GG612
Lecture 3

Strain and Stress

Should complete infinitesimal strain by adding rotation.

Outline

Matrix Operations

Strain

1 General concepts
2 Homogeneous strain
3 Matrix representations
4 Squares of line lengths
5 E (strain matrix)
6 ε (infinitesimal strain)
7 Coaxial finite strain
8 Non-coaxial finite strain

Stress

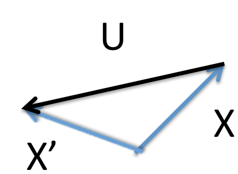
1 Stress vector
2 Stress at a point
3 Principal stresses

Main Theme

- Representation of complicated quantities describing strain and stress at a point in a clear manner

Vector Conventions

- X = initial position
- X' = final position
- U = displacement



Matrix Inverses

- $AA^{-1} = A^{-1}A = [I]$
- $[AB]^{-1} = [B^{-1}][A^{-1}]$ ←

- $ABB^{-1}A^{-1} = A[I]A^{-1} = [I]$
- $[AB][AB]^{-1} = [I]$
- $[B^{-1}A^{-1}] = [AB]^{-1}$

Matrix Inverses and Transposes

- **a•b** = $[a^T][b]$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \ldots + a_nb_n$$

- $[AB]^T = [B^T][A^T]$ ←

$$\underset{nxq}{A} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix};\ \underset{qxm}{B} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_m \end{bmatrix}$$

$$AB = \begin{bmatrix} \vec{a}_1 \bullet \vec{b}_1 & \vec{a}_1 \bullet \vec{b}_2 & \cdots & \vec{a}_1 \bullet \vec{b}_m \\ \vec{a}_2 \bullet \vec{b}_1 & \vec{a}_2 \bullet \vec{b}_2 & \cdots & \vec{a}_2 \bullet \vec{b}_m \\ \vdots & \vdots & \vdots & \vdots \\ \vec{a}_n \bullet \vec{b}_1 & \vec{a}_n \bullet \vec{b}_1 & \cdots & \vec{a}_n \bullet \vec{b}_m \end{bmatrix}$$

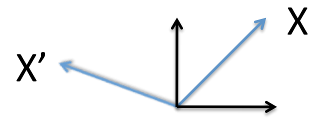
$$[AB]^T = \begin{bmatrix} \vec{a}_1 \bullet \vec{b}_1 & \vec{a}_2 \bullet \vec{b}_1 & \cdots & \vec{a}_n \bullet \vec{b}_1 \\ \vec{a}_1 \bullet \vec{b}_2 & \vec{a}_2 \bullet \vec{b}_2 & \cdots & \vec{a}_n \bullet \vec{b}_1 \\ \vdots & \vdots & \vdots & \vdots \\ \vec{a}_1 \bullet \vec{b}_m & \vec{a}_2 \bullet \vec{b}_m & \cdots & \vec{a}_n \bullet \vec{b}_m \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_m \end{bmatrix}\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} = \begin{bmatrix} \vec{b}_1 \bullet \vec{a}_1 & \vec{b}_1 \bullet \vec{a}_2 & \cdots & \vec{b}_1 \bullet \vec{a}_n \\ \vec{b}_2 \bullet \vec{a}_1 & \vec{b}_2 \bullet \vec{a}_2 & \cdots & \vec{b}_2 \bullet \vec{a}_n \\ \vdots & \vdots & \vdots & \vdots \\ \vec{b}_m \bullet \vec{a}_1 & \vec{b}_m \bullet \vec{a}_1 & \cdots & \vec{b}_m \bullet \vec{a}_1 \end{bmatrix}$$

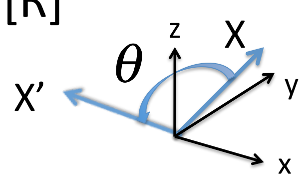
$$[AB]^T = B^TA^T$$

Rotation Matrix [R]

- Rotations change the orientations of vectors but not their lengths
- $X \bullet X = |X||X|\cos\theta_{XX}$
- $X \bullet X = X' \bullet X'$
- $X' = RX$
- $X \bullet X = [RX] \bullet [RX]$
- $X \bullet X = [RX]^T[RX]$
- $X \bullet X = [X^T R^T]\,[RX]$
- $[X^T]\,[X] = [X^T R^T]\,[RX]$
- $[X^T][I][X] = [X^T][R^T]\,[R][X]$
- $[I] = [R^T]\,[R]$
- But $[I] = [R^{-1}]\,[R]$, so
- $[R^T] = [R^{-1}]$



Rotation Matrix [R] 2D Example



$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix};\ [X'] = [R][X]$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{matrix} x' = \cos\theta x + \sin\theta y \\ y' = -\sin\theta x + \cos\theta y \end{matrix} \rightarrow x'^2 + y'^2 = x^2 + y^2$$

$$R^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$RR^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R^T = R^{-1}$$

General Concepts

Deformation = Rigid body motion + Strain

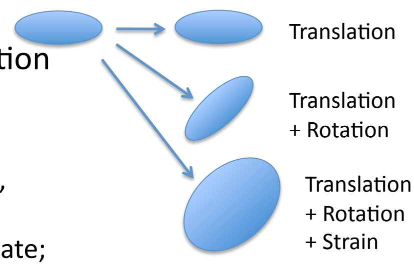
Rigid body motion

Rigid body translation

- Treated by matrix addition
  [X'] = [X] + [U]

Rigid body rotation

- Changes orientation of lines, but not their length
- Axis of rotation does not rotate; it is an eigenvector
- Treated by matrix multiplication
  [X'] = [R] [X]

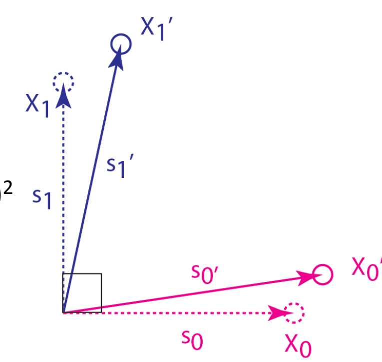


General Concepts

- Normal strains

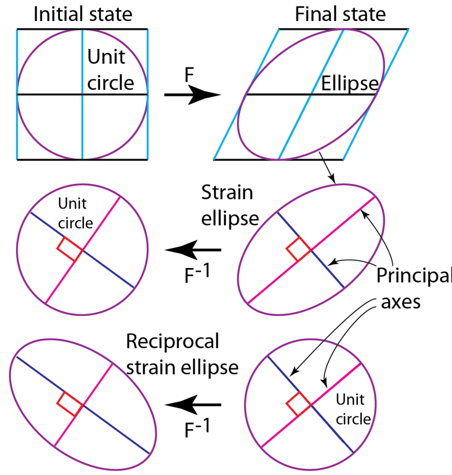
  Change in line length
  - Extension (elongation) = $\Delta s/s_0$
  - Stretch = S = s'/s_0
  - Quadratic elongation = Q = $(s'/s_0)^2$
- Shear strains

  Change in right angles
- Dimensions: <u>Dimensionless</u>



Homogeneous strain

- Parallel lines to parallel lines (2D and 3D)
- Circle to ellipse (2D)
- Sphere to ellipsoid (3D)



