THE WAVE EQUATION (30)

I Main Topics

A Assumptions and boundary conditions used in 2-D small wave theory

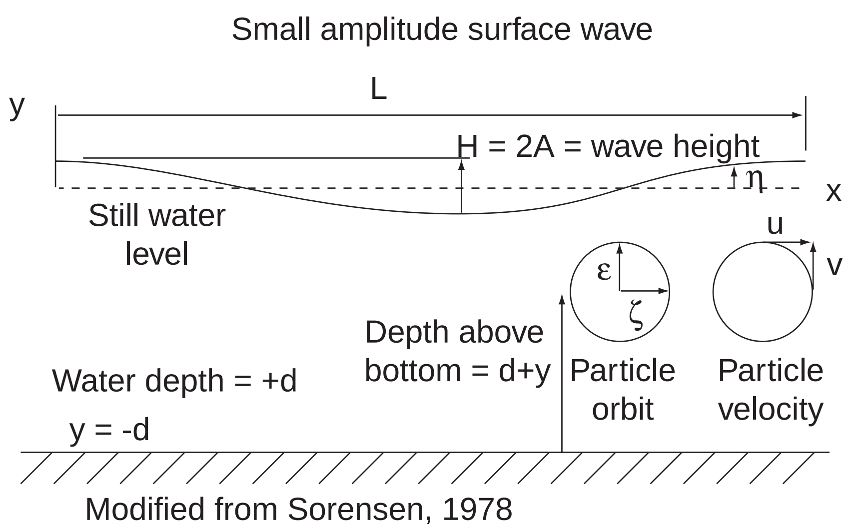
B The Laplace equation and fluid potential

C Solution of the wave equation

D Energy in a wavelength

E Shoaling of waves

II Assumptions and boundary conditions used in 2-D small wave theory



A No geometry changes parallel to wave crest (2-D assumption)

B Wave amplitude is small relative to wave length and water depth; it will follow that particle velocities are small relative to wave speed

C Water is homogeneous, incompressible, and surface tension is nil.

D The bottom is not moving, is impermeable, and is horizontal

E Pressure along air-sea interface is constant

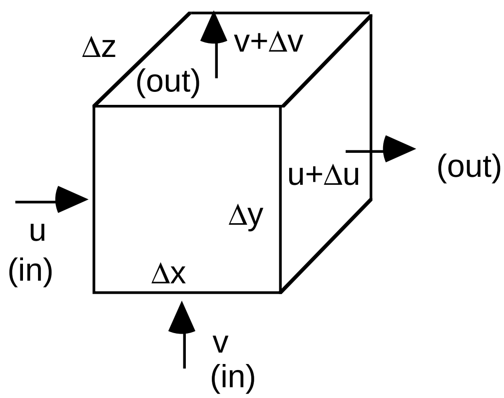
F The water surface has the form of a cosine wave

$$\eta = \frac{H}{2}\cos 2\pi\left(\frac{x}{L} - \frac{t}{T}\right)$$

III The Laplace equation and fluid potential

A Conservation of mass

Consider a box the shape of a cube, with fluid flowing in and out or it.



The mass flow rate **in** the left side of the box is:

1a $$\frac{\Delta m_1}{\Delta t} = \frac{\Delta \rho V}{\Delta t} = \rho \frac{\Delta V}{\Delta t} = \rho \frac{\Delta x \Delta y \Delta z}{\Delta t} = \rho \Delta y \Delta z \frac{\Delta x}{\Delta t} = (\rho)(\Delta y \Delta z)(u)$$

The mass flow rate **out** the right side is

1b $$\frac{\Delta m_2}{\Delta t} = (\rho)(\Delta y \Delta z)(u + \Delta u).$$

For the bottom of the box, mass flow rate **in** is

1c $$\frac{\Delta m_3}{\Delta t} = (\rho)(\Delta x \Delta z)(v),$$

and the mass flow rate **out** the top is

1d $$\frac{\Delta m_4}{\Delta t} = (\rho)(\Delta x \Delta z)(v + \Delta v)$$

If the fluid is incompressible (so no fluid can be compressed and stored in the box), then: a) the fluid mass flowing into the box in a given increment of time must equal the fluid mass flowing out of the box in that same increment of time, and b) the density of the fluid (ρ) is a constant. Remember that u and v are velocities in the x and y directions, respectively. So, **what goes in equals what comes out**:

