STRESS AT A POINT I (11)

I Main Topics

- A Stress vector (traction) on a plane
- B Stress at a point
- C Principal stresses
- D Transformation of principal stresses to tractions in 2D

STRESS AT A POINT

http://hvo.wr.usgs.gov/kilauea/update/images.html

II Stress vector (traction) on a plane

A $\vec{\tau} = \lim_{A \to 0} \vec{F} / A$

B Traction vectors can be added as vectors

C A traction vector can be resolved into normal (τ_n) and shear (τ_s) components

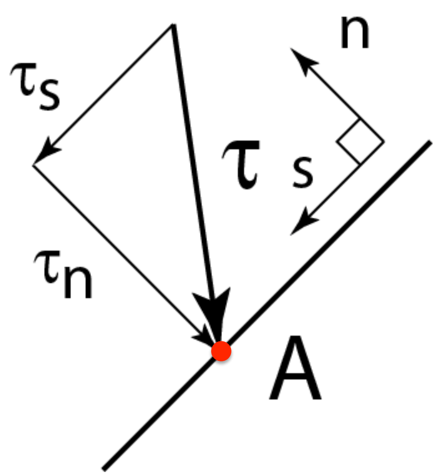
  1 A normal traction (τ_n) acts perpendicular to a plane

  2 A shear traction (τ_s) acts parallel to a plane

D Local reference frame

  1 The n-axis is normal to the plane

  2 The s-axis is parallel to the plane



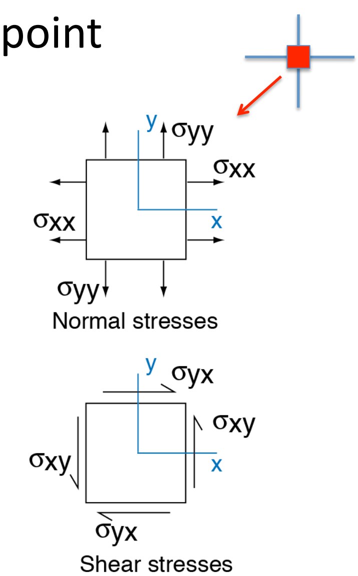
III Stress at a point

A Stresses refer to balanced internal "forces (per unit area)". They differ from force vectors, which, if unbalanced, cause accelerations

B "On -in convention": The stress component σ_{ij} acts on the plane normal to the i-direction and acts in the j-direction

  1 Normal stresses: i=j

  2 Shear stresses: i≠j

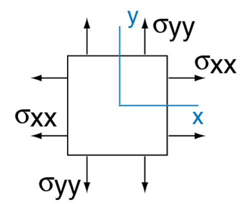


III Stress at a point

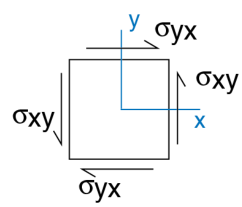
C Dimensions of stress: force/unit area

D Convention for stresses

1. Tension is positive
2. Compression is negative
3. Follows from on-in convention
4. Consistent with most mechanics books
5. Counter to most geology books



Normal stresses



Shear stresses

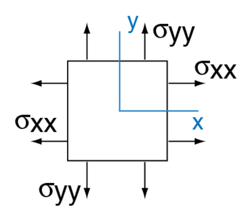
III Stress at a point

C $\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}$ 2-D 4 components

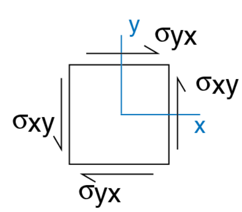
D $\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$ 3-D 9 components

E In nature, the state of stress can (and usually does) vary from point to point

F For rotational equilibrium, $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, $\sigma_{yz} = \sigma_{zy}$



Normal stresses



Shear stresses

IV Principal Stresses (these have magnitudes and orientations)

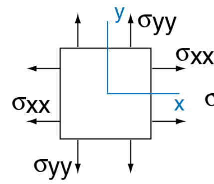
A Principal stresses act on planes which feel no shear stress

B The principal stresses are normal stresses.