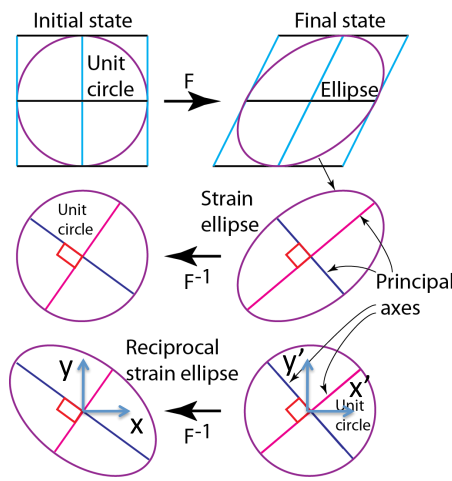
Homogeneous strain
Matrix Representation (2D)

$$[X'] = [F][X]$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[X] = [F]^{-1}[X']$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$$



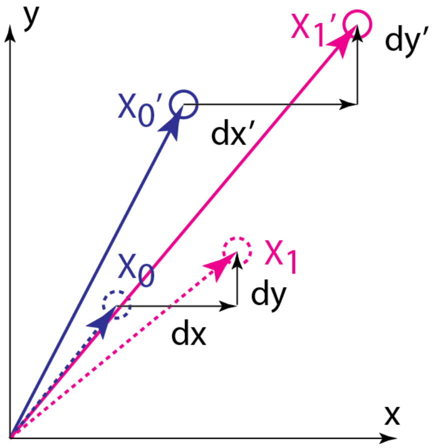
Matrix Representations: Positions (2D)

$$dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy$$

$$dy' = \frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy$$

$$\begin{bmatrix} dx' \\ dy' \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$[dX'] = [F][dX]$$



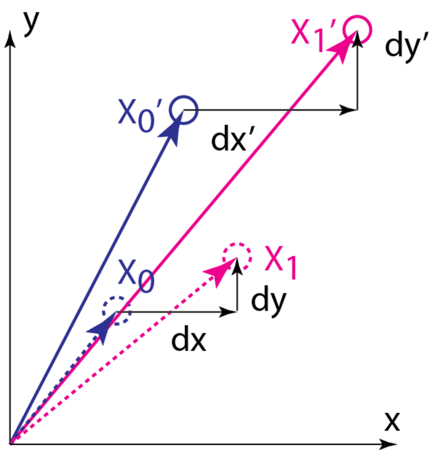
Matrix Representations: Positions (2D)

$$\begin{bmatrix} dx' \\ dy' \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

If derivatives are constant (e.g., at a point)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[X'] = [F][X]$$



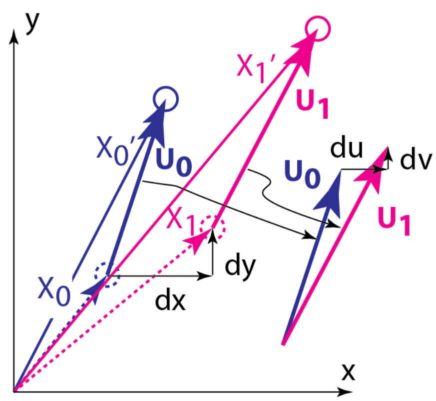
Matrix Representations Displacements (2D)

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$[dU] = [J_u][dX]$$



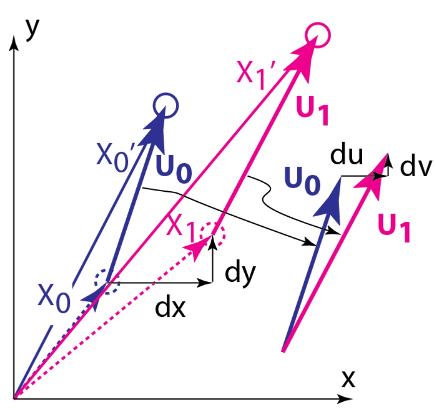
Matrix Representations Displacements (2D)

$$u = \frac{\partial u}{\partial x} x + \frac{\partial u}{\partial y} y$$

$$v = \frac{\partial v}{\partial x} x + \frac{\partial v}{\partial y} y$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[U] = [J_u][X]$$



If derivatives are constant (e.g., at a point)

Matrix Representations
Positions and Displacements (2D)

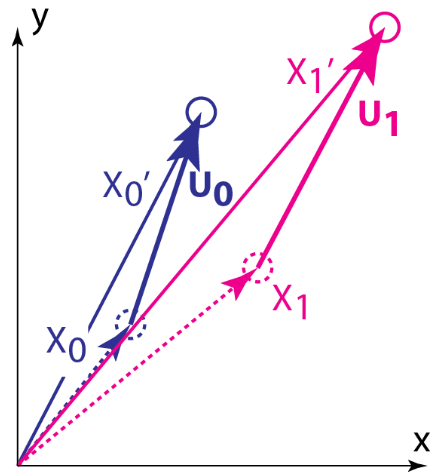
$$U = X' - X$$

$$U = FX - X = FX - IX$$

$$U = [F - I]X$$

$$[F - I] \equiv J_u$$

$$[J_u] = \begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix}$$



Matrix Representations
Positions and Displacements

Lagrangian: f(X)

$$[X'] = [F][X]$$

$$U = X' - X = FX - X$$

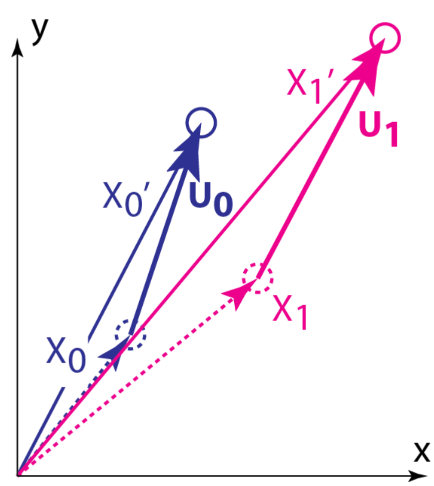
$$U = FX - IX = [F - I]X$$

Eulerian: g(X')

$$[X] = [F^{-1}][X']$$

$$U = X' - X = X' - F^{-1}X'$$

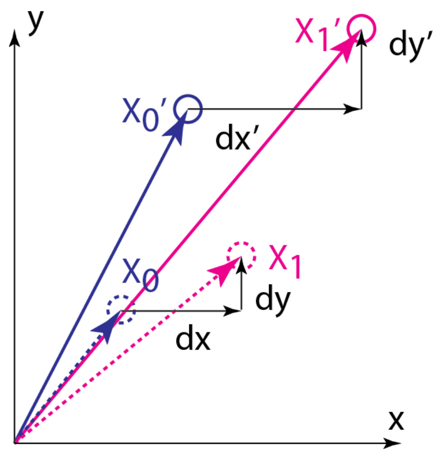
$$U = [I - F^{-1}]X'$$



Squares of Line Lengths

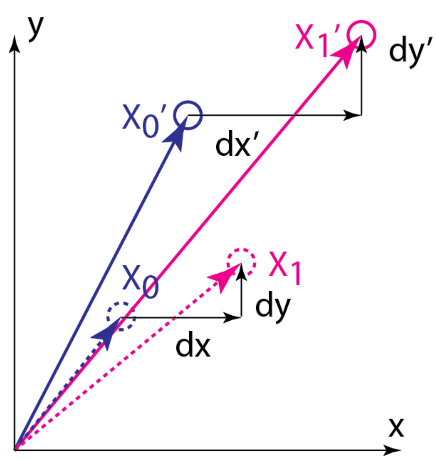
$$s^2 = |\vec{X}||\vec{X}|\cos(\theta_{\vec{X}\vec{X}})$$
$$s^2 = \vec{X} \bullet \vec{X} = X^T X$$
$$s^2 = X^T X$$

$$s'^{\,2} = \vec{X}' \bullet \vec{X}'$$
$$s'^{\,2} = [FX]^T [FX]$$
$$s'^{\,2} = X^T F^T F X$$



