{(1)} = F_{x'}^{(1)} / A_{x'}$ .

The unknown quantities  $F_{x'}^{(1)}$  and  $A_{x'}$  must be found from the known quantities  $\sigma_{xx}$  and  $\theta$ .

To do this we first find the force  $F_x^{(1)}$  associated with  $\sigma_{xx}$ :

Force = (stress)(area)

$$F_x^{(1)} = \sigma_{xx} A_x$$

The component of  $F_x^{(1)}$  that acts along the  $x'$ -direction is  $F_x^{(1)} \cos \theta_{x'x}$ .

$$F_{x'}^{(1)} = F_x^{(1)} \cos \theta_{x'x}$$

As can be seen from the diagram atop the page  $A_x = A_{x'} \cos \theta_{x'x}$ , so

$$A_{x'} = A_x / \cos \theta_{x'x}$$

So the contribution of  $\sigma_{xx}$  to  $\sigma_{x'x'}$  is:

$$\sigma_{x'x'}^{(1)} = F_{x'}^{(1)} / A_{x'} = F_x^{(1)} \cos \theta_{x'x} / (A_x / \cos \theta_{x'x}) = (F_x^{(1)} / A_x) \cos \theta_{x'x} \cos \theta_{x'x}$$

$$\sigma_{x'x'}^{(1)} = a_{x'x} a_{x'x} \sigma_{xx}$$

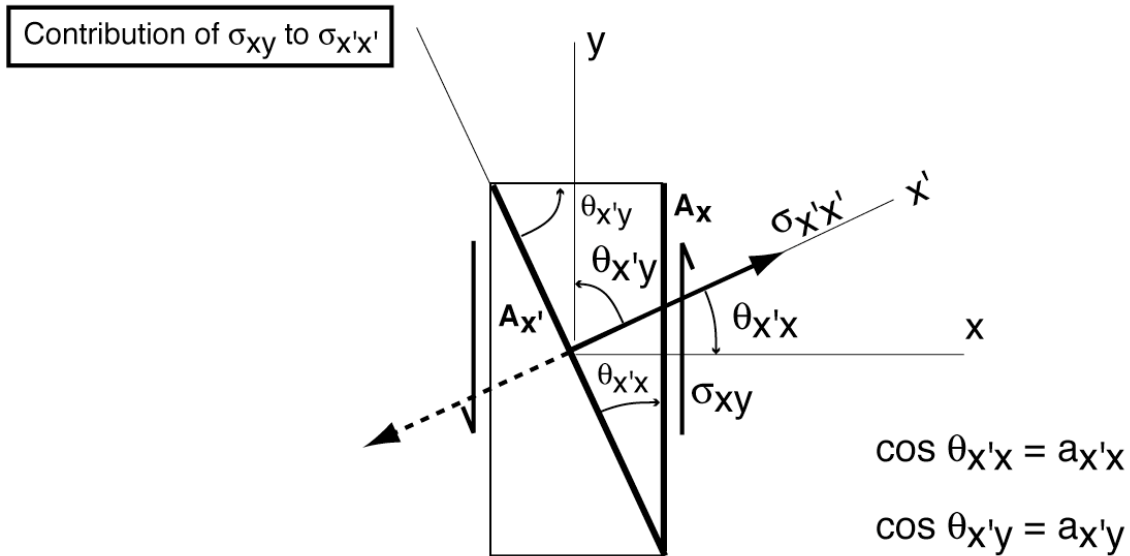
← Contribution of  $\sigma_{xx}$  to  $\sigma_{x'x'}$

$$\frac{F_{x'}^{(1)}}{A_{x'}} = \frac{A_x}{A_{x'}} \frac{F_x^{(1)}}{A_x}$$

← How the components project

$$A_{x'} \quad A_{x'} \quad F_x \quad A_x$$

Fig. 3.2



What does  $\sigma_{xy}$  on face  $A_x$  of area  $A_x$  contribute to  $\sigma_{x'x'}$  on face  $A_{x'}$  of area  $A_{x'}$ ?