2a $$\frac{\Delta m_1}{\Delta t} + \frac{\Delta m_3}{\Delta t} = \frac{\Delta m_2}{\Delta t} + \frac{\Delta m_4}{\Delta t}.$$

or

2b $$\frac{\Delta m_1}{\Delta t} - \frac{\Delta m_2}{\Delta t} + \frac{\Delta m_3}{\Delta t} - \frac{\Delta m_4}{\Delta t} = 0.$$

Substituting equations (1) into (2b)

3 $$(\rho)(\Delta y\Delta z)(u)-(\rho)(\Delta y\Delta z)(u+\Delta u)+(\rho)(\Delta x\Delta z)(v)-(\rho)(\Delta x\Delta z)(v+\Delta v)=0.$$

Dividing out the terms ρ and Δz

4 $$(\Delta y)(u)-(\Delta y)(u+\Delta u)+(\Delta x)(v)-(\Delta x)(v+\Delta v)=0.$$

Now let Δu = Δx [∂u/∂x] and Δv = Δy [∂v/∂y]

5 $$(\Delta y)(u)-(\Delta y)(u+\Delta x[\partial u/\partial x])+(\Delta x)(v)-(\Delta x)(v+\Delta y[\partial v/\partial y])=0.$$

This simplifies to

6 $$-(\Delta y)(\Delta x\frac{\partial u}{\partial x})-(\Delta x)(\Delta y\frac{\partial v}{\partial y})=0.$$

After dividing out the Δx and Δy terms

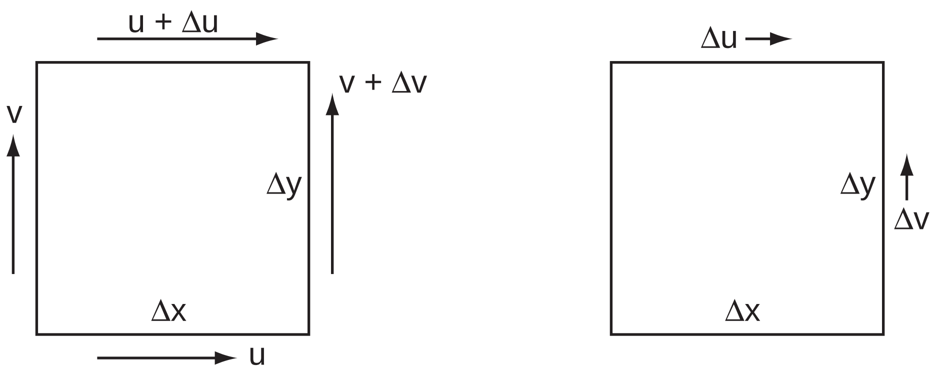
7a $-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0$, or 7b $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$, or 7c $\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y}$.

Equation 7c states that a change in flow in the x direction must be compensated for by an opposite change in flow in the y-direction if mass is to be conserved.

B Conservation of angular momentum

[Conditions of irrotational flow (vorticity = 0)]

We don't want our box to rotate. Experiments with submerged floats beneath waves show that the floats do not spin.



Flow velocities on box sides

If there is no rotation, then there can be no moment:

8 $$(Force)_y(lever \quad arm)_y = (Force)_x(lever \quad arm)_x$$

The shear force is related to the shear stress (τ) as follows:

9 $$F_{shear} = (\tau)(area)$$

Substituting equation (9) into equation (8) yields.