E (strain matrix)

$$\frac{s'^{\,2} - s^2}{2} = \frac{dX^T \left[F^T F - I\right] dX}{2}$$
$$\frac{s'^{\,2} - s^2}{2} = \frac{dX^T [E] dX}{2}$$
$$E \equiv \frac{\left[F^T F - I\right]}{2}$$



ε (Infinitesimal Strain Matrix, 2D)

$$E \equiv \left[F^T F - I\right] = \frac{1}{2}\left[\left[J_u + I\right]^T \left[J_u + I\right] - I\right]$$

$$E = \frac{1}{2}\left[\begin{bmatrix} \frac{\partial u}{\partial x}+1 & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}+1 \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial x}+1 & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}+1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

If partial derivatives << 1, their squares can be dropped to obtain the infinitesimal strain matrix ε

$$\varepsilon = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\right) & \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial y}\right) \end{bmatrix}$$

ε (Infinitesimal Strain Matrix, 2D)

$$J_u = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$\varepsilon = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\right) & \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & \left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial y}\right) \end{bmatrix} = \frac{1}{2}\left[J_u + J_u^T\right]$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\right) & \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & \left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial y}\right) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}\right) & \left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right) \\ \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) & \left(\frac{\partial v}{\partial y}-\frac{\partial v}{\partial y}\right) \end{bmatrix}$$

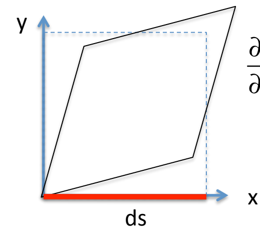
$$J_u \quad = \quad \varepsilon \quad + \quad \omega$$

ε is symmetric
ω is anti-symmetric
Linear superposition

ε (Infinitesimal Strain Matrix, 2D) Meaning of components

$$\varepsilon = \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \left(\frac{\partial v}{\partial y}\right) \end{bmatrix}$$

Pure strain without rotation



$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

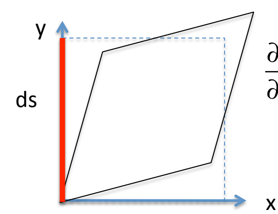
$$\begin{bmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yx} \end{bmatrix}$$

First column in ε: relative displacement vector for unit element in x-direction
ε_{yx} is displacement in the y-direction of right end of unit element in x-direction

ε (Infinitesimal Strain Matrix, 2D) Meaning of components

$$\varepsilon = \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \left(\frac{\partial v}{\partial y}\right) \end{bmatrix}$$

Pure strain without rotation



$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

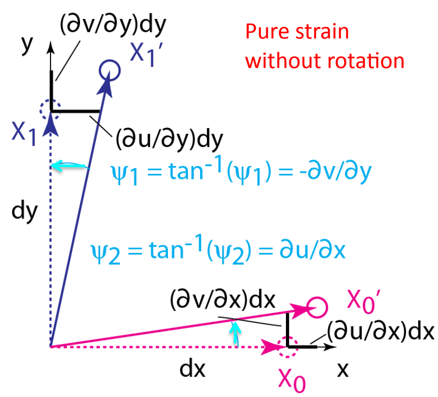
$$\begin{bmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \varepsilon_{xy} \\ \varepsilon_{yy} \end{bmatrix}$$

Second column in ε: relative displacement vector for unit element in y-direction
ε_{xy} is displacement in the x-direction of upper end of unit element in y-direction

ε (Infinitesimal Strain Matrix, 2D) Meaning of components

$$\varepsilon = \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \left(\frac{\partial v}{\partial y}\right) \end{bmatrix}$$

Pure strain without rotation



$\varepsilon_{11} = \varepsilon_{xx}$ = elongation of line parallel to x-axis
$\varepsilon_{12} = \varepsilon_{xy} \approx (\Delta\theta)/2$
$\varepsilon_{21} = \varepsilon_{yx} \approx (\Delta\theta)/2$
$\varepsilon_{22} = \varepsilon_{yy}$ = elongation of line parallel to y-axis

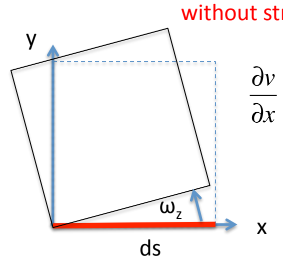
$$\frac{\Delta\theta}{2} = \frac{(\psi_2 - \psi_1)}{2} = \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

Shear strain > 0 if angle between +x and +y axes decreases

ω (Infinitesimal Strain Matrix, 2D) Meaning of components

$$\omega = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) & 0 \end{bmatrix}$$

Pure rotation without strain



$$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

$$\begin{bmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{bmatrix} \approx \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_z \end{bmatrix}$$

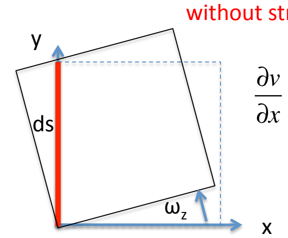
ω_z << 1 radian

First column in ω: relative displacement vector for unit element in x-direction
ω_{xy} is displacement in the y-direction of right end of unit element in x-direction

ω (Infinitesimal Strain Matrix, 2D) Meaning of components

Pure rotation without strain

$$\omega = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 \end{bmatrix}$$



$$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

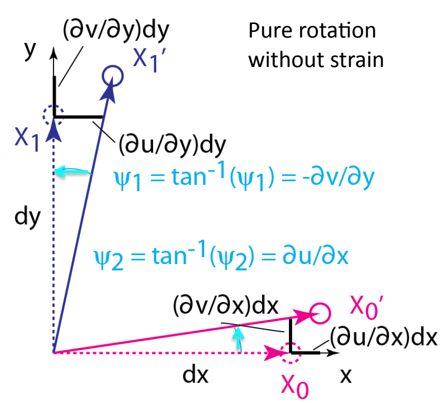
$$\begin{bmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{bmatrix} \approx \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_z \end{bmatrix}$$

$\omega_z << 1$ radian

Second column in ω: relative displacement vector for unit element in y-direction
ω_{yx} is displacement in the ***negative*** x-direction of upper end of unit element in y-direction

ε (Infinitesimal Strain Matrix, 2D) Meaning of components

$$\varepsilon = \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \left(\frac{\partial v}{\partial y}\right) \end{bmatrix}$$



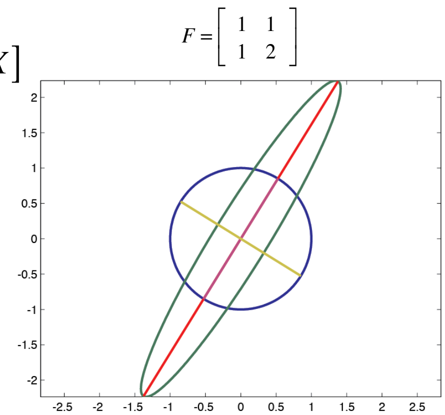
$\varepsilon_{11} = \varepsilon_{xx}$ = elongation of line parallel to x-axis
$\varepsilon_{12} = \varepsilon_{xy} \approx (\Delta\theta)/2$
$\varepsilon_{21} = \varepsilon_{yx} \approx (\Delta\theta)/2$
$\varepsilon_{22} = \varepsilon_{yy}$ = elongation of line parallel to y-axis

$$\frac{\Delta\theta}{2} = \frac{(\psi_2 - \psi_1)}{2} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

