STANDARDS OF ACCURACY

A. ROUNDED NUMBERS

In calculations, measured numbers are usually rounded off to the degree of accuracy required by the problem. The number of people in a room, some numerical constants, conversion factors established by law are called exact numbers. Exact numbers have unlimited accuracy and may be taken to any degree of accuracy required by the nature of the problem. However, measured numbers cannot be any more accurate than the instruments or the care taken by the persons who measured them. Thus measured numbers, and results obtained by using measured numbers must be rounded off to the degree of accuracy that will most honestly report the answer. If we round off 66.6253 to three decimal places, it becomes 66.625; to two decimals 66.63; to one decimal 66.6; to no decimal places, it becomes 67; to the nearest ten, 70. This same procedure is used whether rounding off to the left or the right of the decimal point. When the last digit to be rounded off is greater than 5, the next digit is taken to the next higher number. 7.651 rounded to one decimal place is 7.7; 7.65001 to one decimal place is still 7.7. When the last digit to be rounded off is less than 5, the last digit is dropped and no change is made in the next digit. 7.649 rounded to one decimal place is 7.6.

When the digit to be rounded off is exactly 5 such as 7.65000 and the 5 is to be dropped by rounding to one decimal place, then the next digit is either kept or raised to one higher depending on which is the more convenient. That is, to divide the rounded number by 2, for example, it would be more convenient to merely drop the 5 so that the division is $\frac{7.6}{2}$ rather than $\frac{7.7}{2}$. If the intention is to divide the rounded number by 11, then raising the last digit would be more convenient, so that the division is $\frac{7.7}{11}$. When the digit to be dropped is exactly 5, there can be no arbitrary rule for rounding off. It actually makes no difference. Convenience here should be the governing factor.

B. SIGNIFICANT FIGURES

There is a difference between the meanings of decimal places and significant figures. Heretofore, we have been using the term, decimal places. That is, the zeroes and numbers to the right of the decimal point. Significant figures refer to all numbers both to the right and to the left of the decimal point which establish the true picture of the accuracy of a number resulting from measurement. The number of significant figures depends primarily on the accuracy of the measuring instrument, ability of the observer, and the purpose for which the figures are to be used. If we judge the length of a fast moving automobile, we might estimate it to be 10 ft. long. To determine whether or not it will fit in our garage, we might measure it with a tape measure and find it to be 10.6 ft. long. An engineer with more accurate measuring instruments might wish to use his figures to build tools with which he can construct the automobile and measures it to be 10.647 ft. long.

10, 10.6, 10.647 indicate an increasing degree of accuracy and are dependent on the instruments used, and the care of the observer using them. If the digits in a number, (a) have a definite purpose; (b) are a result of a definite measurement; (c) have a definite meaning, the digits are called significant figures. The measurements .002, .02, .2, 2, 20 and 200 have one significant figure. The measurements .0020, .020, .0022, 2.0, 2.2, 22, 220 and 20. have two significant figures. The measurements .00200, .00220, 0200, 2.00, 2.20, 220., 222 have three significant figures.

Many times there is a misunderstanding as to why numbers such as .0002 is significant to only one place and not to four places where the zeroes are considered to be significant. Suppose we make a measurement of 2 mm. There is no doubt that this is significant to only one place. Now let us determine the value of 2 mm in terms of cm, the 2 mm = .2 cm, still significant to only one place. Let us determine the value of 2 mm in terms of meters or km, then 2 mm = .002 m or 2 mm = .000002 km. In either case, the .002 m, and .000002 km is still the same value we measured to only one significant figure when we expressed it as 2 mm. Thus converting from one unit to another does not change the number of significant figures. That can only be changed by changing the accuracy of our measuring instruments or the accuracy of observation. If we wish to show a higher degree of accuracy than one place, we place the necessary number of zeroes to the right of the significant figure. Thus 2.0 mm = .0020 m are significant to two figures. The purpose of the zeroes to the left of the two merely show the placement of the decimal point, while those to the right increase the degree of accuracy. If a digit occurs to the left of the decimal point, then the zeroes may become significant. For example, 1.002 has four significant figures but 0.002 has only one significant figure, and to increase the number of significant figures, zeroes would be added to the right of the significant number. To show .002 accurate to three significant figures would be written .00200 rather than as 0.002. If the zeroes are used only to locate a decimal point, they are not significant. If they actually were measured to be zeroes, they are significant.

We will observe that if the decimal point is used in a number such as 1000., it establishes a definite degree of accuracy--in this case, significant figures. If it is omitted and is merely understood to be there, it establishes only one figure. Sometimes an ambiguity arises here which cannot be resolved by using or omitting a decimal point. If we round off 5999 to one significant figure, we will have 6000 without a decimal point. If we round off 6000.12 to four places we will have 6000. with a decimal point. If we wish to round off 6001. to three significant figures by using or omitting the decimal point, then 6000 indicates one significant figure, and in no way can we indicate three significant figures even though some of the zeroes are significant. We can resolve this ambiguity by using scientific notation.

