SUBSIDENCE IN THREE DIMENSIONS: CENTER OF DILATION (MOGI SOURCE) (42)

I Main Topics

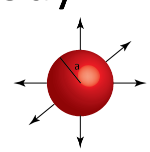
A Deformation about a pressurized spherical cavity in an infinite body

B Center of dilation (contraction) in full-space

C References

* Patterned after Segall, 2010

II Deformation about a pressurized spherical cavity in an infinite body



A Radial displacement u_R about a cavity of radius a

The equilibrium (force balance) equation in the radial direction is

$$\frac{d^2u_R}{dR^2} + \frac{2}{R}\frac{du_R}{dR} - \frac{2}{R^2}u_R = 0$$

Express u_R in the form of a power series (any continuous function can be expressed that way). Also, we expect u_R to decrease with distance from the center, so $u_R \rightarrow 0$ as $R \rightarrow \infty$. The powers of R cannot be positive, otherwise $u_R \rightarrow \pm\infty$ as $R \rightarrow \infty$. So u_R and its first and second derivatives can be expressed as

$$u_R = \ldots + C_{-3}R^{-3} + C_{-2}R^{-2} + C_{-1}R^{-1} + C_0R^0$$

$$\frac{du_R}{dR} = \ldots - 3C_{-3}R^{-4} - 2C_{-2}R^{-3} - 1C_{-1}R^{-2} + 0C_0$$

$$\frac{d^2u_R}{dR^2} = \ldots 12C_{-3}R^{-5} + 6C_{-2}R^{-4} + 2C_{-1}R^{-3} + 0C_0$$

The terms C_{-i} are coefficients

II Deformation about a pressurized spherical cavity in an infinite body

$$\frac{d^2u_R}{dR^2}+\frac{2}{R}\frac{du_R}{dR}-\frac{2}{R^2}u_R=0$$

Inserting the expressions for u_R and its derivatives into the equilibrium equation yields

$$\ldots \quad 12C_{-3}R^{-5}+6C_{-2}R^{-4}+2C_{-1}R^{-3}+0C_0$$

$$\ldots+\frac{2}{R}\left(-3C_{-3}R^{-4}-2C_{-2}R^{-3}-1C_{-1}R^{-2}+0C_0\right)$$

$$\ldots-\frac{2}{R^2}\left(C_{-3}R^{-3}+\ C_{-2}R^{-2}+\ C_{-1}R^{-1}+\ C_0R^0\right)=0$$

Now multiply by the leading radial terms

$$\ldots 12C_{-3}R^{-5}\ +6C_{-2}R^{-4}+2C_{-1}R^{-3}+0C_0$$

$$\ldots-6C_{-3}R^{-5}-4C_{-2}R^{-4}-2C_{-1}R^{-3}+0C_0$$

$$\ldots-2C_{-3}R^{-5}-2C_{-2}R^{-4}-2C_{-1}R^{-3}-2C_0=0$$

Sum terms of like powers of R

$$\ldots 4C_{-3}R^{-5}+0C_{-2}R^{-4}-2C_{-1}R^{-3}-2C_0=0$$

5/6/15 GG454 3

II Deformation about a pressurized spherical cavity in an infinite body

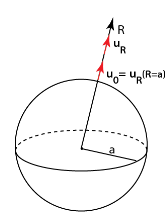
$$\ldots 4C_{-3}R^{-5}+0C_{-2}R^{-4}-2C_{-1}R^{-3}-2C_0R^0=0$$

The equilibrium equation holds for all values of R. Since all the terms in the power series were linearly independent (one term cannot be expressed as combinations of the others), the only way the equation above can hold for all values of R is if each term equals zero. That means all the coefficients except C_{-2} must equal zero. Hence,

$$u_r\ =C_{-2}R^{-2}$$

If the radial displacement at the wall of the hole, where $R = a$, is u_0, then

$$u_R\left(R=a\right)=u_0=C_{-2}a^{-2}$$



Solving for C-2 yields $$C_{-2}=u_0a^2$$

Hence $$u_R=u_0a^2R^{-2}=u_0\frac{a^2}{R^2}$$

5/6/15 GG454 4

II Deformation about a pressurized spherical cavity in an infinite body

B Principal strains and principal stresses in terms of u_0

$$u_R = \frac{u_0 a^2}{R^2}$$

$$\varepsilon_{RR} = \frac{\partial u_R}{\partial R} = \frac{-2u_0 a^2}{R^3}$$

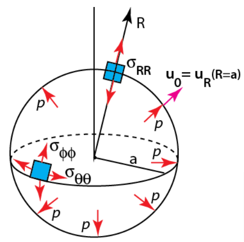
$$\varepsilon_{\phi\phi} = \varepsilon_{\theta\theta} = \frac{u_R}{R} = \frac{u_0 a^2}{R^3}$$

