SUBSIDENCE MECHANICS: HEAT FLOW ANALOG (38)

I Main Topics

- A Motivation: Why investigate heat flow?
- B Development of 1-D heat flow equation as analog for consolidation
- C Finite-difference interpretation of heat flow equation
- D Dimensional analysis

II Motivation

- A Heat flow equation has the same form as the consolidation equation but is easier to grasp
- B Diffusion of heat analogous to diffusion of excess pore pressure
- C Many analytic solutions for heat flow (e.g., Carslaw and Jaeger, 1984)
- D Many analogous equations of great use

E Flow analogs

Flowing quantity	Incompressible Fluid	Heat	Chemical Species
Conserved quantity	Mass	Heat Energy	Molecules
1-D flux law	Darcy's law $q=-k\frac{\partial H}{\partial x}$	Fourier's law $q=-k\frac{\partial T}{\partial x}$	Fick's law $J=-D\frac{\partial c}{\partial x}$
Flux term	q= volume flux density $m^3/(m^2 \bullet sec)$	q= heat flux density joules/$(m^2 \bullet sec)$	J = diffusion flux moles/$(m^2 \bullet sec)$
Coefficient	k = hydraulic conductivity m/sec	k = thermal conductivity Joules/(m•°K•sec)	D = diffusivity m^2/sec
Potential term	H = head (m)	T = temperature (°K)	c = concentration (moles/m^3)
1-D diffusion law	$\frac{\partial H}{\partial t}=\alpha\frac{\partial^2 H}{\partial x^2}$	$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2}$	$\frac{\partial c}{\partial t}=D\frac{\partial^2 c}{\partial x^2}$
Coefficient	α = hydraulic diffusivity m^2/sec	α = thermal diffusivity m^2/sec	D = diffusivity m^2/sec
Steady state flow (Term on left side of diffusion law = 0)	$\nabla^2 H=0$	$\nabla^2 T=0$	$\nabla^2 c=0$

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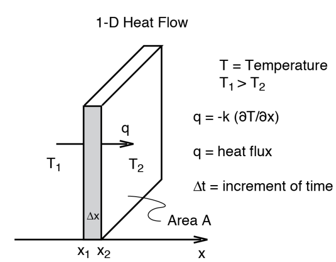
III Development of 1-D heat flow equation as analog for consolidation

A <u>Isotropic, uniform material</u>

B Definition of terms

1. U = heat energy (joules)
2. x = position (meters)
2. t = time (seconds)
3. q = heat flux (joules/(meter^2 sec)
   a. Rate of heat energy transfer per unit area per unit time
   b. Heat flux can vary with time and position, so q = q(x,t)
4. T = temperature (°)
   Temperature can vary with position and time, so T = T(x,t)



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C Fourier's Law of Heat Conduction (1-D)

$$q = -k\frac{\partial T}{\partial x}$$

1 Heat flow (q) scales with the temperature gradient (∂T/∂x)

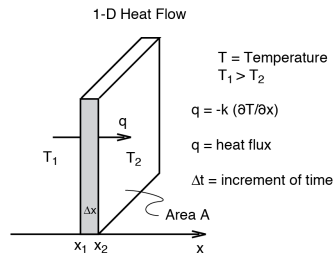
2 k = coefficient of thermal conductivity

  a Dimensions: Joules sec^{-1} m^{-1} K^{-1}

  b k assumed to be constant

3 Dimension check

$$\frac{Joules}{m^2 \sec} = \frac{Joules}{m \sec {}^\circ K}\frac{{}^\circ K}{m}$$



C Fourier's Law of Heat Conduction (1-D)

$$q = -k\frac{\partial T}{\partial x}$$

4 The minus sign

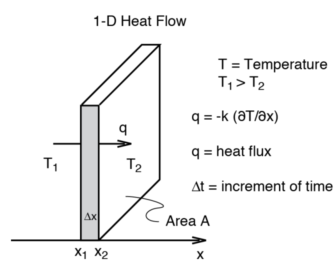
  a For heat to flow from x_1 to x_2 , where $x_1 < x_2$, $T(x_1) > T(x_2)$.

  b Positive heat flow corresponds to a drop in temperature, requiring k to be negative

