STRESSES AROUND A HOLE (II)

I Main Topics

A General Solution for a plane strain case

B Boundary conditions and specific solution

C Significance of solution

II General Solution for a plane strain case

Here is the governing equation:

1 $$0 = \frac{d^2 u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2}$$

Consider a power series solution; this is a very general solution

2 $$u_r = \ldots C_{-3}r^{-3} + C_{-2}r^{-2} + C_{-1}r^{-1} + C_0 r^0 + C_1 r^1 + C_2 r^2 + C_3 r^3 + \ldots$$

We expect that the displacement will not increase with distance from the hole, so we can anticipate that the coefficients for positive exponents will equal zero. Derivatives of (1) yield:

3 $$\frac{du_r}{dr} = \ldots - 3C_{-3}r^{-4} - 2C_{-2}r^{-3} - 1C_{-1}r^{-2} + 0C_0 r^{-1} + 1C_1 r^0 + 2C_1 r^1 + 3C_1 r^2 + \ldots$$

4 $$\frac{d^2 u_r}{dr^2} = \ldots 12C_{-3}r^{-5} + 6C_{-2}r^{-6} + 2C_{-1}r^{-3} + 0C_0 r^{-2} + 0C_1 r^{-1} + 2C_1 r^0 + 6C_1 r^1 \ldots$$

Now (2), (3), and (4) can be substituted into (1)

5 $$\begin{aligned} 0 = {} & \ldots 12C_{-3}r^{-5} + 6C_{-2}r^{-4} + 2C_{-1}r^{-3} + 0C_0 r^{-2} + 0C_1 r^{-1} + 2C_1 r^0 + 6C_1 r^1 \ldots \\ & + \frac{1}{r}\left(\ldots - 3C_{-3}r^{-4} - 2C_{-2}r^{-3} - 1C_{-1}r^{-2} + 0C_0 r^{-1} + 1C_1 r^0 + 2C_1 r^1 + 3C_1 r^2 + \ldots\right) \\ & - \frac{1}{r^2}\left(\ldots C_{-3}r^{-3} + C_{-2}r^{-2} + C_{-1}r^{-1} + C_0 r^0 + C_1 r^1 + C_2 r^2 + C_3 r^3 + \ldots\right) \end{aligned}$$

Multiplying the terms of the second and third rows of (5) yields

6 $$\begin{aligned} 0 = {} & \ldots 12C_{-3}r^{-5} + 6C_{-2}r^{-4} + 2C_{-1}r^{-3} + 0C_0 r^{-2} + 0C_1 r^{-1} + 2C_0 r^0 + 6C_1 r^1 + \ldots \\ & + \left(\ldots - 3C_{-3}r^{-5} - 2C_{-2}r^{-4} - 1C_{-1}r^{-3} + 0C_0 r^{-2} + 1C_1 r^{-1} + 2C_0 r^0 + 3C_1 r^1 + \ldots\right) \\ & - \left(\ldots C_{-3}r^{-5} + C_{-2}r^{-4} + C_{-1}r^{-3} + C_0 r^{-2} + C_1 r^{-1} + C_2 r^0 + C_3 r^1 + \ldots\right) \end{aligned}$$

Terms with the same exponent can now be collected

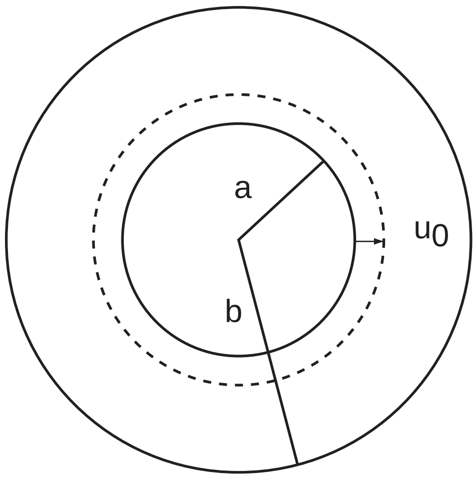
7 $$0 = \ldots 8C_{-3}r^{-5} + 3C_{-2}r^{-4} + 0C_{-1}r^{-3} - 1C_0 r^{-2} + 0C_1 r^{-1} + 3C_0 r^0 + 8C_1 r^1 + \ldots$$

