NUMERICAL SOLUTION OF THE 1-D DIFFUSION EQUATION (39)

I Main Topics

A Motivation for using a numerical technique

B Non-dimensionalizing the diffusion (heat flow) equation

C Finite-difference solution to the 1-D heat equation (diffusion equation)

II Motivation for using a numerical technique

A Insight into the second order PDE governing transient flow

B Insight into effect of initial conditions and boundary conditions on the solution

C To solve for a wide range of initial value/ boundary value combinations and geometries

D Finite-difference method a good learning tool

III Non-dimensionalizing the heat flow equation

A Start with the heat flow equation, where

- T = temperature (variable)
- t = time (variable)
- α =thermal diffusivity (constant)
- x = position (variable)

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

B "Nondimensionalizing" (or scaling) eliminates α

III Non-dimensionalizing the heat flow equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

C Select constant scaling terms

$$x_{\max} \quad (\text{dimensions:length})$$

$$T_{\max} \quad (\text{dimensions:}^{\circ}\text{K})$$

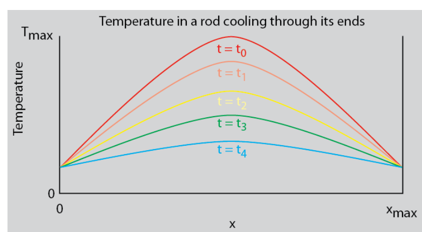
$$\frac{{x_{\max}}^2}{\alpha} \quad (\text{dimensions:time})$$

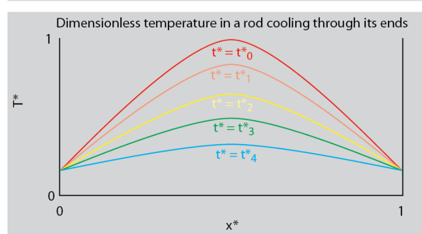
D Define dimensionless terms

$$x^* = x / x_{\max}$$

$$T^* = T / T_{\max}$$

$$t^* = t \Big/ \left(\frac{{x_{\max}}^2}{\alpha} \right)$$





III Non-dimensionalizing the heat flow equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

C Select constant scaling terms

$$x_{max} \quad (\text{dimensions:length})$$
$$T_{max} \quad (\text{dimensions:}^\circ\text{K})$$
$$\frac{x_{max}^{\ 2}}{\alpha} \quad (\text{dimensions:time})$$

D Define dimensionless terms

$$x^* = x/x_{max}$$
$$T^* = T/T_{max}$$
$$t^* = t\Big/\left(\frac{x_{max}^{\ 2}}{\alpha}\right)$$

E Recast dimensioned terms

$$x = x^* x_{max}$$
$$T = T^* T_{max}$$
$$t = t^*\left(\frac{x_{max}^{\ 2}}{\alpha}\right)$$

F Take derivatives of x*, t*, and T

$$dx^*/dx = 1/x_{max}$$
$$dt^*/dt = \frac{\alpha}{x_{max}^{\ 2}}$$
$$dT/dT^* = T_{max}$$

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III Non-dimensionalizing the heat flow equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

F Derivatives of x*, t*, and T (from previous slide)

$$\frac{dx^*}{dx} = \frac{1}{x_{max}} \qquad \frac{dt^*}{dt} = \frac{\alpha}{x_{max}^{\ 2}} \qquad \frac{dT}{dT^*} = T_{max}$$

G Recast derivatives in heat flow equation using chain rule

$$\frac{\partial T}{\partial t} = \frac{\partial T^*}{\partial t^*}\frac{dt^*}{dt}\frac{dT}{dT^*} = \frac{\partial T^*}{\partial t^*}\frac{\alpha}{(x_{max})^2}T_{max}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T^*}{\partial x^*}\frac{dx^*}{dx}\frac{dT}{dT^*} = \frac{\partial T^*}{\partial x^*}\frac{1}{x_{max}}T_{max}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial\left[\frac{\partial T}{\partial x}\right]}{\partial x} = \frac{\partial\left[\frac{\partial T}{\partial x}\right]}{\partial x^*}\frac{dx^*}{dx} = \frac{\partial\left[\frac{\partial T^*}{\partial t^*}\frac{1}{x_{max}}T_{max}\right]}{\partial x^*}\frac{1}{x_{max}} = \frac{\partial^2 T^*}{\partial x^{*2}}\frac{T_{max}}{(x_{max})^2}$$

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III Non-dimensionalizing the heat flow equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

G Dimensioned derivatives in heat flow equation

$$\frac{\partial T}{\partial t} = \frac{\partial T^*}{\partial t^*} T_{\max} \frac{\alpha}{\left(x_{\max}\right)^2} \qquad \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T^*}{\partial x^{*2}} \frac{\alpha T_{\max}}{\left(x_{\max}\right)^2}$$