10 $$\left(\tau_{xy}[\Delta y \Delta z]\right)(\Delta x) = \left(\tau_{yx}[\Delta x \Delta z]\right)(\Delta y)$$

For linear fluids, the shear stress is proportional to the velocity gradient

11a $\tau_{xy} = \mu \frac{du}{dy}$ 11b $\tau_{yx} = \mu \frac{dv}{dx}$ (μ = viscosity)

Substituting equations (11) into (10) then gives:

12 $$\left(\mu \frac{\partial u}{\partial y}[\Delta y \Delta z]\right)(\Delta x) = \left(\mu \frac{\partial v}{\partial x}[\Delta x \Delta z]\right)(\Delta y)$$

Hence

13a $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ or 13b $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$ (vorticity = 0)

C Irrotational potential flow and the Laplace equation

The Laplace equation is one of the most common equations in physics. It describes how the second partial derivatives of a function (ϕ) are related:

14a $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$ or 14b $$\nabla^2 \phi = 0.$$

In the context of waves, ϕ will represent a fluid potential function. You are already familiar with gravitational potential energy U:

15 $$U = mgy$$ y = height

<u>Partial derivatives</u> of potential functions, taken with respect to position, yield measurable physical quantities (e.g., gravitational potential U):

16 $$-\frac{\partial U}{\partial y} = -mg = F_y$$ F_y= gravitational force (Note that ∇^2U = 0).

Darcy's law for one-dimensional fluid flow can be written in terms of fluid potential ϕ:

17 $$q_x = -k_x \left(\frac{\rho}{\mu}\right) \frac{\partial \phi}{\partial x}$$

where q_x = flux (m/s), k_x = permeability in the x-direction (m^2), ρ = density (kg/m^3), μ = fluid viscosity (kg m^{-1} s^{-1}), and x = position (m).

Suppose that a potential function ϕ exists that satisfies the Laplace equation, and that the following conditions apply:

18a $$\frac{\partial \phi}{\partial x} = u,$$ where u =horizontal component of particle velocity

18b $$\frac{\partial \phi}{\partial y} = v,$$ where v =vertical component of particle velocity.

Now let us substitute the expressions of equations (18) into the Laplace equation (equation 14a); this yields the continuity equation (7b).

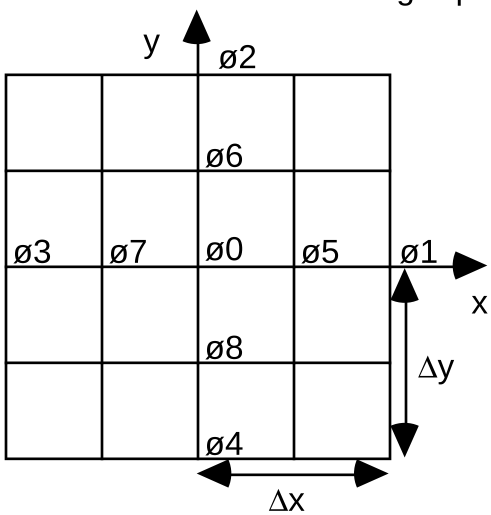
19 $$\frac{\partial\left(\frac{\partial \phi}{\partial x}\right)}{\partial x} + \frac{\partial\left(\frac{\partial \phi}{\partial y}\right)}{\partial y} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Does the vorticity condition hold? Inserting eqs. (18) into eq. (13b):

20 $$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=\frac{\partial\left(\frac{\partial\phi}{\partial x}\right)}{\partial y}-\frac{\partial\left(\frac{\partial\phi}{\partial y}\right)}{\partial x}=\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}=0$$ So vorticity = 0.

So the Laplace equation can be used to study water waves.

Solutions to the Laplace equation also obey an averaging procedure, where the value of the function at a point on a square grid is the average of the values at the nearest four gridpoints.



$\phi_0 = 1/4\ [\ \phi_1 + \phi_2 + \phi_3 + \phi_4]$

IV Solutions for wave speed and particle velocities
(see appendix for derivation)

A General solutions

$$\phi = \frac{H}{2}\frac{g \cosh([2\pi / L][d+y] \sin(2\pi x / L - 2\pi t / T)}{(2\pi / T)\cosh(2\pi d / L)}$$

1 Wave speed or wave celerity (C)
$C = (g\mathbf{T}/2\pi) \tanh(2\pi \mathbf{d/L})$ **T = wave period = constant**
Function of wave period and relative water depth