$$\Delta = \varepsilon_{RR} + \varepsilon_{\phi\phi} + \varepsilon_{\theta\theta} = 0$$

$$\sigma_{RR} = \frac{2Gv}{1-2v}\Delta + 2G\varepsilon_{RR} = \frac{-4Gu_0 a^2}{R^3}$$

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} = \frac{2Gv}{1-2v}\Delta + 2G\varepsilon_{\phi\phi} = \frac{-1}{2}\sigma_{RR} = \frac{2Gu_0 a^2}{R^3}$$

$$\boxed{G = \text{shear modulus} = \frac{E}{2(1+v)}}$$



If p is the pressure in the cavity, then at the wall of the cavity

$$\sigma_{RR}(R=a) = \frac{-4Gu_0 a^2}{a^3} = \frac{-4Gu_0}{a} = -p$$

$$\boxed{u_0 = \frac{p}{4G}a}$$

This allows the displacement, strains, and stresses to be expressed in terms of *p*.

II Deformation about a pressurized spherical cavity in an infinite body

C Principal strains and principal stresses in terms of p
From the previous slide, $u_0 = pa/4G$

$$u_R = \frac{u_0 a^2}{R^2} = \frac{pa^3}{4GR^2}$$

$$\varepsilon_{RR} = \frac{\partial u_R}{\partial R} = \frac{-2pa^3}{4GR^3} = \frac{-pa^3}{2GR^3}$$

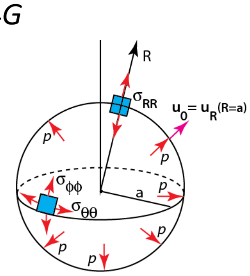
$$\varepsilon_{\phi\phi} = \varepsilon_{\theta\theta} = \frac{u_R}{R} = \frac{pa^3}{2GR^3}$$

$$\Delta = \varepsilon_{RR} + \varepsilon_{\phi\phi} + \varepsilon_{\theta\theta} = 0$$

$$\sigma_{RR} = \frac{2Gv}{1-2v}\Delta + 2G\varepsilon_{RR} = \frac{-pa^3}{R^3}$$

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} = \frac{2Gv}{1-2v}\Delta + 2G\varepsilon_{\phi\phi} = \frac{-1}{2}\sigma_{RR} = \frac{pa^3}{2R^3}$$

$$\boxed{u_0 = \frac{p}{4G}a}$$



p is the pressure in the cavity at its walls

II Deformation about a pressurized spherical cavity in an infinite body

D Principal strains and principal stresses in terms of ΔV

The change in volume of the sphere is $\Delta V = 4\pi a^2 u_0 = \pi a^3 p/G$

$$u_R = \frac{u_0 a^2}{R^2} = \frac{\Delta V}{4\pi R^2}$$

$$\boxed{u_0 = \frac{p}{4G}a = \frac{\Delta V}{4\pi a^2}}$$

$$\varepsilon_{RR} = \frac{\partial u_R}{\partial R} = \frac{-\Delta V}{2\pi R^3}$$

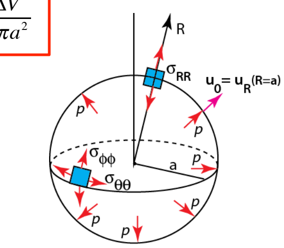
$$\varepsilon_{\phi\phi} = \varepsilon_{\theta\theta} = \frac{u_R}{R} = \frac{\Delta V}{4\pi R^3}$$

$$\Delta = \varepsilon_{RR} + \varepsilon_{\phi\phi} + \varepsilon_{\theta\theta} = 0$$

$$\sigma_{RR} = \frac{2Gv}{1-2v}\Delta + 2G\varepsilon_{RR} = \frac{-G\Delta V}{\pi R^3}$$

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} = \frac{2Gv}{1-2v}\Delta + 2G\varepsilon_{\phi\phi} = \frac{-1}{2}\sigma_{RR} = \frac{-G\Delta V}{2\pi R^3}$$

These solutions with ΔV are independent of a and apply as $a \rightarrow 0$



ΔV is the change in volume of the cavity resulting from the radial displacement of its walls