Start with the definition of stress:  $\sigma_{x'x'}^{(2)} = F_{x'}^{(2)} / A_{x'}$ .

The unknown quantities  $F_{x'}^{(1)}$  and  $A_{x'}$  must be found from the known quantities  $\sigma_{xy}$  and  $\theta$ .

To do this we first find the force  $F_y^{(2)}$  associated with  $\sigma_{xy}$ :

Force = (stress)(area)

$$F_y^{(2)} = \sigma_{xy} A_x$$

The component of  $F_y^{(2)}$  that acts along the  $x'$ -direction is  $F_y^{(2)} \cos \theta_{x'y}$ .

$$F_{x'}^{(2)} = F_y^{(2)} \cos \theta_{x'y}$$

As can be seen from the diagram atop the page  $A_x = A_{x'} \cos \theta_{x'x}$ , so

$$A_{x'} = A_x / \cos \theta_{x'x}$$

So the contribution of  $\sigma_{xy}$  to  $\sigma_{x'x'}$  is:

$$\sigma_{x'x'}^{(2)} = F_{x'}^{(2)} / A_{x'} = F_y \cos \theta_{x'y} / (A_x / \cos \theta_{x'x}) = (F_y / A_x) \cos \theta_{x'x} \cos \theta_{x'y}$$

$$\sigma_{x'x'}^{(2)} = a_{x'x} a_{x'y} \sigma_{xy}$$

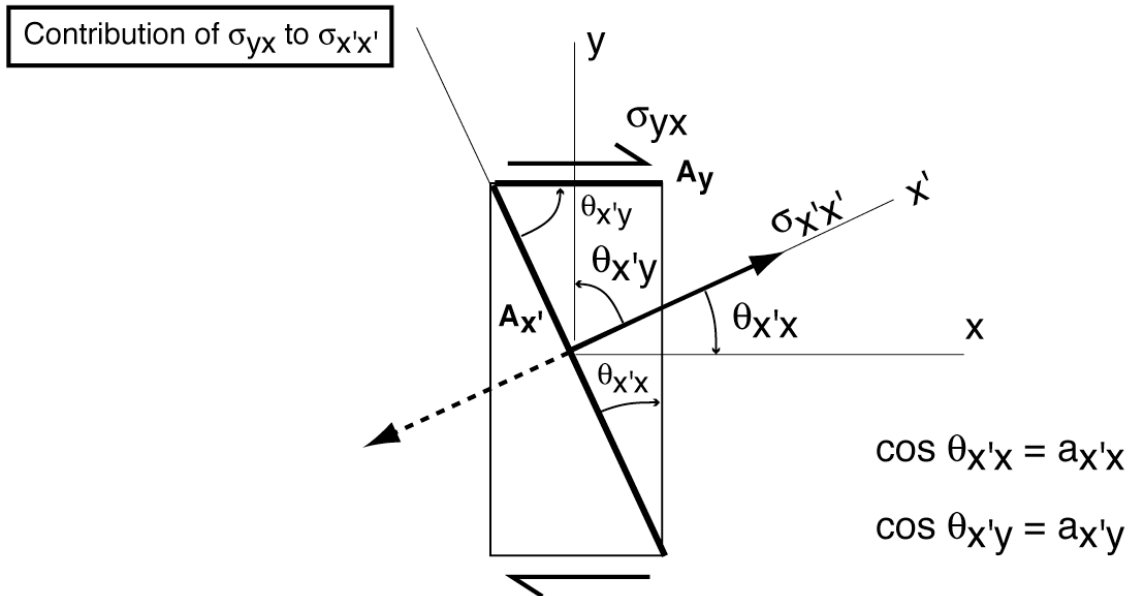
← Contribution of  $\sigma_{xy}$  to  $\sigma_{x'x'}$

$$\frac{F_{x'}^{(2)}}{A_{x'}} = \frac{A_x}{A_{x'}} \frac{F_y}{A_x} \frac{F_y^{(2)}}{A_x}$$

← How the components project

$$A_{x'} \quad A_{x'} \quad F_y \quad A_x$$

Fig. 3.3



What does  $\sigma_{yx}$  on face  $A_y$  of area  $A_y$  contribute to  $\sigma_{x'x'}$  on face  $A_{x'}$  of area  $A_{x'}$ ?