EXAMPLE: Round 6001. to three places -- Answer 6.00×10^3. The zeroes after the decimal indicate the degree of accuracy while the power of ten locates the decimal point.

ACCURACY OF A PRODUCT

Suppose we wish to find the area of a table top by using a meter stick. We measure its length and find that it is somewhere between 221.5 cm and 222.5 cm. A good estimate is 222 cm. Measuring its width, we find it is somewhere between 21.5 cm and 22.5 cm, and as a good estimate we accept 22 cm as the width. The area will be 222 x 22 = 4884. sq. cm, or will it? Suppose we had chosen 221.5 cm as the best estimate of the length, and 21.5 cm as the width. The area would have been 221.5 x 21.5 = 4762.25 sq. cm. Or if we had chosen 222.5 cm as the length and 22.5 cm as the width, the area would have been 222.5 x 22.5 = 5006.25 sq. cm. We see there is a difference between 4762.25 sq. cm and 5006.25 sq. cm. The true answer may lie anywhere in between the two limits, but we don't know where. The best we can do is accept a value halfway in between and hope it is near the true answer. Thus we should accept 4900 sq. cm rather than exactly 4884. sq. cm as the best answer.

We will notice that in our answer 4900 sq. cm, we have an accuracy of 2 significant figures. Let us see why this is true. We accepted as a good estimate of the length and width, 22 cm and 22 cm respectively. However, the last digit to the right of each number is only approximated. It was a compromise between accepting 221.5 or 222.5 and 21.5 or 22.5. The last digit, then, does not have as high a degree of accuracy as do the others, and we may call its accuracy doubtful. To indicate all doubtful figures in the discussion, we will draw a line through them. Thus $\cancel{2}$ is doubtful. Let us multiply $22\cancel{2}$ x $2\cancel{2}$. When we multiply doubtful figures by doubtful figures, the products are doubtful. Multiplying a doubtful figure by an accurate figure, the product is still doubtful. If we add a doubtful figure to another doubtful figure or to an accurate figure, the sum is doubtful. Indicating all doubtful figures we have:

$$\begin{array}{r} 22\cancel{2} \\ \underline{2\cancel{2}} \\ \cancel{4}\cancel{4}\cancel{4} \\ \underline{44\cancel{4}} \\ 4\cancel{8}\cancel{8}\cancel{4} \end{array}$$

The answer $4\cancel{8}\cancel{8}\cancel{4}$ indicates one accurate figure, and three doubtful figures. Since we kept only one doubtful figure in both $22\cancel{2}$ and $2\cancel{2}$, we will keep one doubtful figure in the answer; and even though it is a doubtful figure (not accurate), it is, nevertheless, significant (has meaning). We should round off to $4\cancel{9}\cancel{0}\cancel{0}$. There are two significant figures in the answer, the same number of significant figures as are in $2\cancel{2}$. If we multiply two quantities with different numbers of significant figures, the answer will have as many significant figures as the least accurate of the quantities we multiplied. We might think of it as a chain being as strong as its weakest link. A product is as accurate as the least accurate quantity used.

A few examples to this point:

1000 x 5.5 = 5000	since the least accurate figure is 1000 (only one significant figure)
1000. x 5.5 = 5000	since the least accurate number is 5.5 only two significant figures are kept.
625.1 x .001 = .6	since .001 has only one significant figure
3000. x π = 9425	since π can be taken to any number of figures necessary, 3000 is least accurate

ACCURACY OF A SQUARE ROOT

We may follow the same procedure as we did with products to determine the number of significant figures in a square root. Suppose we wish to find the length of one side of a square whose area is 61504 sq. cm, we need only to determine the square root of 61504. The 61504 sq. cm may have been determined by squaring one side or by counting units squares. In any case, the last digit would have been approximate; thus we may call it doubtful and designate the fact with a line through the last digit. Extracting the square root by conventional methods, we obtain 248. If the last digit in 248 were doubtful, then squaring 248 we would obtain the number 61500 -- significant to only three figures. To obtain a number, 6150$\cancel{4}$, significant to five figures, we would necessarily have squared a number which had five significant figures. Therefore, the square root of 6150$\cancel{4}$ is 248.0$\cancel{0}$ significant to five figures.

RULE: The number of significant figures in a square root is equal to the number of significant figures in its square.

ACCURACY OF A QUOTIENT

To determine the number of significant figures in a quotient, we may proceed in the same manner as we did before with the products. Again we will use the line through a number to indicate that its accuracy is in doubt. Remember that if we multiply, add, or subtract doubtful numbers with other doubtful or accurate numbers, the result is still doubtful.

Let us divide 222̸ by 22̸ using long division.