II Deformation about a pressurized spherical cavity in an infinite body

E Displacements and strains in terms of ΔV in cylindrical coordinates along a plane above a pressurized cavity

$$u_z(z=d) = u_R(z=d)\frac{d}{R} = \frac{\Delta V}{4\pi R^2}\frac{z}{R} = \frac{\Delta V}{4\pi}\frac{z}{\left(r^2+z^2\right)^{3/2}} = \frac{\Delta V}{4\pi}\frac{d}{\left(r^2+d^2\right)^{3/2}}$$

$$u_r(z=d) = u_R(z=d)\frac{r}{R} = \frac{\Delta V}{4\pi R^2}\frac{r}{R} = \frac{\Delta V}{4\pi}\frac{r}{\left(r^2+z^2\right)^{3/2}} = \frac{\Delta V}{4\pi}\frac{r}{\left(r^2+d^2\right)^{3/2}}$$

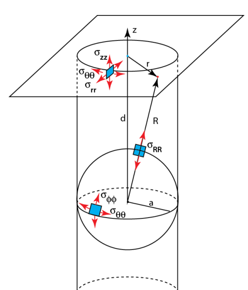
$$\varepsilon_{zz}(z=d) = \frac{\partial u_z}{\partial z} = \frac{\Delta V}{4\pi}\left[\frac{-3z^2}{\left(r^2+z^2\right)^{5/2}} + \frac{1}{\left(r^2+z^2\right)^{3/2}}\right] = \frac{\Delta V}{4\pi}\left[\frac{-3d^2}{\left(r^2+d^2\right)^{5/2}} + \frac{1}{\left(r^2+d^2\right)^{3/2}}\right]$$

$$\varepsilon_{rr}(z=d) = \frac{\partial u_r}{\partial r} = \frac{\Delta V}{4\pi}\left[\frac{-3r^2}{\left(r^2+z^2\right)^{5/2}} + \frac{1}{\left(r^2+z^2\right)^{3/2}}\right] = \frac{\Delta V}{4\pi}\left[\frac{-3r^2}{\left(r^2+d^2\right)^{5/2}} + \frac{1}{\left(r^2+d^2\right)^{3/2}}\right]$$

$$\varepsilon_{\theta\theta}(z=d) = \frac{u_r}{r} = \frac{\Delta V}{4\pi}\left[\frac{1}{\left(r^2+d^2\right)^{3/2}}\right]$$

$$\Delta(z=d) = \varepsilon_{zz} + \varepsilon_{rr} + \varepsilon_{\theta\theta} = 0$$

Cylindrical coordinates



These solutions with ΔV are independent of a and apply as $a \rightarrow 0$

II Deformation about a pressurized spherical cavity in an infinite body

E Strains and stresses in terms of *ΔV* in cylindrical coordinates along a plane above a pressurized cavity (cont.)

$$\varepsilon_{zz}(z=d)=\frac{\partial u_z}{\partial z}=\frac{\Delta V}{4\pi}\left[\frac{-3d^2}{\left(r^2+d^2\right)^{5/2}}+\frac{1}{\left(r^2+d^2\right)^{3/2}}\right]$$

$$\varepsilon_{rr}(z=d)=\frac{\partial u_r}{\partial r}=\frac{\Delta V}{4\pi}\left[\frac{-3r^2}{\left(r^2+z^2\right)^{5/2}}+\frac{1}{\left(r^2+d^2\right)^{3/2}}\right]$$

$$\varepsilon_{\theta\theta}(z=d)=\frac{u_r}{r}=\frac{\Delta V}{4\pi}\left[\frac{1}{\left(r^2+d^2\right)^{3/2}}\right]$$

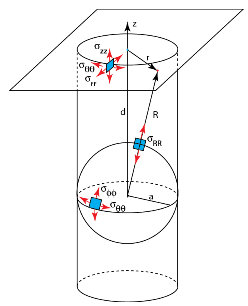
$$\Delta(z=d)=\varepsilon_{zz}+\varepsilon_{rr}+\varepsilon_{\theta\theta}=0$$

$$\sigma_{zz}(z=d)=\frac{2G\nu}{1-2\nu}\Delta+2G\varepsilon_{zz}=\frac{G\Delta V}{2\pi}\left[\frac{-3d^2}{\left(r^2+d^2\right)^{5/2}}+\frac{1}{\left(r^2+d^2\right)^{3/2}}\right]$$

$$\sigma_{rr}(z=d)=\frac{2G\nu}{1-2\nu}\Delta+2G\varepsilon_{rr}=\frac{G\Delta V}{2\pi}\left[\frac{-3r^2}{\left(r^2+d^2\right)^{5/2}}+\frac{1}{\left(r^2+d^2\right)^{3/2}}\right]$$

$$\sigma_{\theta\theta}(z=d)=\frac{2G\nu}{1-2\nu}\Delta+2G\varepsilon_{\theta\theta}=\frac{G\Delta V}{2\pi}\left[\frac{1}{\left(r^2+d^2\right)^{3/2}}\right]$$