H Substitute for dimensioned derivatives in heat flow equation to obtain dimensionless form

$$\frac{\partial T^*}{\partial t^*} T_{\max} \frac{\alpha}{\left(x_{\max}\right)^2} = \alpha \frac{\partial^2 T^*}{\partial x^{*2}} \frac{T_{\max}}{\left(x_{\max}\right)^2}$$

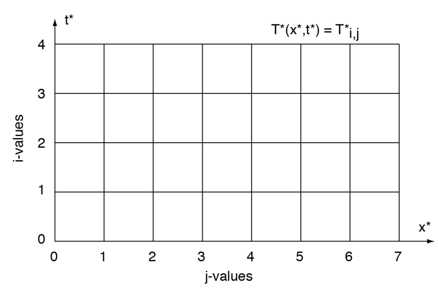
$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}}$$

All terms here are dimensionless

III Finite-difference solution to the 1-D heat equation (diffusion equation)

A Set up a dimensionless grid

1. Let Δx* = Δt* =1
2. The row number I gives the time step
3. The column number j gives the position



III Finite-difference solution to the 1-D heat equation (diffusion equation)

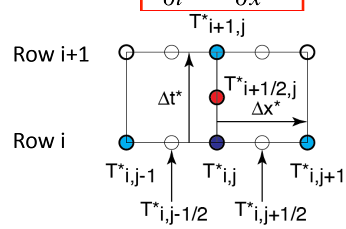
$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}}$$

B Explicit method

1 Approximate ∂T*/∂t*

$$\frac{\partial T^*}{\partial t^*} \approx \frac{[T^*(x^*, t^* + \Delta t^*) - T^*(x^*, t^*)]}{\Delta t^*}$$

$$\left.\frac{\partial T^*}{\partial t^*}\right|_{i+1/2,j} \approx \frac{[T^*_{i+1,j} - T^*_{i,j}]}{\Delta t^*}$$



2 Approximate ∂²T*/∂x*²

$$\frac{\partial^2 T^*}{\partial x^{*2}} = \frac{\partial\left(\frac{\partial T^*}{\partial x^*}\right)}{\partial x^*} \approx \frac{\partial\left(\frac{\partial T^*}{\partial x^*}\right)_{i,j+1/2} - \partial\left(\frac{\partial T^*}{\partial x^*}\right)_{i,j-1/2}}{\Delta x^*}$$

$$\left.\frac{\partial^2 T^*}{\partial x^{*2}}\right|_{i,j} \approx \frac{\frac{T^*_{i,j+1} - T^*_{i,j}}{\Delta x^*} - \frac{T^*_{i,j} - T^*_{i,j-1}}{\Delta x^*}}{\Delta x^*} = \frac{T^*_{i,j+1} - 2T^*_{i,j} - T^*_{i,j-1}}{(\Delta x^*)^2}$$

Problem: approximations are at two different points, so equating them is problematic.

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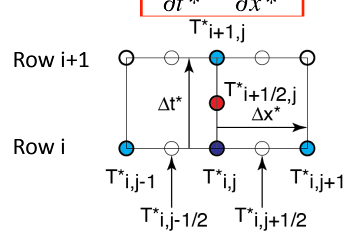
III Finite-difference solution to the 1-D heat equation (diffusion equation)

$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}}$$

B Explicit method

1 Approximate ∂T*/∂t*

$$\left.\frac{\partial T^*}{\partial t^*}\right|_{i+1/2,j} \approx \frac{[T^*_{i+1,j} - T^*_{i,j}]}{\Delta t^*}$$



2 Approximate ∂²T*/∂x*2

$$\left.\frac{\partial^2 T^*}{\partial x^{*2}}\right|_{i,j} \approx \frac{T^*_{i,j+1} - 2T^*_{i,j} - T^*_{i,j-1}}{(\Delta x^*)^2}$$

3 Set the derivatives equal

4 Solve for $T^*_{i+1,j}$

5 Solution prone to numerical error: approximations are at two different points

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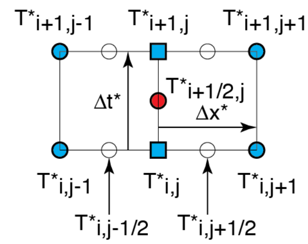
III Finite-difference solution to the 1-D heat equation (diffusion equation)

C Crank-Nicolson method

1 Evaluate $\partial^2 T/\partial x^2$ at • by *averaging* the second derivatives at ■

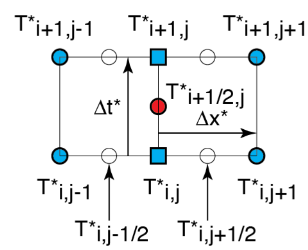


III Finite-difference solution to the 1-D heat equation (diffusion equation)

2 Approximate $\partial T^*/\partial t^*$ at •

$$\left.\frac{\partial T^*}{\partial t^*}\right|_{i+1/2,j} \approx \frac{T^*_{i+1,j} + T^*_{i,j}}{\Delta t^*}$$