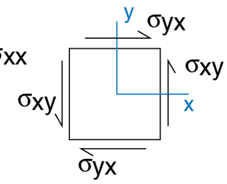
C Principal stresses act on perpendicular planes

D The maximum, intermediate, and minimum principal stresses are usually designated σ_1, σ_2, and σ_3, respectively.

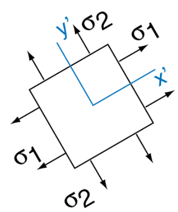
E Principal stresses have a single subscript.



Normal stresses



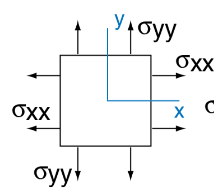
Shear stresses



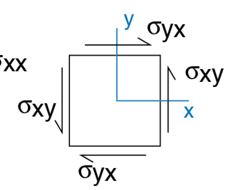
Principal stresses

IV Principal Stresses (these have magnitudes and orientations)

F <u>Principal stresses represent the stress state most simply</u>



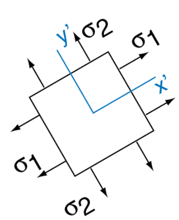
Normal stresses



Shear stresses

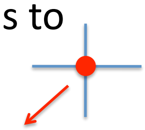
G $\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ 2-D 2 components

H $\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$ 3-D 3 components



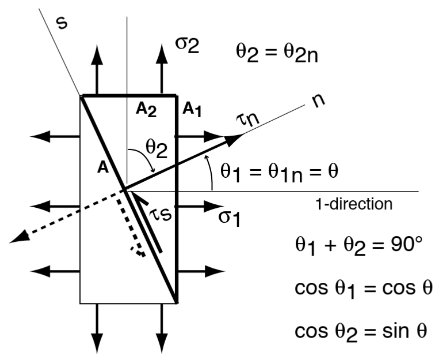
Principal stresses

V Transformation of principal stresses to tractions in 2D



A Description of terms

1. Three planes A, A_1, and A_2 form the sides of a triangular prism; these have normals in the n-, 1-, and -2-directions, respectively.
2. Plane A_1 is acted on by known normal stress σ_1.
3. Plane A_2 is acted on by known normal stress σ_2.
4. The n-direction is at angle θ_1 (=θ) with respect to the 1-direction, and at angle θ_2 with respect to the 2-direction.
5. The s-direction is at a counter-clockwise 90° angle relative to the n-direction (like y and x).



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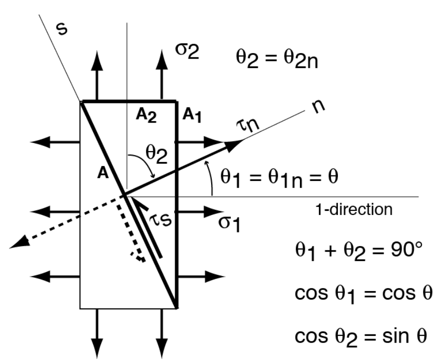
V Transformation of principal stresses to tractions in 2D

B Approach

Find weighting factors that determine contributions of known stresses to desired tractions and sum contributions

1. $\tau_n = w_{n1}\,\sigma_1 + w_{n2}\,\sigma_2$
2. $\tau_s = w_{s1}\,\sigma_1 + w_{s2}\,\sigma_2$



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V Transformation of principal stresses to tractions in 2D

C Contribution of σ_1 (on face A_1 of area A_1) to τ_n (on face A of area A)

Start with the definition of traction:

1 $\tau_n^{(1)} = F_n^{(1)}/A$

Find unknowns $F_n^{(1)}$ and A from knowns σ_1 and θ.