5 Partial derivative used because T is a function of x and t.

6 Finite difference approximation:

$$q = -k\frac{\Delta T}{\Delta x}$$



D Fluid flow analog (slow laminar flow)

$$q = -k\frac{\partial H}{\partial x}$$

1 Volumetric flux (q) scales with the head gradient (∂H/∂x)

2 k = hydraulic conductivity

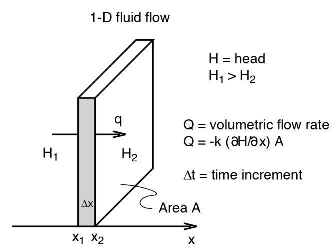
  a Dimensions: m/sec

  b k assumed to be constant

  c k depends on the intrinsic permeability of the material, the degree of saturation, and on the density and viscosity of the fluid

3 Dimension check

$$\frac{m^3}{m^2\,\text{sec}} = \frac{m}{\text{sec}}\frac{m}{m}$$



E Heat Flow Equation (Conservation of energy)

- Change in heat energy = heat in – heat out

$$\Delta U_{heat} = (\Delta T)(mass)(specific\,heat)$$

$$\Delta U_{heat} = (A)(\Delta t)\left[q(x=x_1) - q(x=x_2)\right]$$

$$(\Delta T)(mass)(specific\,heat) = (A)(\Delta t)\left[-\Delta q\right]$$

$$(\Delta T)(\rho A\Delta x)(c) = (A)(\Delta t)\left[\frac{-\partial q}{\partial x}\Delta x\right]$$

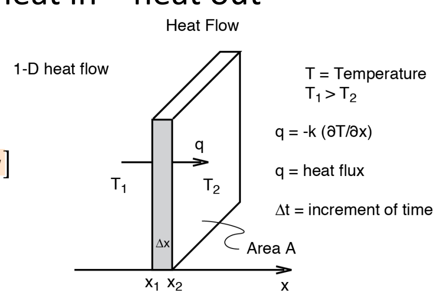
$$\frac{(\Delta T)(\rho)(c)}{(\Delta t)} = \left[\frac{-\partial\left(-k\frac{\partial T}{\partial x}\right)}{\partial x}\right]$$

q stated using Fourier's Law

$$\frac{(\Delta T)(\rho)(c)}{(\Delta t)k} = \frac{\partial^2 T}{\partial x^2}$$

$$\boxed{\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}}$$

α = thermal diffusivity = k/ρc



F Heat Flow Equation

- 1-D form: $K\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}$ parabolic differential equation
- 2-D form: $K\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}$
- 3-D form: $K\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}$
-

G Laplace equation

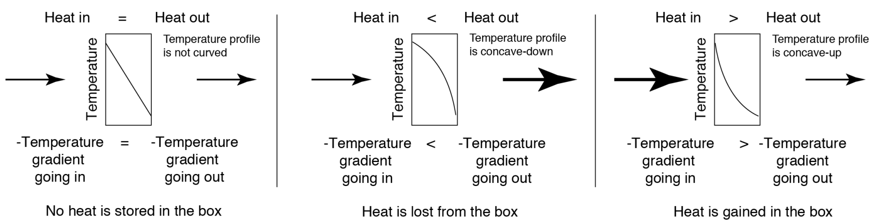
- Applies to steady state distribution of temperature
- Temperature does not change as a function of time
- $\frac{\partial T}{\partial t}=0$
- 1-D: $\frac{\partial^2 T}{\partial x^2}=0$
- 2-D: $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$
- 3-D: $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0$
- General: $\nabla^2 T=0$

$$Curvature(T_{1-D})=\frac{\frac{d^2T}{dx^2}}{\sqrt{1+\left(\frac{dT}{dx}\right)^2}^{3/2}},\ so\ \frac{d^2T}{dx^2}=0\ means\ curvature(T_{1-D})=0$$

H Relationship between 1-D temperature profile and heat change

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

- To a good approximation, the rate of temperature change with time scales with the curvature of the temperature profile
- If the 1-D temperature profile isn't curved, then no change in heat energy occurs in the slab (i.e., steady state exists)



I Fluid Flow Equation

(Conservation of mass for incompressible fluid)

- Change in fluid mass = mass in – mass out

$$\Delta mass = \left(\frac{\Delta volume}{\Delta H}\right)(density)(\Delta H)$$

$$\Delta mass = (A)(\Delta t)(density)\left[q(x = x_1) - q(x = x_2)\right]$$

$$\left(\frac{\Delta volume}{\Delta H}\right)(density)(\Delta H) = (A)(\Delta t)(density)[-\Delta q]$$