Start with the definition of stress:  $\sigma_{x'x'}^{(3)} = F_{x'}^{(3)} / A_{x'}$ .

The unknown quantities  $F_{x'}^{(3)}$  and  $A_{x'}$  must be found from the known quantities  $\sigma_{yx}$  and  $\theta$ .

To do this we first find the force  $F_x^{(3)}$  associated with  $\sigma_{yx}$ :

Force = (stress)(area)

$$F_x^{(3)} = \sigma_{yx} A_x$$

The component of  $F_x^{(3)}$  that acts along the  $x'$ -direction is  $F_x^{(3)} \cos \theta_{x'x}$ .

$$F_{x'}^{(3)} = F_x^{(3)} \cos \theta_{x'x}$$

As can be seen from the diagram atop the page  $A_y = A_{x'} \cos \theta_{x'y}$ , so

$$A_{x'} = A_y / \cos \theta_{x'y}$$

So the contribution of  $\sigma_{yx}$  to  $\sigma_{x'x'}$  is:

$$\sigma_{x'x'}^{(3)} = F_{x'}^{(3)} / A_{x'} = F_x^{(3)} \cos \theta_{x'x} / (A_y / \cos \theta_{x'y}) = (F_x^{(3)} / A_y) \cos \theta_{x'x} \cos \theta_{x'y}$$

$$\sigma_{x'x'}^{(3)} = a_{x'y} a_{x'x} \sigma_{yx}$$

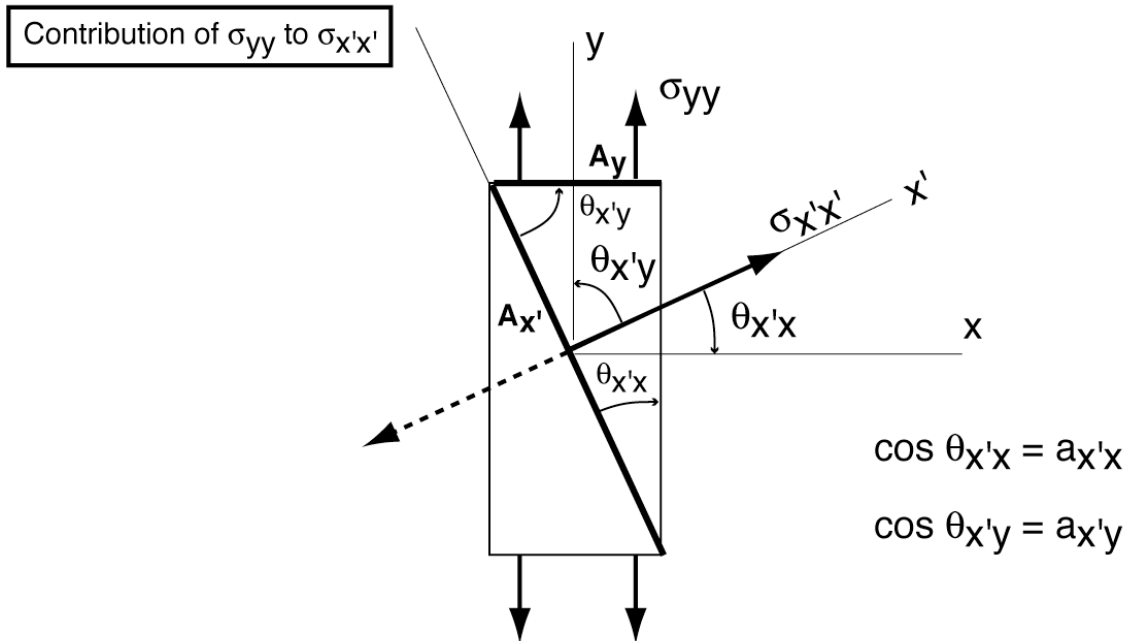
← Contribution of  $\sigma_{yx}$  to  $\sigma_{x'x'}$

