THE WAVE EQUATION (30)

I Main Topics

A The Laplace equation and fluid potential

B Assumptions and boundary conditions of 2D small wave theory

C Solution of the wave equation

D Energy in a wave

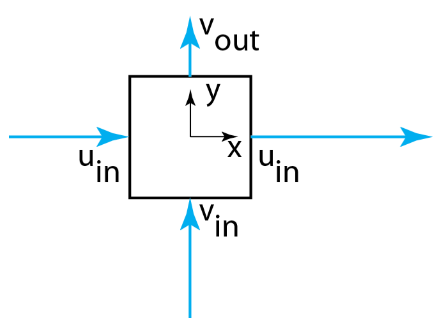
E Shoaling of waves

II TheLaplace equation and potential fluid flow

- Consider a square which fluid is flowing across, with no fluid being stored or lost in the square
- Any increase in the velocity of fluid in the x-direction (u) across the square must be matched by a decrease in velocity in the y-direction (v)

(1) $\partial u/dx = -\partial v/\partial y$

(2) $\partial u/dx + \partial v/\partial y = 0$



II TheLaplace equation and potential fluid flow

$(2)\ \partial u/\partial x + \partial v/\partial y = 0$

- Suppose that the velocities can be given by partial derivatives of a potential function ϕ

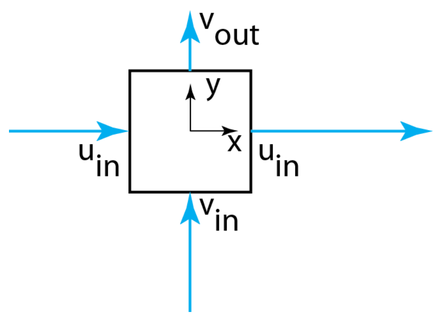
$(3)\ u = \partial \phi/\partial x$

$(4)\ v = \partial \phi/\partial y$

- Substituting (3) and (4) into (2) yields the Laplace equation

$(4)\ \partial^2 \phi/\partial x^2 + \partial^2 \phi/\partial y^2 = 0$

$(5)\ \nabla^2 \phi = 0$



II TheLaplace equation and potential fluid flow

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{\phi 0} \approx \frac{\left.\frac{\partial \phi}{\partial x}\right|_{\phi 5} - \left.\frac{\partial \phi}{\partial x}\right|_{\phi 7}}{\Delta x} \approx \frac{\frac{\phi_1 - \phi_0}{\Delta x} - \frac{\phi_0 - \phi_3}{\Delta x}}{\Delta x} = \frac{-2\phi_0 + \phi_1 + \phi_3}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 \phi}{\partial y^2}\right|_{\phi 0} \approx \frac{\left.\frac{\partial \phi}{\partial y}\right|_{\phi 6} - \left.\frac{\partial \phi}{\partial y}\right|_{\phi 8}}{\Delta y} \approx \frac{\frac{\phi_2 - \phi_0}{\Delta y} - \frac{\phi_0 - \phi_4}{\Delta y}}{\Delta y} = \frac{-2\phi_0 + \phi_2 + \phi_4}{(\Delta y)^2}$$

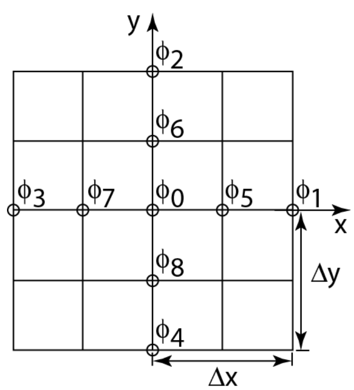
$$\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right]_{\phi 0} \approx \frac{-2\phi_0 + \phi_1 + \phi_3}{(\Delta x)^2} + \frac{-2\phi_0 + \phi_2 + \phi_4}{(\Delta y)^2} = 0$$

If $\Delta x = \Delta y$, *then*

$$\frac{-2\phi_0 + \phi_1 + \phi_3}{(\Delta x)^2} + \frac{-2\phi_0 + \phi_2 + \phi_4}{(\Delta x)^2} = 0$$

$$-4\phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 = 0$$

$$\phi_0 = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4}{4}$$



So a function that satisfies the Laplace equation has values that average those at nearest neighbors

Assuming Δx = Δy

III Assumptions and boundary conditions of 2-D small wave theory

A No geometry changes parallel to wave crest (2-D assumption)

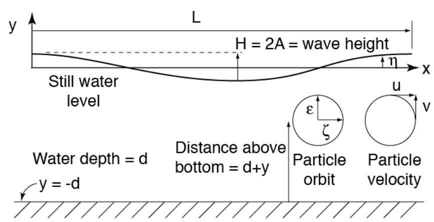
B Wave amplitude is small relative to wave length and water depth

C Water is homogeneous, incompressible, and surface tension is nil.