First find the force F_1 associated with σ_1

2 $F_1 = \sigma_1 A_1$ Force = (stress)(area)

Find $F_n^{(1)}$, the component of F_1 in the n-direction

3 $F_n^{(1)} = F_1 \cos\theta_1$

Find A in terms of A_1

$A_1 = A \cos\theta_1$ (see diagram at right)

4 $A = A_1/\cos\theta_1$

Contribution of σ_1 to τ_n:

5a $\tau_n^{(1)} = F_n^{(1)} / A = F_1 \cos\theta_1 / (A_1/\cos\theta_1)$

5b $\tau_n^{(1)} = (F_1 / A_1) \cos\theta_1 \cos\theta_1 = \sigma_1 \cos\theta_1 \cos\theta_1$

Weighting factor w_{n1}

6 $w_{n1} = \cos\theta_1 \cos\theta_1 = \cos\theta \cos\theta$

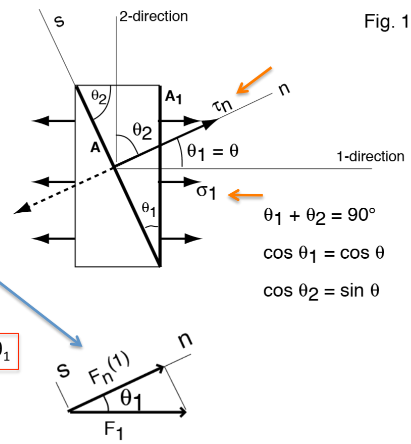


Fig. 16.1

V Transformation of principal stresses to tractions in 2D

D Contribution of σ_2 (on face A_2 of area A_2) to τ_n (on face A of area A)

Start with the definition of traction:

1 $\tau_n^{(2)} = F_n^{(2)}/A$

Find unknowns $F_n^{(2)}$ and A from knowns σ_2 and θ.

First find the force F_2 associated with σ_2

2 $F_2 = \sigma_2 A_2$ Force = (stress)(area)

Find $F_n^{(2)}$, the component of F_2 in the n-direction

3 $F_n^{(2)} = F_2 \cos\theta_2$

Find A in terms of A_2

$A_2 = A \cos\theta_2$ (see diagram at right)

4 $A = A_2/\cos\theta_2$

Contribution of σ_2 to τ_n:

5a $\tau_n^{(2)} = F_n^{(2)} / A = F_2 \cos\theta_2 / (A_2/\cos\theta_2)$

5b $\tau_n^{(2)} = (F_2 / A_2) \cos\theta_2 \cos\theta_2 = \sigma_2 \cos\theta_2 \cos\theta_2$

Weighting factor w_{n2}

6 $w_{n2} = \cos\theta_2 \cos\theta_2 = \sin\theta \sin\theta$

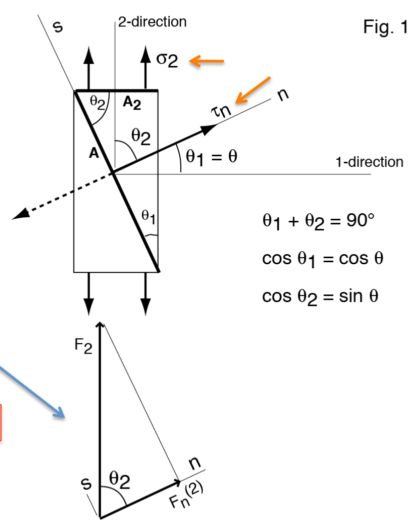
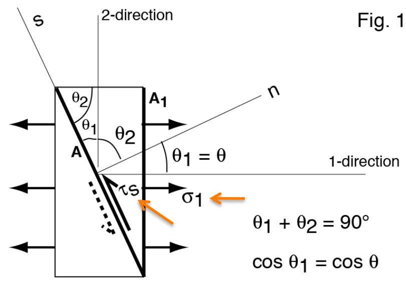


Fig. 16.2

V Transformation of principal stresses to tractions in 2D

E Contribution of σ_1 (on face A_1 of area A_1) to τ_s (on face A of area A)

Fig. 16.3



$\theta_1 + \theta_2 = 90°$

$\cos\theta_1 = \cos\theta$

$\cos\theta_2 = \sin\theta$

Start with the definition of traction:

1 $\tau_s^{(1)} = F_s^{(1)} / A$

Find unknowns $F_s^{(1)}$ and A from knowns σ_1 and θ.