Cylindrical coordinates

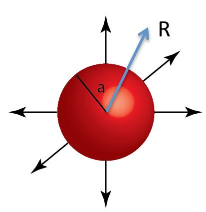


These solutions with *ΔV* are independent of *a* and apply as *a* → 0

II Deformation about a pressurized spherical cavity in an infinite body

F Key points

1. Displacements are radial
2. Displacements decay as $1/R^2$
3. Strains decay as $1/R^3$
4. Stresses decay as $1/R^3$
5. Displacements, strains, and stresses scale with ΔV
6. No volumetric dilation predicted anywhere in the linear elastic full space outside the pressurized sphere

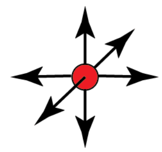


III Center of dilation in full-space

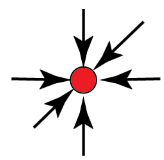
A Center of dilation
   1 A point from which nearby displacements radiate outward equally in all directions
   2 An infinitely small spherical hole ($a \rightarrow 0$) with a singular pressure for which the solutions of the previous slide apply, with $\Delta V > 0$
   3 A "nucleus of strain" obtained by differentiating and superposing the effect of a force at a point
   4 Can represent fluid accumulation at great depth

B Center of contraction
   1 A point from which nearby displacements converge equally in all directions
   2 An infinitely small spherical hole ($a \rightarrow 0$) containing a singular suction for which the solutions of the previous slide apply, with $\Delta V < 0$
   3 Can represent fluid withdrawal at great depth

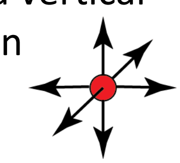
Center of dilation

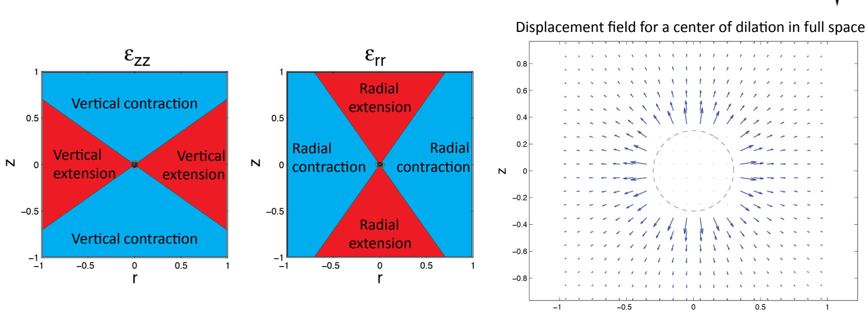


Center of contraction



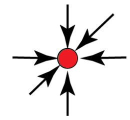
C Normal strains and displacements in a vertical plane through a center of dilation

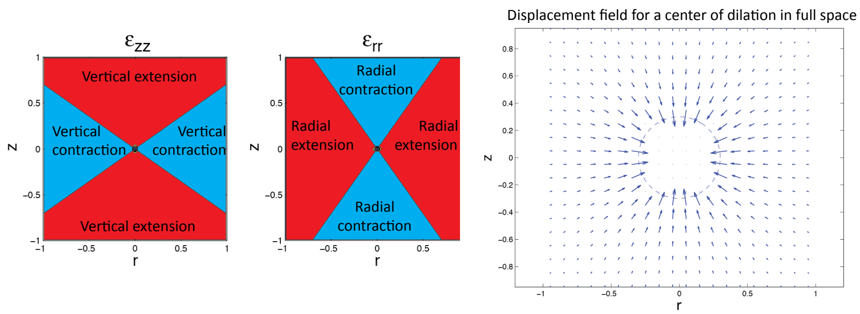




Displacements not shown within dashed circle for diagrammatic reasons

D Normal strains and displacements in a vertical plane through a center of contraction





Displacements not shown within dashed circle for diagrammatic reasons

References

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