2 Wave length (L)
$L = \mathbf{C}T$

3 Horizontal particle velocity amplitude (|u|)
$|u| = (\pi \mathbf{H/T})$ (cosh $[2\pi(\mathbf{d+y})/\mathbf{L}]$)/(sinh $[2\pi \mathbf{d/L}]$)
Function of wave height and relative water depth and relative distance above bottom

4 Vertical particle velocity amplitude (|v|)
$|v| = (\pi \mathbf{H/T})$ (sinh $[2\pi(\mathbf{d+y})/\mathbf{L}]$)/(sinh $[2\pi \mathbf{d/L}]$)

5 Amplitude of horizontal particle displacement (|ζ|)
$|\zeta| = |u| (T/2\pi)$

6 Amplitude of vertical particle displacement (|ε|)
$|\varepsilon| = |v| (T/2\pi)$

B Deep-water solutions (d/L > 0.5, or 2πd/L > π, so tanh(2πd/L) ≈ 1)

1 Wave speed (C)
a $C = g\mathbf{T}/2\pi$ (function of wave period; independent of depth)

2 Wave length (L)
a $L = \mathbf{C}T$
b $L = (g\mathbf{T}^2)/2\pi$ (function of wave period; independent of depth)

3 Horizontal particle velocity amplitude (|u|)
$|u| = (\pi H/T)(\mathbf{e}^{\mathbf{2}\pi \mathbf{y/L}})$.
The horizontal velocities decrease exponentially with depth (y<0)

4 Vertical particle velocity amplitude (|v|)
$|v| = (\pi H/T)(\mathbf{e}^{\mathbf{2}\pi \mathbf{y/L}})$.
The vertical velocities decrease exponentially with depth (y<0)

5 Amplitude of horizontal particle displacement (|ζ|)
$|\zeta| = (\mathbf{H}/2)(\mathbf{e}^{\mathbf{2}\pi \mathbf{y/L}})$. Exponential decay with depth

6 Amplitude of vertical particle displacement (|ε|)
$|\varepsilon| = (\mathbf{H}/2)(\mathbf{e}^{\mathbf{2}\pi \mathbf{y/L}})$. Exponential decay with depth

C Shallow-water solutions ($d/L < 0.05$, or $2\pi d/L < \pi/10$)
Examples: tides, tsunamis,

1 Wave speed (C) [$\tanh(2\pi d/L) \approx 2\pi d/L$]

$C = (gT)(d/L) = (gd)(T/L) = (gd)(1/C) \Rightarrow \mathbf{C = (gd)^{1/2}}$

  i Function of water depth; independent of period

  ii As water depth decreases, wave speed decreases

2 Wave length (L= **C**T)

$L = (gd)^{1/2}T$

  i As water depth decreases, T stays constant, L decreases

  ii Waves will bunch together as they enter shallower water

3 Horizontal particle velocity amplitude (|u|)

$|u| = (\pi H/T)/(\mathbf{2\pi d/L}) = HL/2dT = (L/T)(H/2d) = \mathbf{C\ (H/2d)}$

|u| is independent of distance above bottom; $|u| \neq 0$ at bottom

4 Vertical particle velocity amplitude (|v|)

$|v| = (\pi H/T)\ [\mathbf{(d + y)/d}]$

$|v| = 0$ at bottom and increases linearly to $\pi H/T$ at surface ($y=0$)