D The bottom is not moving, is impermeable, and is horizontal

E Pressure along air-sea interface is constant

F The water surface has the form of a cosine wave



Modified from Sorensen, 1978

$$\eta = A\cos\left[2\pi\left(\frac{x}{L}-\frac{t}{T}\right)\right] = \frac{H}{2}\cos\left[2\pi\left(\frac{x}{L}-\frac{t}{T}\right)\right]$$

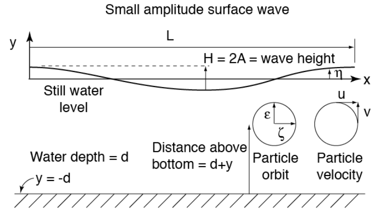
L = wavelength
T = wave period

IV Solution of the wave equation

A General solutions

$$(6)\ \phi = \frac{H}{2}\frac{gT}{2\pi}\frac{\cosh\left[\left(\frac{2\pi x}{L}\right)(d+y)\right]}{\cosh\left(\frac{2\pi d}{L}\right)}\sin\left(\frac{2\pi x}{L}-\frac{2\pi t}{T}\right)$$

- H = wave height; L = wavelength;
- d = water depth; C = wave speed;
- t = time; T = wave period (constant);
- x = horizontal position; y = vertical position



Modified from Sorensen, 1978

$$1\ C = \left(\frac{gT}{2\pi}\right)\tanh\left(\frac{2\pi d}{L}\right)$$

$$2\ L = CT$$

$$3\ |u| = \left(\frac{\pi H}{T}\right)\left\{\cosh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$$

$$4\ |v| = \left(\frac{\pi H}{T}\right)\left\{\sinh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$$

$$5\ |\zeta| = \left(\frac{H}{2}\right)\left\{\cosh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$$

$$6\ |\varepsilon| = \left(\frac{H}{2}\right)\left\{\sinh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$$

|u| = horizontal H_20 particle velocity amplitude*
|v| = vertical H_20 particle velocity amplitude*
|ζ| = horizontal H_20 particle displacement amplitude*
|ε| = verticalH_20 particle displacement amplitude*

*Function of wave height, wave period, wavelength, water depth, and distance above bottom (d+y)

IV Solution of the wave equation

A General solutions

1 $C = \left(\frac{gT}{2\pi}\right)\tanh\left(\frac{2\pi d}{L}\right)$

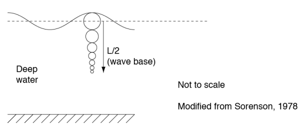
2 $L = CT$

3 $|u| = \left(\frac{\pi H}{T}\right)\left\{\cosh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$

4 $|v| = \left(\frac{\pi H}{T}\right)\left\{\sinh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$

5 $|\zeta| = \left(\frac{H}{2}\right)\left\{\cosh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$

6 $|\varepsilon| = \left(\frac{H}{2}\right)\left\{\sinh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$



B Deep-water solutions (d/L > 0.5)
tanh(2πd/L) ≈1

1 $C = \frac{gT}{2\pi}$

2 $L = CT = \frac{gT^2}{2\pi}$

3 $|u| = \left(\frac{\pi H}{T}\right)e^{2\pi y/L}$

4 $|v| = \left(\frac{\pi H}{T}\right)e^{2\pi y/L}$

5 $|\zeta| = \left(\frac{H}{2}\right)e^{2\pi y/L}$

6 $|\varepsilon| = \left(\frac{H}{2}\right)e^{2\pi y/L}$

* C and L depend on T, not water depth d
* Amplitudes decrease exponentially with depth (y<0)
* Wave base: y = -L/2 ($e^{-\pi}$ = 0.04)