Shear strain > 0 if angle between +x and +y axes decreases

Coaxial Finite Strain

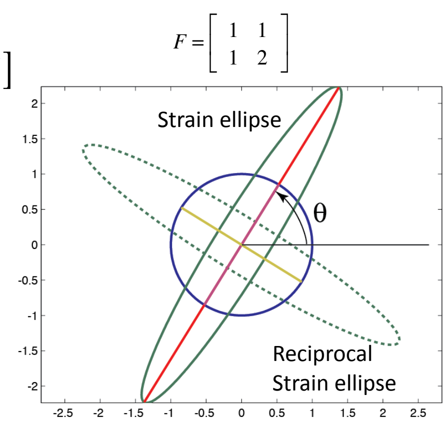
$$F=\begin{bmatrix} a & b \\ b & d \end{bmatrix}; \quad [F][X]=\lambda[X]$$



- $F = F^T$
- All values of X'•X' are positive if X'≠0
- F is positive definite
  - F has an inverse
  - Eigenvalues > 0
  - F has a square root

Coaxial Finite Strain

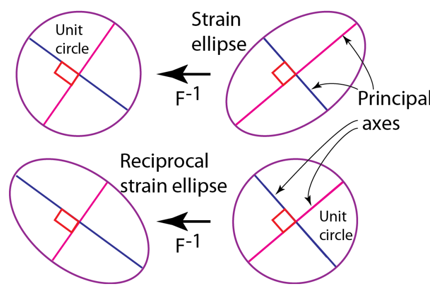
$$F=\begin{bmatrix} a & b \\ b & d \end{bmatrix}; \quad [F][X]=\lambda[X]$$



1 Eigenvectors (**X**) of F are perpendicular because F is symmetric ($\mathbf{X}_1 \bullet \mathbf{X}_2 = 0$)
2 $\mathbf{X}_1$, $\mathbf{X}_2$ solve d(**X'**•**X'**)/dθ = 0
3 $\mathbf{X}_1$, $\mathbf{X}_2$ along major axes of strain ellipse
4 $\mathbf{X}_1=\mathbf{X}_{1'}$; $\mathbf{X}_2=\mathbf{X}_{2'}$
5 Principal strain axes do not rotate

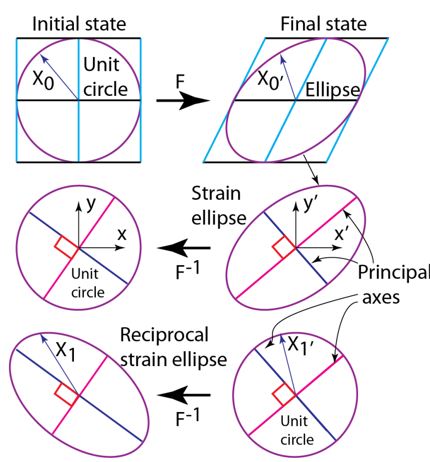
Non-coaxial Finite Strain

- The vectors that transform *from* the axes of the reciprocal strain ellipse *to* the principal axes of the strain ellipse rotate
- The rotation is given by the matrix that rotates the principal axes of the reciprocal strain ellipse to those of the strain ellipse



Non-coaxial Finite Strain

1. [X’]=[F][X]
2. X’•X’=[X][F^TF][X]
3. [F^TF] is symmetric
4. Eigenvectors of [F^TF] give principal strain directions
5. Square roots of eigenvalues of [F^TF] give principal stretches
6. [X]=[F-1][X’]
7. X•X=[X’][F^{-1}]^T[F^{-1}][X’]
8. [F^{-1}]^T[F^{-1}] is symmetric
9. Eigenvectors of [F^{-1}]^T[F^{-1}]give principal strain directions
10. Square roots of eigenvalues of [F^{-1}]^T[F^{-1}] give (reciprocal) principal stretches

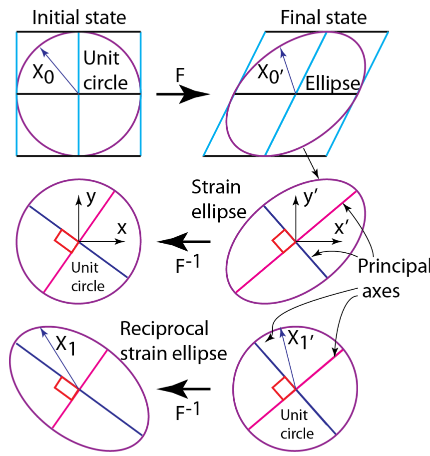


Non-coaxial Finite Strain

1 The strain ellipse and the reciprocal strain ellipse have the same eigenvalues but different eigenvectors.

2 $[F^TF]=[[F^{-1}]^T[F^{-1}]]^{-1}$

3 $[[F^{-1}]^T[F^{-1}]]^{-1} = [[F^{-1}]^{-1}[F^{-1}]^T]^{-1}]=FF^T$.



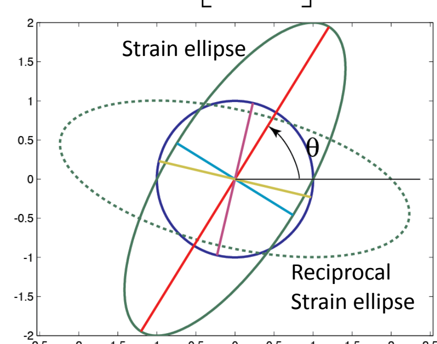
Non-coaxial Finite Strain

1 Strain ellipse and reciprocal strain ellipse have equal eigenvalues, different eigenvectors.

2 $[F^TF]=[[F^{-1}]^T[F^{-1}]]^{-1}$

3 $[[F^{-1}]^T[F^{-1}]]^{-1} =$
$[[F^{-1}]^{-1}[F^{-1}]^T]^{-1}] = FF^T$.

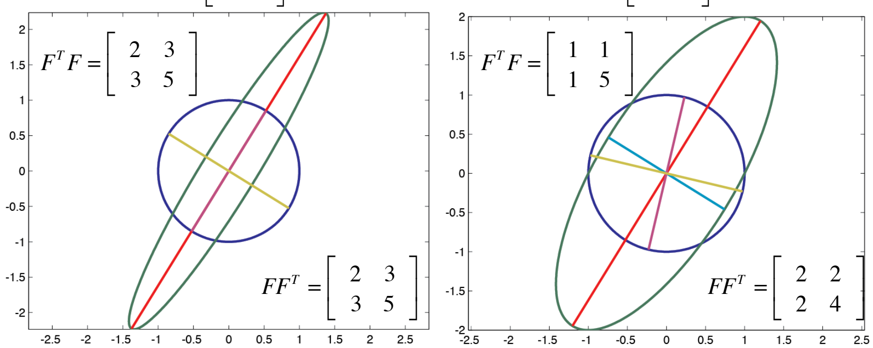
$$F = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$



$$F^T F = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}; FF^T = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

Coaxial vs. Non-coaxial Strain

$F = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $F = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$




```
>> [X,L] = eig(F)
X =
  -0.8507   0.5257
   0.5257   0.8507
L =
   0.3820        0
        0   2.6180

>> [X1,L1] = eig(F*F)
X1 =
  -0.8507   0.5257
   0.5257   0.8507
L1 =
   0.1459        0
        0   6.8541

>> [X,L0]=eig(F)
X =
   1.0000   0.7071
        0   0.7071
L0 =
        1        0
        0        2

>> [X2,L1]=eig(F'*F)
X2 =
  -0.9732   0.2298
   0.2298   0.9732
L1 =
   0.7639        0
        0   5.2361

>> [X1,L2]=eig(F*F')
X1 =
  -0.8507   0.5257
   0.5257   0.8507
L2 =
   0.7639        0
        0   5.2361
```