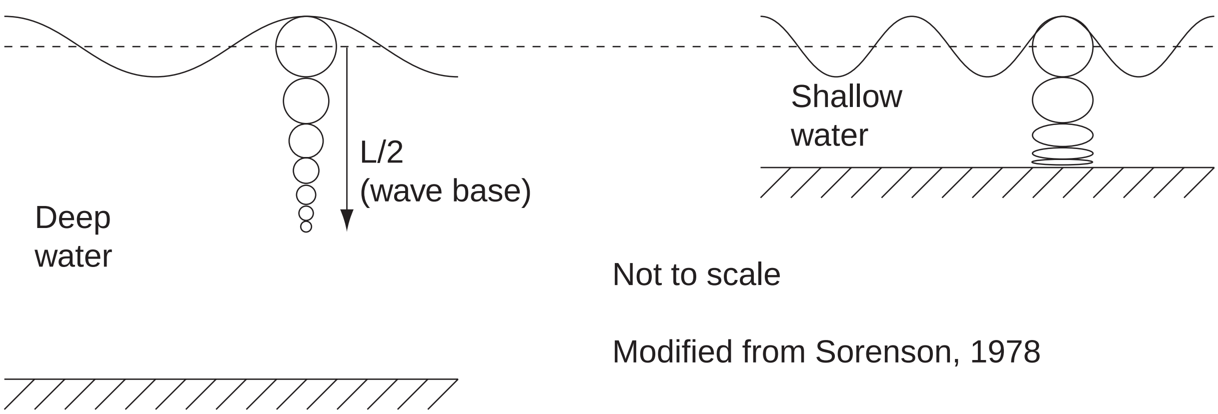
5 Amplitude of horizontal particle displacement ($|\zeta|$)

$|\zeta| = (H/2)/(\mathbf{2\pi d/L}) = \mathbf{(L/\pi)(H/d)}$ Independent of y

6 Amplitude of vertical particle displacement ($|\varepsilon|$)

$|\varepsilon| = (H/2)\ [\mathbf{(d + y)/d}]$ Decays linearly with y

7 Wave base: $y = -L/2$ ($e^{-\pi} = 0.04$)



Not to scale

Modified from Sorenson, 1978

V Energy in a wavelength (per unit length along wave crest)

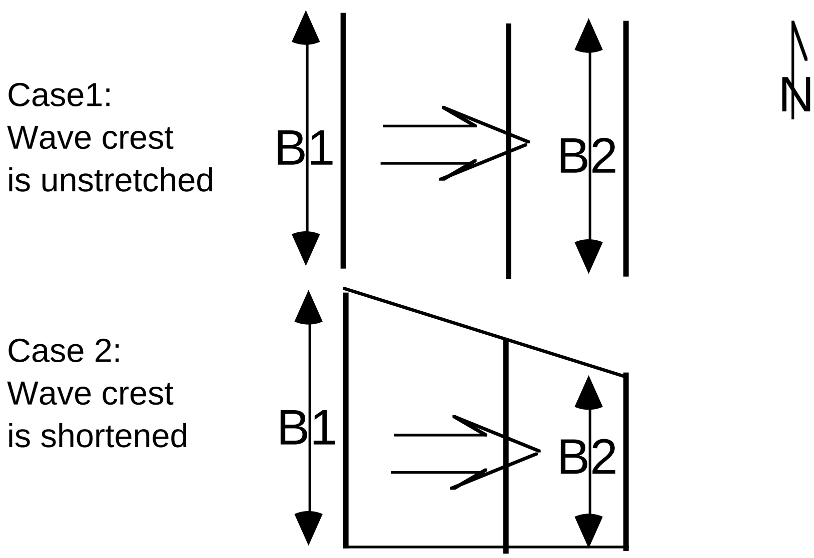
A Kinetic energy = $(E_k)/z = \rho g H^2 L/16$

B Potential energy = $(E_p)/z = \rho g H^2 L/16$ So $E_k = E_p$!

C Total energy = $(E_T)/z = (E_k+E_p)/z = \rho g H^2 L/8$

VI Shoaling of waves

<u>Assuming no energy loss as a wave shoals</u>*



A $E_1 = E_2$

B $(B_1 \rho g H_1^2 L_1)/8 = (B_2 \rho g H_2^2 L_2)/8$

C $(H_2/H_1) = (L_1/L_2)^{1/2} (B_1/B_2)^{1/2}$

1 <u>As L decreases, H increases</u>

2 <u>As B decreases, H increases</u>

D Wave steepness = H/L

E Waves get taller and steeper as they shoal because:
L decreases and H increases (conservation of energy)

F Waves break when $(H/L) = 1/7 \tanh(2\pi d/L)$

Hyperbolic Functions

$$\sinh(\beta) = \frac{e^{\beta} - e^{-\beta}}{2} = \beta + \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \ldots$$

Shallow water: As ß → 0, sinh (ß) → ß (from series expansion)