IV Solution of the wave equation

A General solutions

1 $C = \left(\frac{gT}{2\pi}\right)\tanh\left(\frac{2\pi d}{L}\right)$

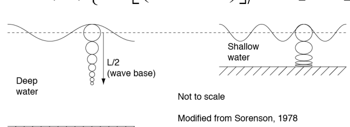
2 $L = CT$

3 $|u| = \left(\frac{\pi H}{T}\right)\left\{\cosh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$

4 $|v| = \left(\frac{\pi H}{T}\right)\left\{\sinh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$

5 $|\zeta| = \left(\frac{H}{2}\right)\left\{\cosh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$

6 $|\varepsilon| = \left(\frac{H}{2}\right)\left\{\sinh\left[\left(\frac{2\pi(d+y)}{L}\right)\right]\Big/\sinh\left[\frac{2\pi d}{L}\right]\right\}$



B Shallow-water solutions (d/L< 0.05)
As ω → 0, cosh(ω) → tanh(ω) → ω

1 $C = gd/C = \sqrt{gd}$

2 $L = CT = \sqrt{gd}T$

3 $|u| = \left(\frac{\pi H}{T}\right)\left(\frac{L}{2\pi d}\right) = \left(\frac{L}{T}\right)\left(\frac{\pi H}{2\pi d}\right) = C\left(\frac{H}{2d}\right) = C\frac{A}{d}$

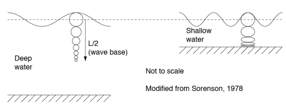
4 $|v| = \left(\frac{\pi H}{T}\right)\left(\frac{d+y}{d}\right)$

5 $|\zeta| = \left(\frac{H}{2}\right)\left(\frac{L}{2\pi d}\right)$

6 $|\varepsilon| = \left(\frac{H}{2}\right)\left(\frac{d+y}{d}\right)$

* C and L decrease as d decreases
* |v| and |ε| → 0 as y → -d
* |u| and |ζ| do not change with y

IV Solution of the wave equation



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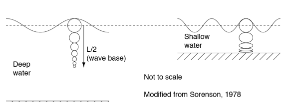
IV Solution of the wave equation



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V Energy in a wave

A Potential energy in excess of static situation per unit horizontal area

1 Express excess in terms of the potential energy density ρgy

$$E_P = \rho g \int_0^z \left[\int_{-d}^{\eta} \left(\int_0^L y\,dx \right) dy \right] dz - \rho g \int_0^z \left[\int_{-d}^{0} \left(\int_0^L y\,dx \right) dy \right] dz = \rho g \int_0^z \left[\int_{0}^{\eta} \left(\int_0^L y\,dx \right) dy \right] dz$$

2 Integrate the energy density ρgy over the height range 0→η, and then average that over a wavelength to find average excess potential energy per unit horizontal area

$$\overline{E_P} = \frac{1}{L}\int_0^L \left(\int_0^{\eta} \rho g y\,dy \right) dx = \rho g \frac{1}{L}\int_0^L \frac{\eta^2}{2} dx = \frac{\rho g}{2}\frac{1}{L}\int_0^L \eta^2\,dx = \frac{\rho g}{2}\overline{\eta^2}$$

$\overline{\eta^2}$ = mean squared displacement

3 Now express this in terms of wave amplitude A

$$\eta = A\cos(2\pi x/L)$$

$$\overline{E_P} = \frac{\rho g}{2}\frac{1}{L}\int_0^L \left(A\cos\frac{2\pi x}{L} \right)^2 dx = \frac{\rho g}{2}\frac{1}{L} A^2 \int_0^L \cos^2\left(\frac{2\pi x}{L} \right) dx = \frac{\rho g}{2}\frac{1}{L} A^2 \left(\frac{L}{2} \right) = \frac{\rho g}{4} A^2$$

V Energy in a wave

B Kinetic energy in excess of static situation per unit horizontal area

1 Express excess in terms of the kinetic energy density ρg velocity²/2

$$E_K = \int \frac{1}{2}\left(\rho\,\text{velocity}^2\,dV \right) = \frac{\rho}{2}\int_0^z \int_{-d}^0 \int_0^L \left(u^2 + v^2 \right) dx\,dy\,dz$$