First find the force F_1 associated with σ_1

2 $F_1 = \sigma_1 A_1$ Force = (stress)(area)

Find $F_s^{(1)}$, the component of F_1 in the s-direction

3 $F_s^{(1)} = -F_1 \cos\theta_2$

Find A in terms of A_2

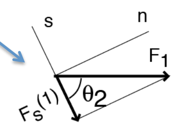
$A_1 = A \cos\theta_1$ (see diagram at right)

4 $A = A_1/\cos\theta_1$

Contribution of σ_1 to τ_n of σ_1:

5a $\tau_s^{(1)} = F_s^{(1)} / A = -F_1 \cos\theta_2 / (A_1/\cos\theta_1)$

5b $\tau_s^{(1)} = -(F_1 / A_1) \cos\theta_2 \cos\theta_1 = -\sigma_1 \cos\theta_2 \cos\theta_1$



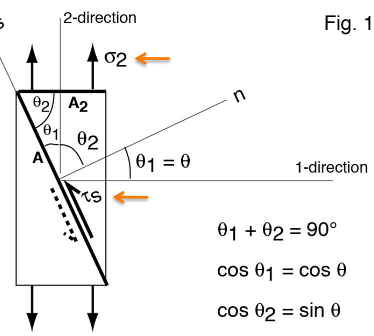
Weighting factor w_{s1}

6 $w_{s1} = -\cos\theta_2 \cos\theta_1 = -\sin\theta \cos\theta$

V Transformation of principal stresses to tractions in 2D

F Contribution of σ_2 (on face A_1 of area A_1) to τ_s (on face A of area A)

Fig. 16.4



$\theta_1 + \theta_2 = 90°$

$\cos\theta_1 = \cos\theta$

$\cos\theta_2 = \sin\theta$

Start with the definition of traction:

1 $\tau_s^{(2)} = F_s^{(2)} / A$

Find unknowns $F_s^{2)}$ and A from knowns σ_2 and θ.

First find the force F_2 associated with σ_2

2 $F_2 = \sigma_2 A_2$ Force = (stress)(area)

Find $F_s^{(2)}$, the component of F_2 in the s-direction

3 $F_s^{(2)} = F_2 \cos\theta_1$

Find A in terms of A_1

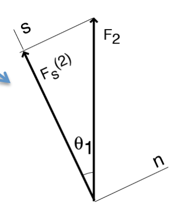
$A_1 = A \cos\theta_1$ (see diagram at right)

4 $A = A_2/\cos\theta_2$

Contribution of σ_1 to τ_n of σ_1:

5a $\tau_s^{(2)} = F_s^{(2)} / A = F_2 \cos\theta_1 / (A_2/\cos\theta_2)$

5b $\tau_s^{(2)} = (F_2 / A_2) \cos\theta_1 \cos\theta_2 = \sigma_2 \cos\theta_1 \cos\theta_2$



Weighting factor w_{s2}

6 $w_{s2} = \cos\theta_1 \cos\theta_2 = \cos\theta \sin\theta$

V Transformation of principal stresses to tractions in 2D

G Original equations

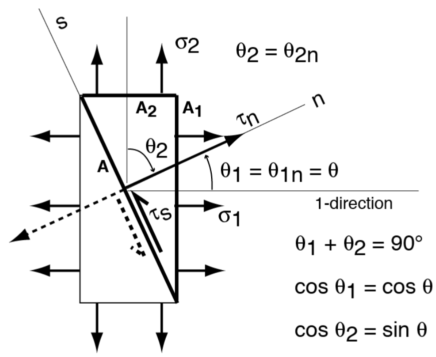
1 $\tau_n = w_{n1}\, \sigma_1 + w_{n1}\, \sigma_2$

2 $\tau_s = w_{s1}\, \sigma_1 + w_{s1}\, \sigma_2$

H Revised equations

1 $\tau_n = \cos\theta\cos\theta\, \sigma_1$
$+ \sin\theta\sin\theta\, \sigma_2$

2 $\tau_s = -\sin\theta\cos\theta\, \sigma_1$
$+ \sin\theta\cos\theta\, \sigma_2$



Weighting factors are products of two direction cosines