Deep water: As ß → ∞, sinh (ß) → $(e^{ß})/2$ (from definition)

sinh (π) = 11.549

$$\cosh(\beta) = \frac{e^{\beta} + e^{-\beta}}{2} = 1 + \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \ldots$$

Shallow water: As ß → 0, cosh (ß) → 1 (from definition)

Deep water: As ß → ∞, cosh (ß) → $(e^{ß})/2$ (from definition)

cosh (π) = 11.592

$$\tanh(\beta) = \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}} = \beta - \frac{\beta^3}{3} + \frac{2\beta^5}{15} + \ldots \qquad \textit{for } |\beta| < \frac{\pi}{2}$$

Shallow water: As ß → 0, tanh (ß) → ß (from series expansion)

Deep water: As ß → ∞, tanh (ß) → 1 (from definition)

tanh (π) = 0.9963

In the expressions below, k = 2π/L

$$\frac{\cosh k(d+y)}{\sinh kd} = \frac{\dfrac{e^{k(d+y)} + e^{-k(d+y)}}{2}}{\dfrac{e^{kd} - e^{-kd}}{2}} = \frac{e^{kd}e^{ky} + e^{-kd}e^{-ky}}{e^{kd} - e^{-kd}}$$

Shallow water: As kd→0, cosh k(d+y)→1, sinh kd→kd, so...

[cosh k(d+y)]/sinh kd → 1/kd

Deep water: As kd→π , [cosh k(d+y)]/sinh kd → e^{ky}

$$\frac{\sinh k(d+y)}{\sinh kd} = \frac{\dfrac{e^{k(d+y)} - e^{-k(d+y)}}{2}}{\dfrac{e^{kd} - e^{-kd}}{2}} = \frac{e^{kd}e^{ky} + e^{-kd}e^{-ky}}{e^{kd} - e^{-kd}}$$

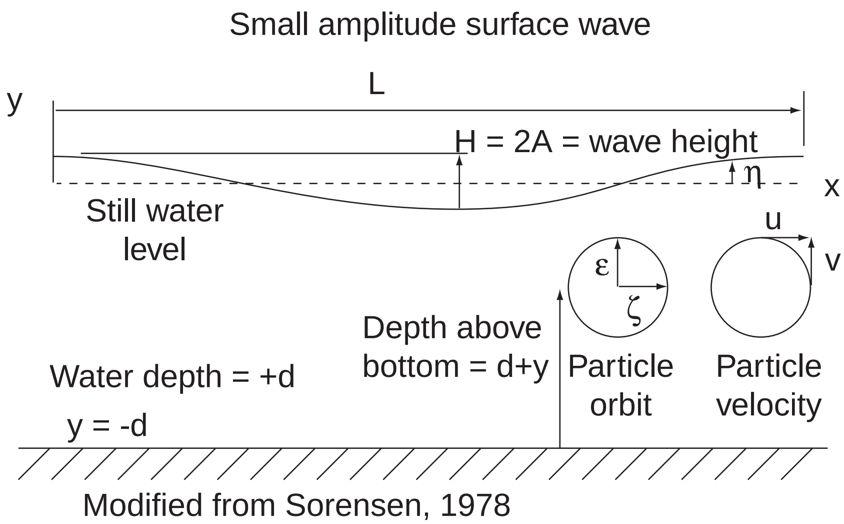
Shallow water: As kd→0, sinh k(d+y)→k(d+y), sinh kd→kd, so...

[sinh k(d+y)]/sinh kd → (d + y)/d = height above bottom/depth

Deep water: As kd→π , [sinh k(d+y)]/sinh kd → e^{ky}

Appendix

Derivation of the small amplitude wave equation
(from Sorenson, R.M., 1978, Basic coastal engineering: Wiley, New York, 227 p.)