Coaxial vs. Non-coaxial Strain

Coaxial

- $F = F^T$ (F is symmetric)
- $FF^T = F^TF = F^2$ (F^2 is symmetric)
- $FX = \lambda X$
- $[FF^T]X = \lambda^2 X$
- $[F^TF]X = \lambda^2 X$
- $F = U = V$

F=[1 1; 1 2]; F^2=[2 3; 3 5];

```
>> [X,L] = eig(F)
X =
  -0.8507   0.5257
   0.5257   0.8507
L =
   0.3820        0
        0   2.6180

>> [X1,L1] = eig(F*F)
X1 =
  -0.8507   0.5257
   0.5257   0.8507
L1 =
   0.1459        0
        0   6.8541
```

Non-coaxial

- $F \neq F^T$ (F is not symmetric)
- $F^TF \neq F^TF$ (but both symmetric)
- $FX = \lambda X$
- $[FF^T]X_1 = \lambda_1^2 X_1$; $\lambda_1 = \lambda_2 \neq \lambda$
- $[FF^T]X_2 = \lambda_2^2 X_2$; $X \neq X_1 \neq X_2$
- $F = RU = R[F^TF]^{1/2} = VR = [F^TF]^{1/2} R$

F=[1 1; 0 2]; F^TF=[1 1; 1 5]; FF^T=[2 2; 2 4];

```
>> [X,L0]=eig(F)
X =
   1.0000   0.7071
        0   0.7071
L0 =
        1        0
        0        2

>> [X2,L2]=eig(F'*F)
X2 =
  -0.9732   0.2298
   0.2298   0.9732
L2 =
   0.7639        0
        0   5.2361

>> [X1,L1]=eig(F*F')
X1 =
  -0.8507   0.5257
   0.5257   0.8507
L1 =
   0.7639        0
        0   5.2361
```

Polar Decomposition Theorem

Suppose

(1) $[F] = [R][U]$,

where R is a rotation matrix and U is a symmetric stretch matrix. Then

(2) $F^TF = [RU]^T[RU] = U^TR^TRU$
$= U^TR^{-1}RU = U^TU$

However, U is postulated to be positive definite, so

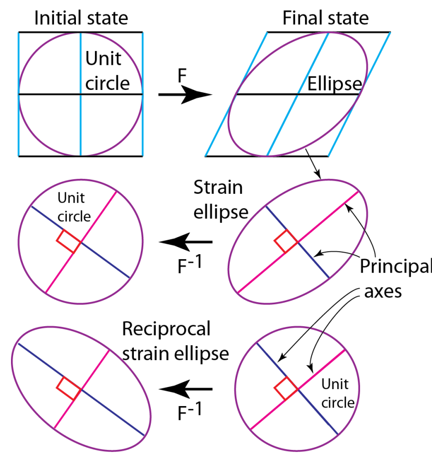
(3) $U^TU = U^2 = F^TF$

Since F^TF gives squares of line lengths, if U gives strains without rotations, it too should give the same squares of line lengths. Hence

(4) $U = [F^TF]^{1/2}$

From equation (1):

(5) $R = FU^{-1}$



Polar Decomposition Theorem

Suppose

(1) $[F] = [V][R^*]$,

where R^* is a rotation matrix and V is a symmetric stretch matrix. Then

(2) $FF^T = [VR^*][VR^*]^T = VR^*R^{*T}V^T$
$= VR^*R^{*-1}V^T = VV^T$

However, V is postulated to be positive definite, so

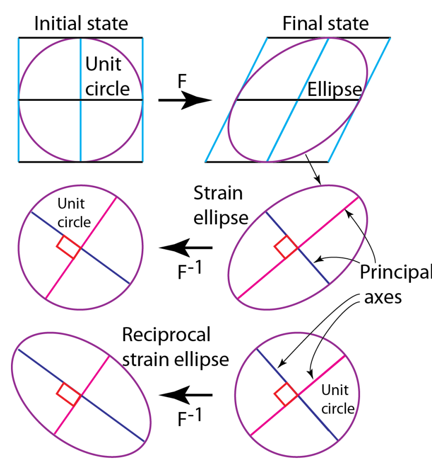
(3) $VV^T = V^2 = FF^T$

Since FF^T gives squares of line lengths, if V gives strains without rotations, it too should give the same squares of line lengths. Hence

(4) $V = [FF^T]^{1/2}$

From equation (1):

(5) $R^* = V^{-1}F$



Polar Decomposition Theorem

Proof that the polar decompositions are unique.

Suppose different decompositions exist

$$F = R_1 U_1 = R_2 U_2$$

$$X' \bullet X' = [FX] \bullet [FX] = [FX]^T [FX] = X^T F^T F X$$

$$F^T F = [R_1 U_1]^T [R_1 U_1] = U_1^T R_1^T R_1 U_1 = U_1^T R_1^{-1} R_1 U_1$$

$$= U_1^T I U_1 = U_1^T U_1 = U_1 U_1 = U_1^2$$

$$F^T F = [R_2 U_2]^T [R_2 U_2] = U_2^T R_2^T R_2 U_2 = U_2^T R_2^{-1} R_2 U_2$$

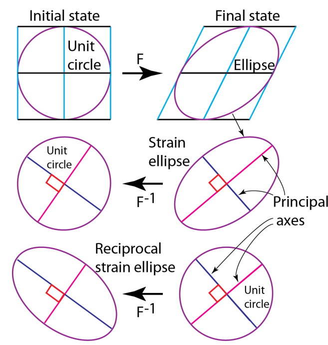
$$= U_2^T I U_2 = U_2^T U_2 = U_2 U_2 = U_2^2$$

$$U_1^2 = U_2^2$$

$$U_1 = U_2 = U$$

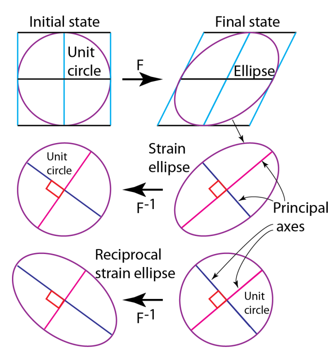
$$F = R_1 U_1 = R_2 U_1$$

$$R_1 = R_2 = R$$



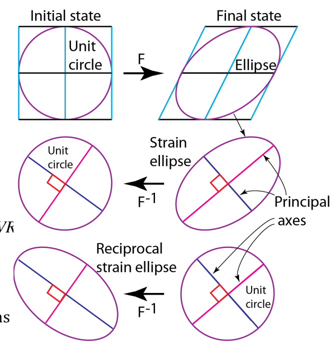
Polar Decomposition Theorem

•The same procedure can be followed to show that the decomposition F=VR* is unique. These results are very important: F can be decomposed into only one symmetric matrix that is pre-multiplied by a unique rotation matrix, and F can be decomposed into only one symmetric matrix that is post-multiplied by a unique rotation matrix.



Polar Decomposition Theorem

Proof that F = RU = VR



Intuitively, we might expect that R = R*. This is straightforward to show.