$$F_{x'}^{(3)} = \frac{A_x}{A_{x'}} \frac{F_x^{(3)}}{A_x}$$

← How the components project

$$A_{y'}$$

Fig. 3.4



What does  $\sigma_{yy}$  on face  $A_y$  of area  $A_y$  contribute to  $\sigma_{x'x'}$  on face  $A_{x'}$  of area  $A_{x'}$ ?

Start with the definition of stress:  $\sigma_{x'x'}^{(4)} = F_{x'}^{(4)} / A_{x'}$ .

The unknown quantities  $F_{x'}^{(4)}$  and  $A_{x'}$  must be found from the known quantities  $\sigma_{yy}$  and  $\theta$ .

To do this we first find the force  $F_y^{(4)}$  associated with  $\sigma_{yy}$ :

Force = (stress)(area)

$$F_y^{(4)} = \sigma_{yy} A_y$$

The component of  $F_y^{(4)}$  that acts along the  $x'$ -direction is  $F_y^{(4)} \cos \theta_{x'y}$ .

$$F_{x'}^{(4)} = F_y^{(4)} \cos \theta_{x'y}$$

As can be seen from the diagram atop the page  $A_y = A_{x'} \cos \theta_{x'y}$ , so

$$A_{x'} = A_y / \cos \theta_{x'y}$$

So the contribution of  $\sigma_{yy}^{(4)}$  to  $\sigma_{x'x'}^{(4)}$  is:

$$\sigma_{x'x'}^{(4)} = F_{x'}^{(4)} / A_{x'} = F_y^{(4)} \cos \theta_{x'y} / (A_y / \cos \theta_{x'y}) = (F_y^{(4)} / A_y) \cos \theta_{x'y} \cos \theta_{x'y}$$

$$\sigma_{x'x'}^{(4)} = a_{x'y} a_{x'y} \sigma_{yy}^{(4)}$$

← Contribution of  $\sigma_{yy}$  to  $\sigma_{x'x'}$

$$\frac{F_{x'}^{(4)}}{A_{x'}} = \frac{A_y}{A_{x'}} \frac{F_{x'}}{F_y} \frac{F_y^{(4)}}{A_y}$$

← How the components project

$$A_{x'} \quad A_{x'} \quad F_y \quad A_y$$

The matrix equations used to transform the stresses are:

In 2-D

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

or

$$[\sigma'_{ij}] = [a] [\sigma] [a]^T$$

In 3-D

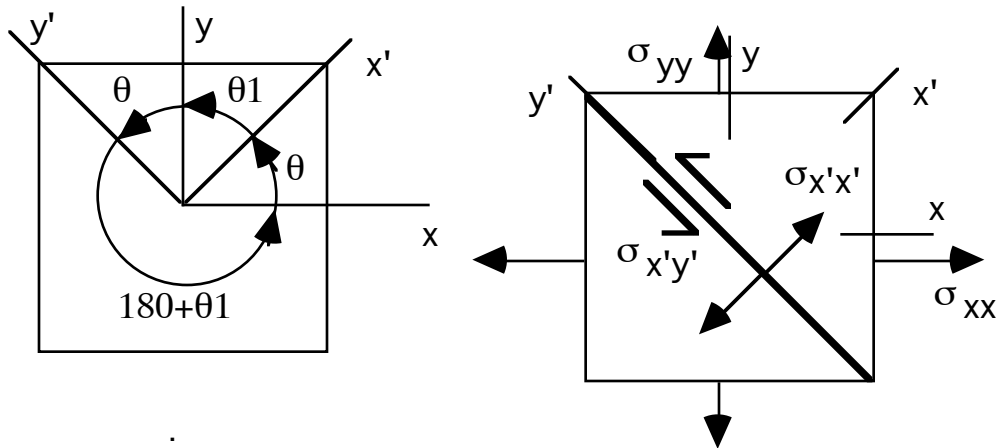
$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