The original solution is attributed to Airy (Airy, C.B., 1845, On tides and waves, in Encyclopedia Metropolitana, London, p. 241-396).

$$\eta = \frac{H}{2}\cos 2\pi\left(\frac{x}{L} - \frac{t}{T}\right) \tag{30A.1}$$

or

$$\eta = \frac{H}{2}\cos 2\pi(kx - \sigma t) \tag{30A.2}$$

where

$$k = \frac{2\pi}{L} \; (wave \;\; number) \tag{30A.3}$$

$$\sigma = \frac{2\pi}{T} \; (wave \;\; angular \;\; frequency) \tag{30A.4}$$

The flow normal to the sea bed is zero, so

$$v = \frac{\partial\phi}{\partial y} = 0 \;\; at \;\; y = -d \tag{30A.5}$$

This is the first boundary condition.

The *unsteady* Bernoulli equation for irrotational flow is

$$\frac{1}{2}\left(u^2 + v^2\right) + gy + \frac{p}{\rho} + \frac{\partial\phi}{\partial t} = 0 \tag{30A.6}$$

where g = gravitational acceleration, p is the pressure, ρ is fluid density, and the last term is a dynamic pressure term associated with accelerations. If the squares of the velocity terms are assumed to be small relative to the other terms, and if the particle velocities are small relative to the wave speed, then at the surface (i.e., at y = η), where the pressure is taken as zero, the unsteady Bernoulli equation yields

$$y = \frac{-1}{g}\frac{\partial\phi}{\partial t} \quad at \quad y = \eta \tag{30A.7}$$

This yields the second boundary condition is at y=0,

$$\eta = \frac{-1}{g}\frac{\partial\phi}{\partial t} \quad at \quad y = 0 \tag{30A.8}$$

The velocity potential should vary with depth, and should have the same cycle as the wave. If the depth contribution (Y) can be separated from the cyclic contribution (a common assumption in solving differential equations), then the velocity potential ϕ would have the following form:

$$\phi = Y\sin(kx - \sigma t) \tag{30A.9}$$

where Y = Y(y). Upon insertion of this function into the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{30A.10}$$

one obtains

$$\frac{\partial^2(Y\sin(kx-\sigma t))}{\partial x^2} + \frac{\partial^2(Y\sin(kx-\sigma t))}{\partial y^2} = 0 \tag{30A.11}$$

This simplifies first to

$$Y\frac{\partial^2(\sin(kx-\sigma t))}{\partial x^2} + \sin(kx-\sigma t)\frac{\partial^2 Y}{\partial y^2} = 0 \tag{30A.12}$$

and then to

$$-k^2 Y\sin(kx-\sigma t) + \sin(kx-\sigma t)\frac{\partial^2 Y}{\partial y^2} = -k^2 Y + \frac{\partial^2 Y}{\partial y^2} = 0 \tag{30A.13}$$

The solution to this differential equation is well known

$$Y = Ae^{ky} + Be^{-ky} \tag{30A.14}$$

and can be verified by substitution into (30A.13). Substituting this into (30A.9) yields a general solution of the Laplace equation .

$$\phi = \left(Ae^{ky} + Be^{-ky}\right)\sin(kx-\sigma t) \tag{30A.15}$$

The two constants A and B now need to be solved for using the two boundary conditions (30A.5) and (30A.8). Inserting (30A.15) into (30A.5)

$$v = \frac{\partial\left\{\left(Ae^{ky} + Be^{-ky}\right)\sin(kx-\sigma t)\right.}{\partial y} = 0 \;\; at \;\; y=-d \qquad (30A.16)$$

or

$$v = \frac{k\left\{\left(Ae^{-kd} - Be^{kd}\right)\sin(kx-\sigma t)\right\}}{\partial y} = 0 \qquad (30A.17)$$

The only way this can hold for all values of x and t is if

$$Ae^{-kd} - Be^{kd} = 0 \qquad (30A.18)$$

or

$$A = B\frac{e^{kd}}{e^{-kd}} \qquad (30A.19)$$

Inserting this back into (30A.15) yields ϕ with one unknown constant

$$\phi = \left(B\frac{e^{kd}}{e^{-kd}}e^{ky} + Be^{-ky}\right)\sin(kx-\sigma t) = Be^{kd}\left(\frac{e^{ky}}{e^{-kd}} + \frac{e^{-ky}}{e^{kd}}\right)\sin(kx-\sigma t) \qquad (30A.20)$$

or

$$\phi = Be^{kd}\left(e^{k(y+d)} + e^{-k(y+d)}\right)\sin(kx-\sigma t) \qquad (30A.21)$$