(gg) $F = VR^* = IVR^* = \left[R^* [R^*]^{-1} \right] VR^* = R^* \left[[R^*]^{-1} VR* \right] = R^* \left[[R^*]^T VR^* \right]$

Now consider the character of $\left[[R^*]^T VR^* \right]$ by taking its transpose

$\left[[R^*]^T VR^* \right]^T = [VR^*]^T \left[[R^*]^T \right] = \left[[R^*]^T [V]^T \right] \left[[R^*]^T \right]^T = [R^*]^T [V]^T [R^*] = [R^*]^T [VR$

The transpose of $\left[[R^*]^T VR^* \right]$ equals $\left[[R^*]^T VR^* \right]$, so $\left[[R^*]^T VR^* \right]$ is symmetric (definite-positive) matrix. It also is pre-multiplied by a rotation matrix. That means equation (gg) can be re-written as

$F = R^* U_2$

Equating the two right sides above

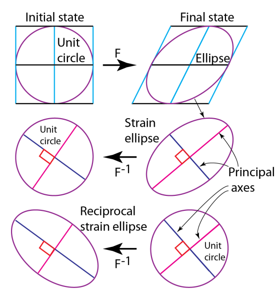
$F = RU = R^* U_2$

The results of (ff) show that the rotation matrix and U-matrix are uniquely defined, so R = R*, hence

$F = RU = VR$.

Polar Decomposition Theorem

Comparison of eigenvectors and eigenvalues



Now compare the eigenvectors and eigenvalues of U and V (see example 3.2.1 of Lai et al.). Suppose X is an eigenvector of U and λ is an eigenvalue of U.

$U\hat{X} = \lambda \hat{X}$

$RU\hat{X} = \lambda R\hat{X}$

$[RU]\hat{X} = \lambda R\hat{X}$

$[RU] = [VR] = F$

$[VR]\hat{X} = \lambda R\hat{X}$

$V\left[R\hat{X} \right] = \lambda \left[R\hat{X} \right]$

So RX is an eigenvector of V, and λ is an eigenvalue of V. Since λ is also an eigenvalue of U (see the first step), that means the eigenvalues of U and V are the same, even though the eigenvectors are not.

The rotation matrix R rotates the eigenvectors of U to the orientation of the eigenvectors of V. This means that the matrix U describes the principal axes of the reciprocal strain ellipse, and the matrix V describes the principal axes of the strain ellipse.

Stress

1 Stress vector
2 Stress state at a point
3 Stress transformations
4 Principal stresses

16. STRESS AT A POINT

http://hvo.wr.usgs.gov/kilauea/update/images.html

16. STRESS AT A POINT

I Stress vector (traction) on a plane

A $\vec{\tau} = \lim_{A \to 0} \vec{F} / A$

B Traction vectors can be added as vectors

C A traction vector can be resolved into normal (τ_n) and shear (τ_s) components

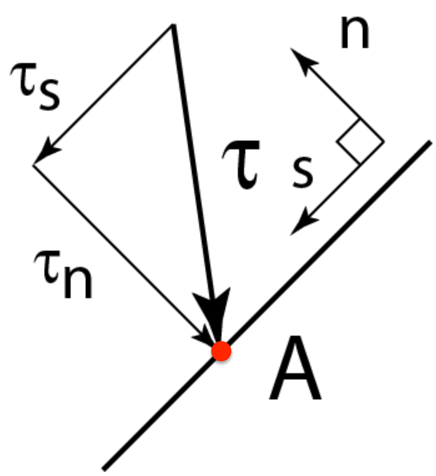
1 A normal traction (τ_n) acts perpendicular to a plane

2 A shear traction (τ_s) acts parallel to a plane

D Local reference frame

1 The n-axis is normal to the plane

2 The s-axis is parallel to the plane



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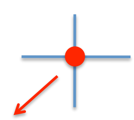
16. STRESS AT A POINT



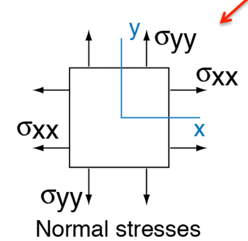
III Stress at a point (cont.)

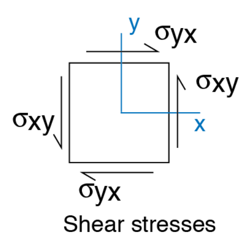
A Stresses refer to balanced internal "forces (per unit area)". They differ from force vectors, which, if unbalanced, cause accelerations

B "On -in convention": The stress component σ_{ij} acts on the plane normal to the i-direction and acts in the j-direction

1 Normal stresses: i=j

2 Shear stresses: i≠j





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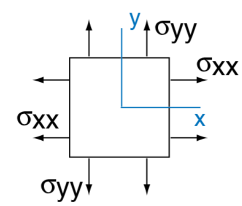
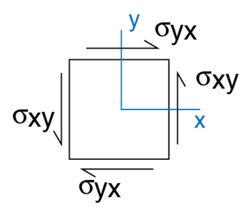
16. STRESS AT A POINT

III Stress at a point

- C Dimensions of stress: force/unit area
- D Convention for stresses
  - 1 Tension is positive
  - 2 Compression is negative
  - 3 Follows from on-in convention
  - 4 Consistent with most mechanics books
  - 5 Counter to most geology books



Normal stresses



Shear stresses

16. STRESS AT A POINT

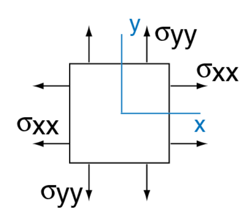
III Stress at a point

C $\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}$ 2-D 4 components

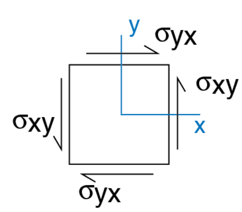
D $\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$ 3-D 9 components

E In nature, the state of stress can (and usually does) vary from point to point

F For rotational equilibrium, $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, $\sigma_{yz} = \sigma_{zy}$



Normal stresses

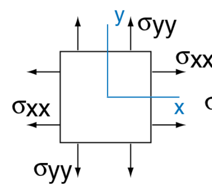


Shear stresses

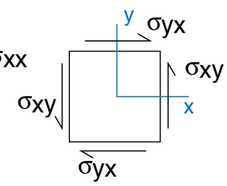
16. STRESS AT A POINT

IV Principal Stresses (these have magnitudes and orientations)

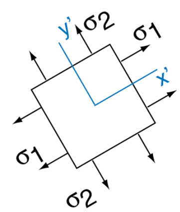
- A Principal stresses act on planes which feel no shear stress
- B The principal stresses are normal stresses.
- C Principal stresses act on perpendicular planes
- D The maximum, intermediate, and minimum principal stresses are usually designated σ_1, σ_2, and σ_3, respectively.
- E Principal stresses have a single subscript.



Normal stresses



Shear stresses

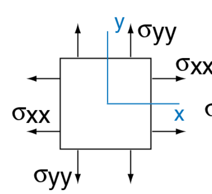


Principal stresses

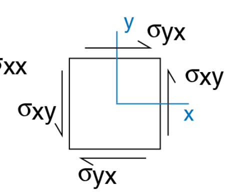
16. STRESS AT A POINT

IV Principal Stresses (cont.)