3 Approximate $\partial^2 T^*/\partial x^{*2}$ at • by averaging values at ■



$$\left.\frac{\partial^2 T^*}{\partial x^{*2}}\right|_{i+1/2,j} \approx \frac{1}{2}\left(\left.\frac{\partial^2 T^*}{\partial x^{*2}}\right|_{i+1,j} + \left.\frac{\partial^2 T^*}{\partial x^{*2}}\right|_{i,j}\right)$$

$$\left.\frac{\partial^2 T^*}{\partial x^{*2}}\right|_{i+1/2,j} \approx \frac{1}{2}\left(\frac{T^*_{i+1,j+1} - 2T^*_{i+1,j} - T^*_{i+1,j-1}}{(\Delta x^*)^2} + \frac{T^*_{i,j+1} - 2T^*_{i,j} - T^*_{i,j-1}}{(\Delta x^*)^2}\right)$$

$$\left.\frac{\partial^2 T^*}{\partial x^{*2}}\right|_{i+1/2,j} \approx \frac{1}{2}\left(\frac{T^*_{i+1,j+1} - 2T^*_{i+1,j} - T^*_{i+1,j-1} + T^*_{i,j+1} - 2T^*_{i,j} - T^*_{i,j-1}}{(\Delta x^*)^2}\right)$$

III Finite-difference solution to the 1-D heat equation (diffusion equation)

$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}}$$

4 Now equate the dimensionless partial derivatives, setting dx* = dt* = 1

$$T^*_{i+1,j} - T^*_{i,j} \approx \frac{1}{2}\left(T^*_{i+1,j+1} - 2T^*_{i+1,j} + T^*_{i+1,j-1} + T^*_{i,j+1} - 2T^*_{i,j} + T^*_{i,j-1}\right)$$

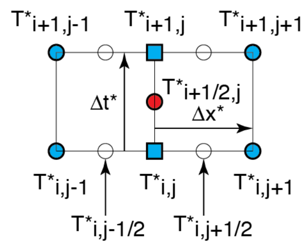
5 Solve for $T^*_{i+1,j}$

$$T^*_{i+1,j} \approx \frac{1}{2}\left(T^*_{i+1,j+1} - 2T^*_{i+1,j} + T^*_{i+1,j-1} + T^*_{i,j+1} + T^*_{i,j-1}\right)$$

$$2T^*_{i+1,j} \approx T^*_{i+1,j+1} - 2T^*_{i+1,j} + T^*_{i+1,j-1} + T^*_{i,j+1} + T^*_{i,j-1}$$

$$4T^*_{i+1,j} \approx T^*_{i+1,j+1} + T^*_{i+1,j-1} + T^*_{i,j+1} + T^*_{i,j-1}$$

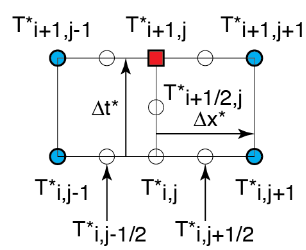
$$T^*_{i+1,j} \approx \frac{T^*_{i+1,j+1} + T^*_{i+1,j-1} + T^*_{i,j+1} + T^*_{i,j-1}}{4}$$



III Finite-difference solution to the 1-D heat equation (diffusion equation)

$$T^*_{i+1,j} \approx \frac{T^*_{i+1,j+1} + T^*_{i+1,j-1} + T^*_{i,j+1} + T^*_{i,j-1}}{4}$$

5 The value of T at any node is equal to the average value of the two adjacent nodes at the same time step and the two nodes at the preceding time step



III Finite-difference solution to the 1-D heat equation (diffusion equation)

D Recap

1. Convert dimensioned problem to dimensionless problem
2. Solve the dimensionless problem
3. Convert dimensionless temperature solution back to dimensioned solutions by multiplying by the scaling factor

$$x^* = x/x_{\max} \quad T^* = T/T_{\max} \quad t^* = t\Big/\left(\frac{x_{\max}{}^2}{\alpha}\right)$$

$$T^*_{i+1,j} \approx \frac{T^*_{i+1,j+1} + T^*_{i+1,j-1} + T^*_{i,j+1} + T^*_{i,j-1}}{4}$$

$$x = x^* x_{\max} \quad T = T^* T_{\max} \quad t = t^*\left(\frac{x_{\max}{}^2}{\alpha}\right)$$

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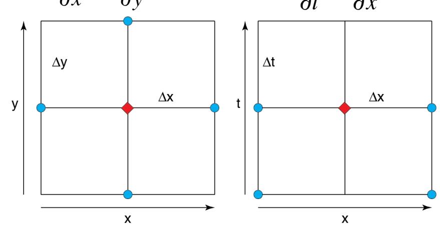
III Finite-difference solution to the 1-D heat equation (diffusion equation)

E Closing comments

1. The finite-difference method is essentially an averaging procedure that describes how "information" propagates within a system
2. Solutions for 2-D Laplace equation and 1-D heat equation have revealing similarities and revealing differences

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$



The value of T at the red node is approximately the average of the T values at the blue nodes

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