2 Substitute expressions for u and v [see eqs. (3), (4), and (5)] and proceed as before by integrating vertically and then averaging horiztonally

$$\overline{E_K} = \frac{\rho}{2}\left[\frac{2\pi}{T\sinh(2\pi H/L)} \right]^2 \left[\frac{1}{L}\int_0^L \left(A\cos\frac{2\pi x}{L} \right)^2 \int_{-d}^L \left(\cosh\frac{2\pi(y+d)}{L} \right)^2 dy\,dx \right]$$
$$+ \frac{\rho}{2}\left[\frac{2\pi}{T\sinh(2\pi H/L)} \right]^2 \left[\frac{1}{L}\int_0^L \left(A\sin\frac{2\pi x}{L} \right)^2 \int_{-d}^L \left(\sinh\frac{2\pi(y+d)}{L} \right)^2 dy\,dx \right]$$

3 After considerable algebra (see Kundu (1990) for guidance)

$$\overline{E_K} = \frac{1}{2}\rho g \overline{\eta^2} = \frac{1}{4}\rho g A^2$$

V Energy in a wave

C Total energy in excess of static situation <u>per unit horizontal area</u>

1 Horizontally averaged excess potential energy and kinetic energy are equal

$$\overline{E_P} = \frac{1}{2}\rho g \overline{\eta^2} = \frac{1}{4}\rho g A^2$$

$$\overline{E_K} = \frac{1}{2}\rho g \overline{\eta^2} = \frac{1}{4}\rho g A^2$$

2 Total horizontally averaged excess energy is evenly split between kinetic and potential energy

$$\overline{E_{total}} = \overline{E_P} + \overline{E_K} = \rho g \overline{\eta^2} = \frac{1}{2}\rho g A^2$$

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VI A shoaling wave

A $\overline{E_{total(1)}} = \overline{E_{total(2)}}$

B $(B_1 L_1)(\rho g A_1{}^2)/2 = (B_2 L_2)(\rho g A_2{}^2)/2$

C $\frac{A_2}{A_1} = \left(\frac{L_1}{L_2}\right)^{1/2}\left(\frac{B_1}{B_2}\right)^{1/2}$

1 As L_2 decreases, A_2 increases

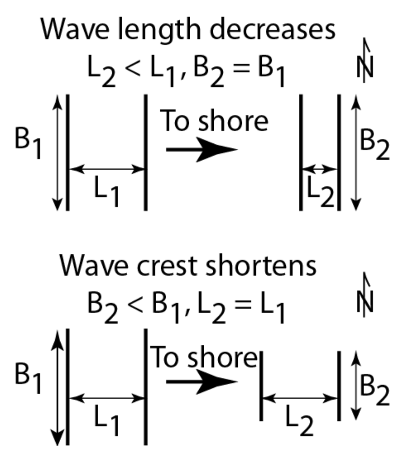
2 As B_2 decreases, A_2 increases

D Wave steepness = 2A/L = H/L

E Waves get taller and steeper as they shoal because $\overline{E_K}$ decreases and $\overline{E_P}$ increases as water depth decreases (conservation of energy)

F Rule of thumb: waves break where (H/L) = 1/7 tanh (2πd/L)

Map view of waves nearing shore



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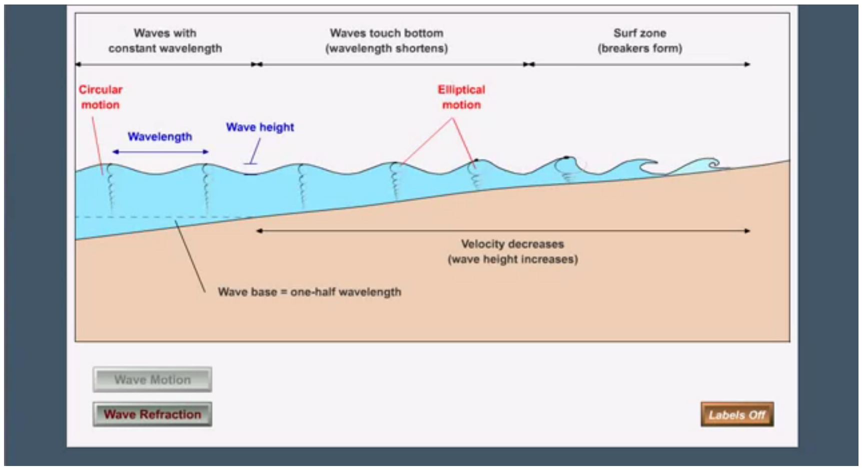
VI A shoaling wave