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         1Ø.ØØ
   22̸   ______
        222̸.ØØ
        22̸
        ___
         Ø2̸
         ØØ
        ___
          2̸Ø
          ØØ
          ___
          2̸ØØ
          1̸9̸8̸
          ___
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In the answer we see that the Ø to the left of the decimal point is doubtful since it was used to determine how many times 22̸ must be multiplied to equal the doubtful number Ø2̸. We keep all the significant figures and the first doubtful figure, in this case, 1Ø. The other doubtful figures are thrown away because their significance depends on how significant the digit to the left of it is. The number to the left of the decimal is doubtful, therefore any others that follow on the right have little or no meaning at all, and we are justified in omitting them.

We find that a rule we can follow with quotients is the same as for products. In division, the quotient contains the same number of significant figures as that of the least accurate number in the divisor or dividend.

Examples:

$\frac{424}{26} = 16$ since the least accurate number is 26 (2 significant figures)

$\frac{424}{26.00} = 16.3$ since 424 is accurate to only 3 significant figures

$\frac{424.0}{26.00} = 16.30$ since both divisor and dividend have 4 significant figures

$\frac{424.0}{3.14} = 135$ since 3.14 has only 3 significant figures

$\frac{424.0}{\pi} = 134.9$ since π can be as accurate as needed, 424.0 is the least accurate number

The number of significant figures in a sum or difference does not depend on the least or the greatest number of significant figures in the quantities added or subtracted, but rather on the accuracy of each digit in each digit position. To make this clear, let us consider the following sums, remembering that our adopted notation "2̸" means a figure of approximation or of doubtful accuracy.

(A)
$$\begin{array}{r} 9222.44\cancel{4} \\ 2.2\cancel{3} \\ \hline 9224.6\cancel{7}\cancel{4} \end{array}$$
should be rounded to 9224.67

(B)
$$\begin{array}{r} 9222.4\cancel{4} \\ 2.2\cancel{3} \\ \hline 9224.6\cancel{7} \end{array}$$
should remain 9224.67

(C)
$$\begin{array}{r} 9222.\cancel{4} \\ 2.2\cancel{3} \\ \hline 9224.\cancel{6}\cancel{3} \end{array}$$
should be rounded to 9224.6

(D)
$$\begin{array}{r} 922\cancel{2} \\ 2.2\cancel{3} \\ \hline 922\cancel{4}.\cancel{2}\cancel{3} \end{array}$$
should be rounded to 9224

(E)
$$\begin{array}{r} 92\cancel{2}\cancel{0} \\ 2.2\cancel{3} \\ \hline 92\cancel{2}\cancel{2}.\cancel{2}\cancel{3} \end{array}$$
should be rounded to 9220

(F)
$$\begin{array}{r} 9200 \\ 2.2\cancel{3} \\ \hline 9\cancel{2}\cancel{0}\cancel{2}.\cancel{2}\cancel{3} \end{array}$$
should be rounded to 9200

(G)
$$\begin{array}{r} \cancel{9}\cancel{0}\cancel{0}\cancel{0} \\ 2.23 \\ \hline \cancel{9}\cancel{0}\cancel{0}\cancel{2}.\cancel{2}\cancel{3} \end{array}$$
should be rounded to 9000

(H)
$$\begin{array}{l} .922\cancel{2} \\ .000027\cancel{3} \\ \hline .922\cancel{2}\cancel{2}\cancel{7}\cancel{3} \end{array}$$
should be rounded to .9222

Notice in examples A through G as the number of significant figures to the right of the decimal point decreases from right to left, the significant figures in the answer decrease from right to left. Example (A) and example (H) show that the number of significant figures in the answer is not necessarily equal to the least or greatest number of significant figures in the numbers added, but rather, is dependent on the position of the first doubtful digit. The thousandths digit is dependent on the accuracy of the hundredths digit, which in turn is dependent on the accuracy of the tenths digit. The tenths digit is dependent on the accuracy of the units digit, the unit digit on the tens, the tens on the hundreds and so forth. Thus as several numbers are added together, there are several digits, falling in the hundredth, tenth, unit, ten, hundred, etc. positions, depending on the size of the numbers and the number of significant figures each contains. Therefore, starting with the position of the highest order of magnitude, the first position to contain a doubtful figure is the position that limits the accuracy of the answer.

This may be easier to apply if it is put in the form of a rule.

RULE: To determine the number of significant figures in either a sum or a difference, start with the digit farthest to the left in the number with the highest order of magnitude, and compare the accuracy of all numbers falling in each digit position.

In the position where the first doubtful figure occurs in any of the numbers added or subtracted, the accuracy ends there. If the accuracy is maintained in each position up to the decimal point, then to the right of the decimal point the same procedure is continued until the first doubtful figure occurs. The accuracy of the answer then ends at that position.