- F <u>Principal stresses represent the stress state most simply</u>



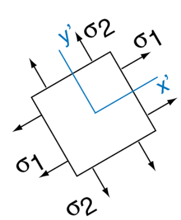
Normal stresses



Shear stresses

G $\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ 2-D 2 components

H $\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$ 3-D 3 components



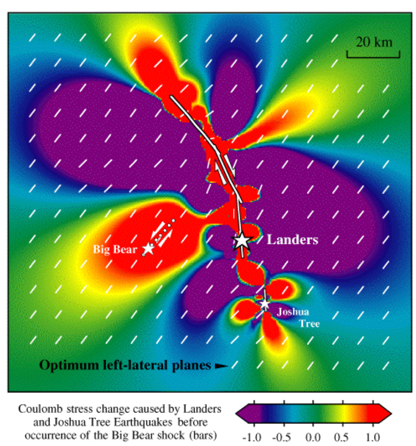
Principal stresses

19. Principal Stresses

http://hvo.wr.usgs.gov/kilauea/update/images.html

17. Mohr Circle for Tractions

- From King et al., 1994 (Fig. 11)
- Coulomb stress change caused by the Landers rupture. The left-lateral ML=6.5 Big Bear rupture occurred along dotted line 3 hr 26 min after the Landers main shock. The Coulomb stress increase at the future Big Bear epicenter is 2.2-2.9 bars.



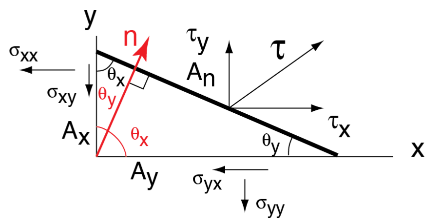
http://earthquake.usgs.gov/research/modeling/papers/landers.php

19. Principal Stresses

II Cauchy's formula

- A Relates traction (stress *vector*) components to stress *tensor* components in the same reference frame
- B 2D and 3D treatments analogous
- C $\tau_i = \sigma_{ij} n_j = n_j \sigma_{ij}$

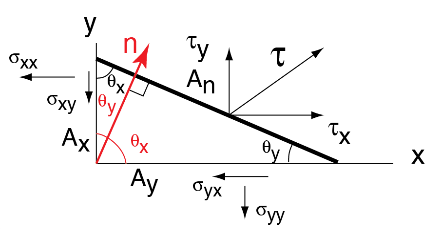
Note: all stress components shown are positive



19. Principal Stresses

II Cauchy's formula (cont.)

- C $\tau_i = n_j \sigma_{ji}$
  - 1 Meaning of terms
    - a τ_i = traction component
    - b n_j = direction cosine of angle between n-direction and j-direction
    - c σ_{ji} = traction component
    - d τ_i and σ_{ji} act *in* the same direction



$n_j = \cos\theta_{nj} = a_{nj}$

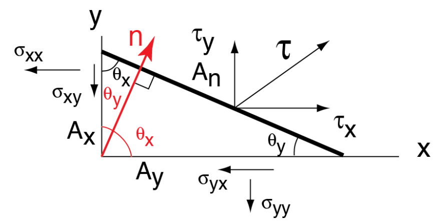
19. Principal Stresses

II Cauchy's formula (cont.)

D Expansion (2D) of $\tau_i = n_j \sigma_{ji}$

1 $\tau_x = n_x \sigma_{xx} + n_y \sigma_{yx}$

2 $\tau_y = n_x \sigma_{xy} + n_y \sigma_{yy}$



$n_j = \cos\theta_{nj} = a_{nj}$

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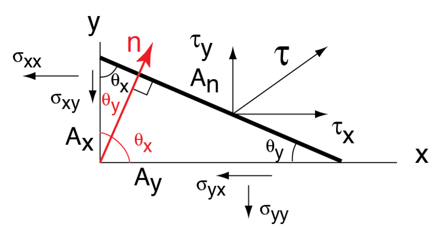
19. Principal Stresses

II Cauchy's formula (cont.)

E Derivation: Contributions to τ_x

Note that all contributions must act in x-direction



1 $\tau_x = w^{(1)}\sigma_{xx} + w^{(2)}\sigma_{yx}$

2 $\frac{F_x}{A_n} = \left(\frac{A_x}{A_n}\right)\frac{F_x^{(1)}}{A_x} + \left(\frac{A_y}{A_n}\right)\frac{F_x^{(2)}}{A_y}$

3 $\tau_x = n_x\sigma_{xx} + n_y\sigma_{yx}$

$n_x = \cos\theta_{nx} = a_{nx}$

$n_y = \cos\theta_{ny} = a_{ny}$

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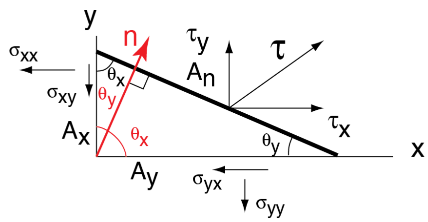
19. Principal Stresses

II Cauchy's formula (cont.)

E Derivation:

Contributions to τ_y

Note that all contributions must act in y-direction



1 $\tau_y = w^{(3)}\sigma_{xy} + w^{(4)}\sigma_{yy}$

2 $\frac{F_y}{A_n} = \left(\frac{A_x}{A_n}\right)\frac{F_y^{(3)}}{A_x} + \left(\frac{A_y}{A_n}\right)\frac{F_y^{(4)}}{A_y}$

3 $\tau_y = n_x\sigma_{xy} + n_y\sigma_{yy}$

$n_x = \cos\theta_{nx} = a_{nx}$

$n_y = \cos\theta_{ny} = a_{ny}$

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19. Principal Stresses

II Cauchy's formula (cont.)

F Alternative forms

1 $\tau_i = n_j\sigma_{ji}$

2 $\tau_i = \sigma_{ji}n_j$

3 $\tau_i = \sigma_{ij}n_j$

4 $\begin{bmatrix}\tau_x\\ \tau_y\\ \tau_z\end{bmatrix} = \begin{bmatrix}\sigma_{xx} & \sigma_{yx} & \sigma_{zx}\\ \sigma_{xy} & \sigma_{yy} & \sigma_{xy}\\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}\end{bmatrix}\begin{bmatrix}n_x\\ n_y\\ n_z\end{bmatrix}$ ← 3D

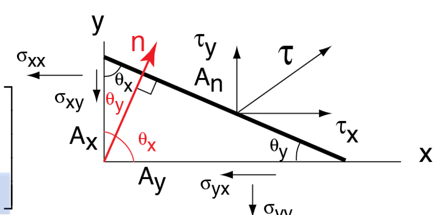
5 Matlab

a t = s'*n

b t = s*n

$\tau_x = n_x\sigma_{xx} + n_y\sigma_{yx}$

$\tau_y = n_x\sigma_{xy} + n_y\sigma_{yy}$



$n_j = \cos\theta_{nj} = a_{nj}$

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19. Principal Stresses

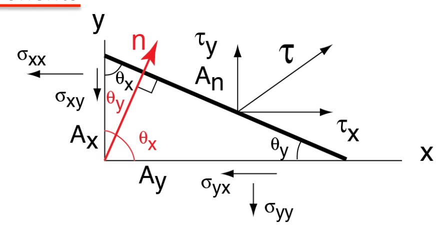
III Principal stresses (eigenvectors and eigenvalues)

A $\begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix}$ Cauchy's Formula

B $\begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} = \left|\vec{\tau}\right| \begin{bmatrix} n_x \\ n_y \end{bmatrix}$ Vector components

$Let\ \lambda = \left|\vec{\tau}\right|$

C $\begin{bmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \lambda \begin{bmatrix} n_x \\ n_y \end{bmatrix}$



The form of (C) is [A][X=λ[X], and [σ] is symmetric

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

From previous notes

III Eigenvalue problems, eigenvectors and eigenvalues (cont.)