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Appendices

Hyperbolic Functions

$$\sinh(\beta)=\frac{e^{\beta}-e^{-\beta}}{2}=\beta+\frac{\beta^3}{3!}+\frac{\beta^5}{5!}+\ldots$$

$$\cosh(\beta)=\frac{e^{\beta}+e^{-\beta}}{2}=1+\frac{\beta^2}{2!}+\frac{\beta^4}{4!}+\ldots$$

$$\tanh(\beta)=\frac{\sinh(\beta)}{\cosh(\beta)}=\frac{e^{\beta}-e^{-\beta}}{e^{\beta}+e^{-\beta}}$$

$$\tanh(\beta)=\beta-\frac{\beta^3}{3}+\frac{2\beta^5}{15}+\ldots \quad for|\beta|<\frac{\pi}{2}$$

$$\frac{\cosh k(d+y)}{\sinh kd}=\frac{\dfrac{e^{k(d+y)}+e^{-k(d+y)}}{2}}{\dfrac{e^{kd}-e^{-kd}}{2}}=\frac{e^{k(d+y)}+e^{-k(d+y)}}{e^{kd}-e^{-kd}}=\frac{e^{kd}e^{ky}+e^{-kd}e^{-ky}}{e^{kd}-e^{-kd}}$$

$$\frac{\sinh k(d+y)}{\sinh kd}=\frac{\dfrac{e^{k(d+y)}-e^{-k(d+y)}}{2}}{\dfrac{e^{kd}-e^{-kd}}{2}}=\frac{e^{k(d+y)}-e^{-k(d+y)}}{e^{kd}-e^{-kd}}=\frac{e^{kd}e^{ky}-e^{-kd}e^{-ky}}{e^{kd}-e^{-kd}}$$

Shallow water and deep water approximations

Shallow water

$$As\,\beta\to 0,\ \sinh(\beta)\to\beta$$
$$As\,\beta\to 0,\ \cosh(\beta)\to 1$$
$$As\,\beta\to 0,\ \tanh(\beta)\to\beta$$
$$\sinh(0)=0$$
$$\cosh(0)=1$$
$$\tanh(0)=0$$

Deep water

$$As\,\beta\to\infty,\ \sinh(\beta)\to e^{\beta}/2$$
$$As\,\beta\to\infty,\ \cosh(\beta)\to e^{\beta}/2$$
$$As\,\beta\to\infty,\ \tanh(\beta)\to 1$$
$$\sinh(\pi)\approx 11.5487$$
$$\cosh(\pi)\approx 11.5920$$
$$\tanh(\pi)\approx 0.9963$$

Shallow water and deep water approximations

Shallow water

$$As\,\beta \to 0,\ \tanh(\beta) \to \beta$$

$$As\,\frac{2\pi d}{L} \to 0,\ \cosh\left(\frac{2\pi d}{L}\right) \to 1$$

$$As\,\frac{2\pi d}{L} \to 0,\ \sinh\left(\frac{2\pi d}{L}\right) \to \frac{2\pi d}{L}$$

$$As\,\frac{2\pi d}{L} \to 0,\ \frac{\cosh\left(\frac{2\pi(d+y)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)} \to \frac{1}{\frac{2\pi d}{L}}$$

$$As\,\frac{2\pi d}{L} \to 0,\ \sinh\left(\frac{2\pi(d+y)}{L}\right) \to \frac{2\pi(d+y)}{L}$$

$$As\,\frac{2\pi d}{L} \to 0,\ \frac{\sinh\left(\frac{2\pi(d+y)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)} \to \frac{d+y}{d}$$

Deep water

$$As\,\beta \to \infty,\ \tanh(\beta) \to 1$$

$$As\,\frac{2\pi d}{L} \to \pi,\ \frac{\cosh\left(\frac{2\pi(d+y)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)} \to e^{2\pi d/L}$$

$$As\,\frac{2\pi d}{L} \to 0,\ \frac{\sinh\left(\frac{2\pi(d+y)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)} \to e^{2\pi d/L}$$

$$\tanh(\pi) \approx 0.9963$$

References

- Kundu, P.K., 1990, Fluid mechanics: Academic Press, San Diego, California, 638 p.