or

$$[\sigma'_{ij}] = [a] [\sigma] [a]^T \quad \text{Exactly the same as 2-D!}$$



Problem: Find  $\sigma_{x'x'}$  and  $\sigma_{y'y'}$  given principal stresses  $\sigma_{xx}$  and  $\sigma_{yy}$



The stresses shown above on the right are **positive** in a tensor notation. We use the general formula  $\sigma_{i'j'} = (a_{i'i})(a_{j'j})\sigma_{ij}$  to first find  $\sigma_{x'x'}$ .

### Table of angles

From x to x'		From x' to x	
xx'	$\theta$	x'x	$-\theta$
xy'	$90^\circ + \theta$	y'x	$-\{90^\circ + \theta\}$
yx'	$360^\circ - \theta_1$	x'y	$\theta_1$
yy'	$\theta$	y'y	$-\theta$

Now  $\sin\theta_1 = \cos\theta$ , and  $\cos\theta_1 = \sin\theta$  because  $\sigma_1 = 90^\circ - \theta$ , so:

$\cos(\theta) = \cos\theta$	$\cos(-\theta) = \cos\theta$
$\cos(90^\circ + \theta) = -\sin\theta$	$\cos(-\{90^\circ + \theta\}) = \cos(90^\circ + \theta) = -\sin\theta$
$\cos(360^\circ - \theta_1) = \cos\theta_1 = \sin\theta$	$\cos(\theta_1) = \sin\theta$
$\cos(\theta) = \cos\theta$	$\cos(-\theta) = \cos\theta$

### Table of direction cosines

$a_{xx'}$	$\cos\theta$	$a_{x'x}$	$\cos\theta$
$a_{xy'}$	$-\sin\theta$	$a_{y'x}$	$-\sin\theta$
$a_{yx'}$	$\sin\theta$	$a_{x'y}$	$\sin\theta$
$a_{yy'}$	$\cos\theta$	$a_{y'y}$	$\cos\theta$

$$\sigma_{x'x'} = (a_{x'x})(a_{x'x})\sigma_{xx} + (a_{x'x})(a_{x'y})\sigma_{xy} + (a_{x'y})(a_{x'x})\sigma_{yx} + (a_{x'y})(a_{x'y})\sigma_{yy}$$

$$\sigma_{x'x'} = (\cos\theta)(\cos\theta)\sigma_{xx} + (\cos\theta)(\sin\theta)\sigma_{xy} + (\sin\theta)(\cos\theta)\sigma_{yx} + (\sin\theta)(\sin\theta)\sigma_{yy}$$

$$\sigma_{x'x'} = (\cos^2\theta)\sigma_{xx} + (\sin^2\theta)\sigma_{yy}$$

Now to get the shear stress  $\sigma_{x'y'}$ . We start again with the general formula:

$$\sigma_{i'j'} = (a_{i'i})(a_{j'j})\sigma_{ij}$$

$$\sigma_{x'y'} = (a_{x'x})(a_{y'x})\sigma_{xx} + (a_{x'x})(a_{y'y})\sigma_{xy} + (a_{x'y})(a_{y'x})\sigma_{yx} + (a_{x'y})(a_{y'y})\sigma_{yy}$$