→J Characteristic equation: |A-Iλ|=0

→3 Eigenvalues of a ***symmetric*** 2x2 matrix $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

a $\lambda_1, \lambda_2 = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - b^2)}}{2}$

b $\lambda_1, \lambda_2 = \dfrac{(a+d) \pm \sqrt{(a+2ad+d)^2 - 4ad + 4b^2}}{2}$

c $\lambda_1, \lambda_2 = \dfrac{(a+d) \pm \sqrt{(a-2ad+d)^2 + 4b^2}}{2}$

→ d $\lambda_1, \lambda_2 = \dfrac{(a+d) \pm \sqrt{(a-d)^2 + 4b^2}}{2}$

Radical term cannot be negative. Eigenvalues are real.

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

From previous notes

L Distinct eigenvectors ($\mathbf{X}_1$, $\mathbf{X}_2$) of a symmetric 2x2 matrix are perpendicular

Since the left sides of (2a) and (2b) are equal, the right sides must be equal too. Hence,

4 $\lambda_1 (\mathbf{X}_2 \bullet \mathbf{X}_1) = \lambda_2 (\mathbf{X}_1 \bullet \mathbf{X}_2)$

Now subtract the right side of (4) from the left

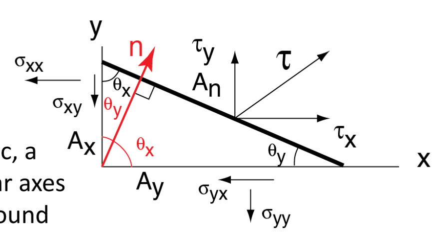
5 $(\lambda_1 - \lambda_2)(\mathbf{X}_2 \bullet \mathbf{X}_1) = 0$

- The eigenvalues generally are different, so $\lambda_1 - \lambda_2 \neq 0$.
- This means for (5) to hold that $\mathbf{X}_2 \bullet \mathbf{X}_1 = 0$.
- → Therefore, the eigenvectors ($\mathbf{X}_1$, $\mathbf{X}_2$) of a symmetric 2x2 matrix are perpendicular

19. Principal Stresses

III Principal stresses (eigenvectors and eigenvalues)

$$\begin{bmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \lambda \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$



D Meaning

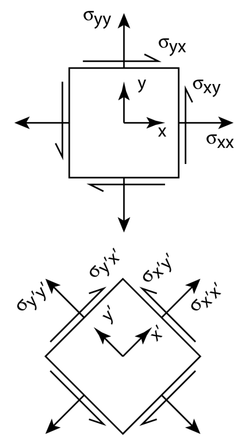
1. Since the stress tensor is symmetric, a reference frame with perpendicular axes defined by n_x and n_y pairs can be found such that the shear stresses are zero
2. This is the only way to satisfy the equation above; otherwise $\sigma_{xy} n_y \neq 0$, and $\sigma_{xy} n_x \neq 0$
3. For different (principal) values of λ, the orientation of the corresponding principal axis is expected to differ

19. Principal Stresses

V Example

Find the principal stresses

given $\sigma_{ij} = \begin{bmatrix} \sigma_{xx} = -4MPa & \sigma_{xy} = -4MPa \\ \sigma_{yx} = -4MPa & \sigma_{yy} = -4MPa \end{bmatrix}$



19. Principal Stresses

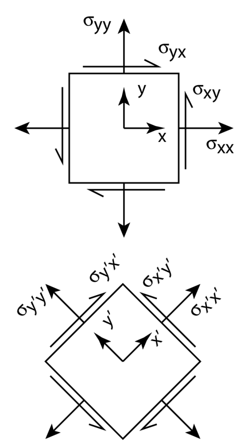
V Example

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} = -4MPa & \sigma_{xy} = -4MPa \\ \sigma_{yx} = -4MPa & \sigma_{yy} = -4MPa \end{bmatrix}$$

First find eigenvalues (in MPa)

$$\lambda_1, \lambda_2 = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4b^2}}{2}$$

$$\lambda_1, \lambda_2 = -4 \pm \frac{\sqrt{64}}{2} = -4 \pm 4 = 0, -8$$



19. Principal Stresses

IV Example

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} = -4MPa & \sigma_{xy} = -4MPa \\ \sigma_{yx} = -4MPa & \sigma_{yy} = -4MPa \end{bmatrix}$$

$$\lambda_1, \lambda_2 = -4 \pm \frac{\sqrt{64}}{2} = -4 \pm 4 = 0, -8$$ ← Eigenvalues (MPa)

Then solve for eigenvectors (X) using [A-Iλ][X]=0

$$For\ \lambda_1 = 0: \begin{bmatrix} -4-0 & -4 \\ -4 & -4-0 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -4n_x - 4n_y = 0 \Rightarrow n_x = -n_y$$

$$For\ \lambda_2 = -8: \begin{bmatrix} -4-(-8) & -4 \\ -4 & \sigma_{yy}-(-8) \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4n_x - 4n_y = 0 \Rightarrow n_x = n_y$$

19. Principal Stresses

IV Example

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} = -4MPa & \sigma_{xy} = -4MPa \\ \sigma_{yx} = -4MPa & \sigma_{yy} = -4MPa \end{bmatrix}$$

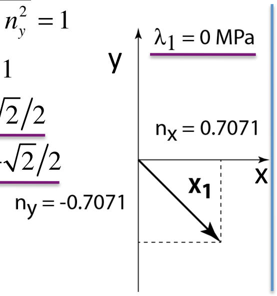
Eigenvalues	Eigenvectors
$\lambda_1 = 0MPa$	$n_x = -n_y$
$\lambda_2 = -8MPa$	$n_x = n_y$

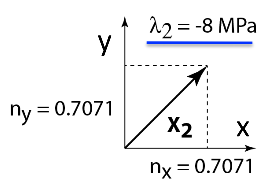
$\sqrt{n_x^2 + n_y^2} = 1$

$2n_x^2 = 1$

$n_x = \sqrt{2}/2$

$n_y = -\sqrt{2}/2$





$\sqrt{n_x^2 + n_y^2} = 1$

$2n_x^2 = 1$

$n_x = \sqrt{2}/2$

$n_y = \sqrt{2}/2$

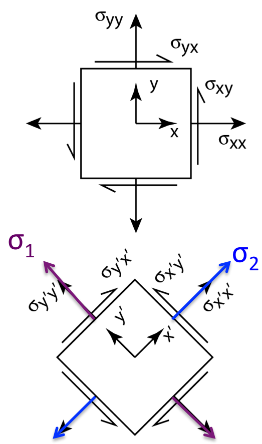
Note that $\mathbf{X_1} \bullet \mathbf{X_2} = 0$

Principal directions *are* perpendicular

19. Principal Stresses

V Example

Matrix form/Matlab




```
>> sij = [-4 -4;-4 -4]
sij =
   -4   -4
   -4   -4
>> [v,d]=eig(sij)
v =
   0.7071  -0.7071
   0.7071   0.7071
d =
   -8   0
    0   0
```

Eigenvectors (in columns)

Corresponding eigenvalues (in columns)

Summary of Strain and Stress

- Different quantities with different dimensions (dimensionless vs. force/unit area)
- Both can be represented by the orientation and magnitude of their principal values
- Strain describes changes in distance between points and changes in right angles
- Matrices of co-axial strain and stress are symmetric: eigenvalues are orthogonal and do not rotate
- Asymmetric strain matrices involve rotation
- Infinitesimal strains can be superposed linearly
- Finite strains involve matrix multiplication