The term in the large parentheses equals 2cosh[k(d+y)], so

$$\phi = Be^{kd}(2\cosh k(y+d))\sin(kx-\sigma t) \qquad (30A.22)$$

Now the form of the water surface is

$$\eta = \frac{H}{2}\cos 2\pi\left(\frac{x}{L} - \frac{t}{T}\right) \qquad (30A.1)$$

So at t=0, x=0

$$\eta = \frac{H}{2} \qquad (30A.23)$$

Now the second boundary condition comes into play.

$$\eta = \frac{-1}{g}\frac{\partial\phi}{\partial t} \;\; at \;\; y=0 \qquad (30A.8)$$

So at t=0, x=0, and y=0

$$\frac{H}{2} = \frac{-1}{g}\frac{\partial\phi}{\partial t} \qquad (30A.24)$$

The derivative on the right side of (30A.24) is found from (30A.22)

$$\frac{\partial\phi}{\partial t} = Be^{kd}(2\cosh(kd))\cos(kx-\sigma t)(-\sigma) \qquad (30A.25)$$

Substituting this back into (30A.24)

$$\frac{H}{2} = \frac{-1}{g}Be^{kd}(2\cosh(kd))\cos(kx-\sigma t)(-\sigma) \qquad (30.A26)$$

This is solved for Be^{kd} most readily where $\cos(kx-\sigma t)=1$:

$$\frac{gH}{2\sigma\left(2\cosh(kd)\right)} = Be^{kd} \tag{30.A27}$$

This goes into (30A.22) to yield the expression for the velocity potential

$$\phi = \frac{gH}{2\cosh[kd]}\left(\cosh k(y+d)\right)\sin\left(kx-\sigma t\right) \tag{30A.28}$$

The wave speed (or celerity) is a key term we wish to find. We find it by evaluating the vertical velocity at the surface in two ways. First, using the expression for the water height above still water level

$$\eta = \frac{-1}{g}\frac{\partial\phi}{\partial t} \quad at \quad y=0 \tag{30A.8}$$

we obtain

$$v = \frac{\partial\eta}{\partial t} = \frac{-1}{g}\frac{\partial^2\phi}{\partial t^2} \quad at \quad y=0 \tag{30A.29}$$

Also the vertical velocity is given by

$$v = \frac{\partial\phi}{\partial y} \tag{30A.30}$$

So at the surface

$$\frac{-1}{g}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial\phi}{\partial y} \quad at \quad y=0 \tag{30A.31}$$

Inserting (30A.28) , gives

$$\frac{-1}{g}\frac{\partial^2 \frac{gH}{2\cosh[kd]}\left(\cosh k(y+d)\right)\sin\left(kx-\sigma t\right)}{\partial t^2} = \frac{\partial \frac{gH}{2\cosh[kd]}\left(\cosh k(y+d)\right)\sin\left(kx-\sigma t\right)}{\partial y} \quad at \quad y=0 \tag{30A.32}$$

Taking the derivatives yields

$$\frac{\sigma^2}{g}\frac{gH}{2\cosh[kd]}\left(\cosh k(y+d)\right)\sin\left(kx-\sigma t\right) = k\frac{gH}{2\cosh[kd]}\left(\sinh k(y+d)\right)\sin\left(kx-\sigma t\right) \quad at \quad y=0 \tag{30A.33}$$

Now set y= 0, and solve for ϕ. Many terms can be dropped from both sides.

$$\sigma^2 = -g\frac{k\sinh(kd)}{\cosh k(d)} \quad = gk\tanh(kd) \tag{30A.34}$$

Now the wave speed C = L/T = σ/k, so

$$C = \frac{\sqrt{gk\tanh(kd)}}{k} = \sqrt{\frac{g}{k}\tanh(kd)} = \sqrt{\frac{gL}{2\pi}\tanh\frac{2\pi d}{L}} \tag{30A.34}$$