**Table of direction cosines**

$a_{xx'}$	$\cos\theta$	$a_{x'x}$	$\cos\theta$
$a_{xy'}$	$-\sin\theta$	$a_{y'x}$	$-\sin\theta$
$a_{yx'}$	$\sin\theta$	$a_{x'y}$	$\sin\theta$
$a_{yy'}$	$\cos\theta$	$a_{y'y}$	$\cos\theta$

$$\sigma_{x'y'} = (\cos\theta)(-\sin\theta)\sigma_{xx} + (\cos\theta)(\cos\theta)\sigma_{xy} + (\sin\theta)(-\sin\theta)\sigma_{yx} + (\sin\theta)(\cos\theta)\sigma_{yy}$$

Again,  $\sigma_{xy}$  and  $\sigma_{yx}$  are zero, so

$$\sigma_{x'y'} = (-\sin\theta\cos\theta)\sigma_{xx} + (\sin\theta\cos\theta)\sigma_{yy} = (\sigma_{yy} - \sigma_{xx})(\sin\theta\cos\theta)$$

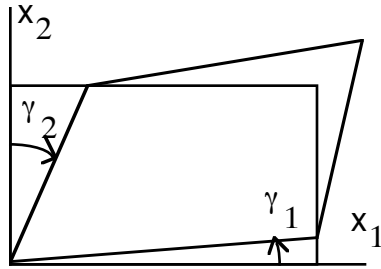
## VI Strain

A Deformation refers to the geometry change of a body

- 1 Change in position: Rigid-body translation
- 2 Change in orientation: Rigid-body rotation
- 3 Change in size and shape: Strain

B Elongation:  $\varepsilon = \Delta L/L_0$

C Engineering shear strain  $\gamma$  : change in angle between originally perpendicular lines



D Infinitesimal strains and displacement gradients (2-D)

- 1 Normal strain: change in line length
- 2 Shear strain : change in angles between lines
- 3  $u_1$  = displacement in the x-direction (or  $x_1$  direction)
- 4  $u_2$  = displacement in the y-direction (or  $x_2$  direction)

5 Displacement gradients:  $\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_1}$

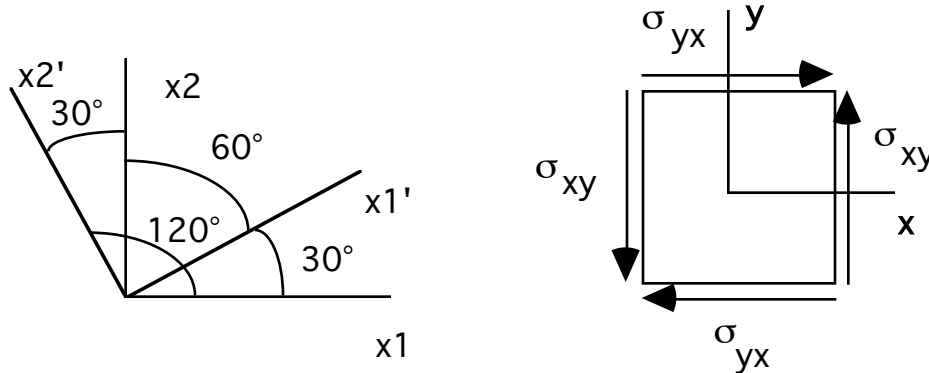
6 Infinitesimal approximations

a Normal strains:  $\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$

b Tensor shear strains  $\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(\gamma_1 + \gamma_2) = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$

## Mohr Circle Exercises

These problems can be solved with either a Mohr circle or stress tensor equations. Two points to remember for the equations. First, they are set up assuming a tensor convention for stresses (tensile stresses positive), so a compressive stress of magnitude 8 MPa would correspond to a tensile stress of -8 MPa. Second, the  $a_{ij}$  terms in those equations are the cosines of the angles between the  $x_i'$  axis and the  $x_j$  axis (e.g., if the  $x_1$  and  $x_1'$  axes differ by  $30^\circ$ , then;  $a_{1'1} = \cos 30^\circ$ ;  $a_{1'2} = \cos 60^\circ$ ;  $a_{2'1} = \cos 120^\circ$ ;  $a_{2'2} = \cos 30^\circ$ ).