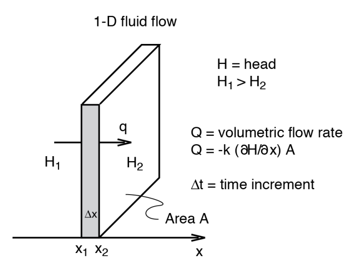
$$\left(\frac{\Delta volume}{A\Delta H}\right)(\Delta H) = (\Delta t)\left[\frac{-\partial q}{\partial x}\Delta x\right]$$

$$S\frac{(\Delta H)}{(\Delta t)} = \left[\frac{-\partial\left(-k\frac{\partial H}{\partial x}\right)}{\partial x}\right]$$

S = storativity

$$S\frac{\Delta H}{\Delta t} = k\frac{\partial^2 H}{\partial x^2} \rightarrow \frac{\partial H}{\partial t} = \alpha\frac{\partial^2 H}{\partial x^2}$$

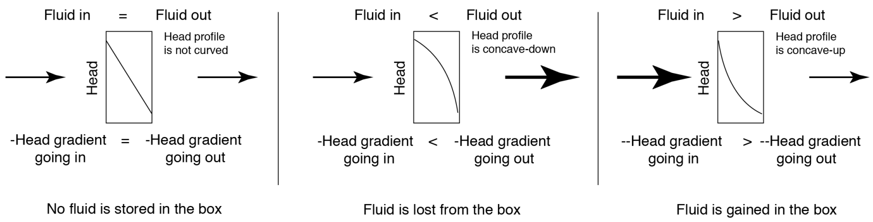
α = hydraulic diffusivity
= k/S



J Relationship between head profile and fluid volume change

$$\frac{\partial H}{\partial t} = \alpha \frac{\partial^2 H}{\partial x^2}$$

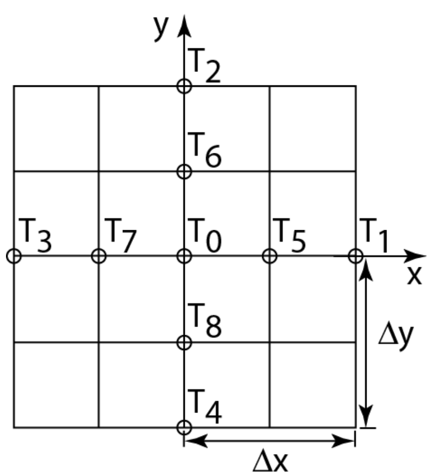
- To a good approximation, the rate of fluid content change with time scales with the curvature of the head profile
- If the head profile isn't curved, then no change in fluid content occurs in the slab



IV Finite difference interpretation of heat flow equation

- $\nabla^2 T = 0$
- The value of T (here T = temperature) at a given point is the average of the values at the nearest neighboring points on a square grid (see notes on wave eqn)

$$T_0 = \frac{T_1 + T_2 + T_3 + T_4}{4}$$



V Dimensional analysis

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

A Question: How does the thickness *H* of a plate control the time the plate takes to cool?

B Consider the dimensions of the terms in the heat flow equation and the plate thickness *H*

$$[t_c] = cooling\,time$$
$$[T] = {}^\circ K$$
$$[\alpha] = (length)^2 (time)^{-1}$$
$$[H] = [x] = length$$

V Dimensional analysis

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

C Now consider the cooling time at a point in the plate

D The equation for the cooling time t_c must be dimensionally consistent, and can only depend on the relevant factors. So

$$t_c = CT^a \alpha^b H^c,$$

$[t_c] = time$ $\quad$ $[T] = {}^\circ K$

$[\alpha] = (length)^2 (time)^{-1}$ $\quad$ $[H] = length$

where *C* is an unknown dimensionless constant

Hence $[t_c] = [T]^a [\alpha]^b [H]^c$

$$(time)^1 = ({}^\circ K)^a \left(\frac{length^2}{time}\right)^b (length)^c$$

$$(time)^1 = ({}^\circ K)^a (length)^{2b+c} (time)^{-b}$$

E By inspection

$a = 0; b = -1; c = 2$

So

$$t_c = C\alpha^{-1} H^2$$

V Dimensional analysis

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

F Meaning of solution $t_c = C\alpha^{-1}H^2$

1. The time for cooling the plate to some fraction of its initial temperature is proportional to $1/H^2$
2. Doubling the plate thickness quadruples the cooling time
3. If $C \approx 1$, then $t_c \approx \frac{H^2}{\alpha}$

   This can be used for rough estimates of the cooling time

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References

- Carslaw, H.S., and Jaeger,J.C., 1984, Conduction of heat in solids: Clarendon Press, Oxford, 510 p.
- Wang, H.F., and Anderson, M.P., 1982, Introduction to groundwater modeling: Academic Press, San Diego, 237 p.

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