Unless stated otherwise, positive stresses in these exercises are considered to be compressive. A positive compressive stress is a negative tensile stress. To evaluate whether a shear stress is positive or negative in the tensor convention, the above figure shows positive shear stresses.

- Suppose the most compressive stress is horizontal, trends east-west, and has a magnitude of 4MPa. The least compressive stress is horizontal, trends north-south (this has to be the case if the most compressive stress trends east east!) and has a magnitude of 2MPa. Find the normal and shear stress that acts on a vertical fault with a strike of N30°E.
- Two locked vertical faults each support a left-lateral shear stress of 8MPa. One fault strikes N30°E, and this fault has a compressive normal stress of 14 MPa. The second fault has a compressive normal stress of 2 MPa. Assume that the stress field around the faults is uniform. (a) Draw the Mohr circle. (b) Determine the magnitude and the orientations of the principal stresses. (c) Determine the strike of the second fault.
- The pressure at the bottom of a swimming pool is  $rg$ , where  $r$  is the density of the water,  $g$  is gravitational acceleration, and  $h$  is the depth of the water. The water is still. (a) What is the pressure (in Pa) at the bottom of a 10m-deep pool? (b) If we call the pressure (stress) on the pool bottom  $s_1 = s_{11}$ , and the pressure (stress) that acts horizontally  $\sigma_3 = \sigma_{33}$ , then  $\sigma_{11} = \sigma_{33}$ . Plot the Mohr circle that describes the state of stress at the bottom of our 10m-deep pool. (This is an example of hydrostatic stress). (c) What is the maximum shear stress at the bottom of the pool? (d) What must the level of the intermediate principal stress  $\sigma_2$  be at the bottom of the pool?
- Two orthogonal planes have the following stresses acting on them: plane 1 ( $\sigma_n = 9$  MPa,  $\tau = 3$  MPa); plane 2 ( $\sigma_n = 1$  MPa,  $\tau = -3$  MPa). These stresses are given for the Mohr circle "compression is positive" convention. (a) Draw the Mohr circle. (b) Calculate the greatest and least compressive stresses, and the maximum shear stress. It should help to look at your Mohr circle while doing the calculations. Remember that the mean normal stress is  $(\sigma_1 + \sigma_3)/2$ , and the radius of the Mohr circle is  $(\sigma_1 - \sigma_3)/2$ . (c) Accurately and neatly draw a square showing  $\sigma_1$  and  $\sigma_3$  acting on the sides of the square, and show the orientation of planes 1 and 2 within the square.

- 5 Find the  $\sigma_n$  and  $\tau$  on a plane whose normal is  $45^\circ$  counterclockwise to the  $\sigma_1$  direction, where  $\sigma_1 = 8 \text{ Mpa}$  (compressive) and  $\sigma_2 = 3 \text{ MPa}$  (compressive).
- 6 Find the magnitudes and orientations of  $\sigma_1$  and  $\sigma_2$  from known  $\sigma_n$  and  $\tau$  on two perpendicular planes P and Q. The known stresses on P and Q are:

Plane	$\sigma_n$	$\tau$
P	2.2 kbar	1.1 kbar
Q	1.2 kbar	-1.1 kbar

- 7 Find the magnitudes and orientations of  $\sigma_1$  and  $\sigma_2$  from known  $\sigma_n$  and  $\tau$  on two perpendicular planes P and Q. The known stresses on P and Q are:

Plane	$\sigma_n$	$\tau$
P	30kbar	8 kbar
Q	0 kbar	-8 kbar

Answer

	$\sigma_1$	$\sigma_2$
Magnitude	34kbar	-2 kbar
Orientation	$14^\circ$ clockwise from normal to plane P	$14^\circ$ clockwise from normal to plane Q

- 8 Find the magnitudes and orientations of  $\sigma_1$  and  $\sigma_2$  from known  $\sigma_n$  and  $\tau$  on two perpendicular planes P and Q. The known stresses on P and Q are:

Plane	$\sigma_n$	$\tau$
P	28kbar	5 kbar
Q	2 kbar	-5 kbar

- 9 Consider the stresses at the surface of the earth in Kansas. Suppose the most compressive horizontal stress trends  $N30^\circ E$  and has a magnitude of 20 PA. Ignoring the weight of the atmosphere, what is the vertical normal stress at the earth's surface? Is this vertical normal stress a principal stress? Why or why not? What would be the normal and shear stresses acting on a plane that strikes  $N60^\circ W$  and dips  $18.5^\circ$  NE? What would be the stress on a plane that strikes  $120^\circ$  and dips  $26.5^\circ$  SW? It might help to note that  $18.5^\circ \approx 0.5 \cdot \tan^{-1}(3/4)$  and  $26.5^\circ \approx 0.5 \cdot \tan^{-1}(4/3)$

- 10 A rock is being prepared for a compression test, with the principal stresses to be applied being  $\sigma_1 = 30 \text{ Mpa}$  (compressive) and  $\sigma_2 = 10 \text{ MPa}$ . Suppose the shear failure criterion for the rock is given by the following equation:  $\tau_f = \sigma_n(\tan 30^\circ)$ . Plot the failure envelope and the Mohr circle describing the state of the stress in the rock. Can the rock sustain the the indicated principal stresses without failing in shear? If the rock will fail in shear, give the orientation of the shear failure plane(s) and the normal and shear stress on the failure plane. If the rock will not fail in shear, give the normal and shear stress on a plane whose normal makes a  $60^\circ$  angle with the  $\sigma_1$  axis.
- 11 A rock is being prepared for a compression test, with the principal stresses to be applied being  $\sigma_1 = 40 \text{ Mpa}$  (compressive) and  $\sigma_2 = 20 \text{ MPa}$ . Suppose the shear failure criterion for the rock is given by the following equation:  $\tau_f = \sigma_n(\tan 30^\circ)$ . What must be the fluid pressure in the rock if it is to fail in shear?

- 12 Sum the expressions for  $\sigma_{x'x'}$  and  $\sigma_{y'y'}$  (assume we are dealing only with a 2-D case). What does the result say about the means normal stress (i.e.,  $1/2[\sigma_{x'x'} + \sigma_{y'y'}]$ ) at a point for any two perpendicular planes?
- 13 Let  $\sigma_{x'x'} = \sigma_1$  be the most compressive principal stress (magnitude of 4MPa) and  $\sigma_{x'x'} = \sigma_2$  be the least compressive principal stress (magnitude of 2MPa). On a piece of graph paper (or using Excel or Matlab) neatly plot the shear stress  $\sigma_{x'y'}$  against the normal stress  $\sigma_{x'x'}$  for the following angles between the  $x$  and  $x'$  axes:  $0^\circ$ ,  $30^\circ$ ,  $40^\circ$ ,  $60^\circ$ ,  $90^\circ$ ,  $120^\circ$ ,  $150^\circ$ , and  $180^\circ$ . What kind of geometric figure do these points appear to lie on?
- 14 Write the expression of the shear stress  $\sigma_{x'y'}$  in terms of principal stresses  $\sigma_1$  and  $\sigma_2$ ; let  $\sigma_{xx} = \sigma_1$  and  $\sigma_{yy} = \sigma_2$ . Take the derivative of the expression for  $\sigma_{x'y'}$  with respect to the angle  $\theta$  between the  $x$  and  $x'$  axes. Setting this derivative equal to zero, determine the orientation of the planes (relative to the  $\sigma_1$  direction) that experience the greatest and least shear stress.