This can hold for all values of r only if <u>each</u> leading term (i.e., nC_m) equals zero. The only C coefficients that need not be zero are those for C_{-1} and C_1, so the general solution in terms of displacements is

8 $$u_r = C_{-1}r^{-1} + C_1 r^1$$

III Boundary conditions and specific solutions

Consider a cylindrical hole in an infinitely large body with a prescribed radial displacement of u_0 at the wall of the hole and a displacement of zero at an infinite distance from the hole:



Boundary Conditions

$u_r|_{r=a} = u_0$

$u_r|_{r=b=\infty} = 0$

Two boundary conditions must be specified to solve our problem because our general solution has two unknown C coefficients: $u_r = C_{-1} r^{-1} + C_1 r^1$

For the radial displacement to go to zero as r goes to infinity, C_1 must equal 0. If $u_r(r=a) = u_0$, then $u_0 = C_{-1}\, a^{-1}$, so $C_{-1} = a\, u_0$.

The solution for the displacements for ***this*** problem is then

9 $u_r = u_0 \dfrac{a}{r}$

Note that the hole radius provides a scale for the problem and the displacements decay with distance from the hole (as suspected).

Problems with different boundary conditions have different solutions.

The strains for this problem are obtained from the displacements.

10 $\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{\partial\left(u_0 a r^{-1}\right)}{\partial r} = u_0 a \frac{\partial\left(r^{-1}\right)}{\partial r} = \frac{-u_0 a}{r^2} = -u_0 a r^{-2}$

11 $\varepsilon_{\theta\theta} = \frac{u_r}{r} = \frac{u_0 a r^{-1}}{r} = \frac{u_0 a}{r^2} = u_0 a r^{-2}$

12 $\varepsilon_{r\theta} = 0$

The stresses (under plain strain conditions) comes from the strains

13 $\sigma_{rr} = \frac{E}{(1+\nu)}\left[\varepsilon_{rr} + \frac{\nu}{(1-2\nu)}\left(\varepsilon_{rr} + \varepsilon_{\theta\theta}\right)\right] = \frac{E}{(1+\nu)}\left[\frac{-u_0 a}{r^2} + \frac{\nu}{(1-2\nu)}\left(\frac{-u_0 a}{r^2} + \frac{u_0 a}{r^2}\right)\right] = \frac{E}{(1+\nu)}\left[\frac{-u_0 a}{r^2}\right]$

14 $\sigma_{\theta\theta} = \frac{E}{(1+\nu)}\left[\varepsilon_{\theta\theta} + \frac{\nu}{(1-2\nu)}\left(\varepsilon_{\theta\theta} + \varepsilon_{rr}\right)\right] = \frac{E}{(1+\nu)}\left[\frac{u_0 a}{r^2} + \frac{\nu}{(1-2\nu)}\left(\frac{u_0 a}{r^2} + \frac{-u_0 a}{r^2}\right)\right] = \frac{E}{(1+\nu)}\left[\frac{u_0 a}{r^2}\right]$

15 $\sigma_{r\theta} = 2G\varepsilon_{r\theta} = 0$

An outward (positive) displacement of the wall of the hole causes a radial compression [from (13)] and a circumferential tension [from (14)] <u>of equal magnitude</u>; each decays as $1/r^2$. For comparison, the displacements decay as 1/r.

Now, from (13) at the wall of the hole (r=a)

$\sigma_{rr}\big|_{r=a} = \frac{E}{(1+\nu)}\left[\frac{-u_0 a}{a^2}\right] = \frac{E}{(1+\nu)}\left[\frac{-u_0}{a}\right] = T$, so $-u_0 = \frac{aT}{\frac{E}{(1+\nu)}}$. Substituting this back into (13) and (14) shows how the stresses outside the hole vary in response to a uniform suction or pressure on the hole walls:

16a $\sigma_{rr} = \frac{E}{(1+\nu)}\left[\frac{\left(aT \Big/ \frac{E}{(1+\nu)}\right)a}{r^2}\right] = T\left(\frac{a}{r}\right)^2$ 16b $\sigma_{rr} = -T\left(\frac{a}{r}\right)^2$

IV Significance

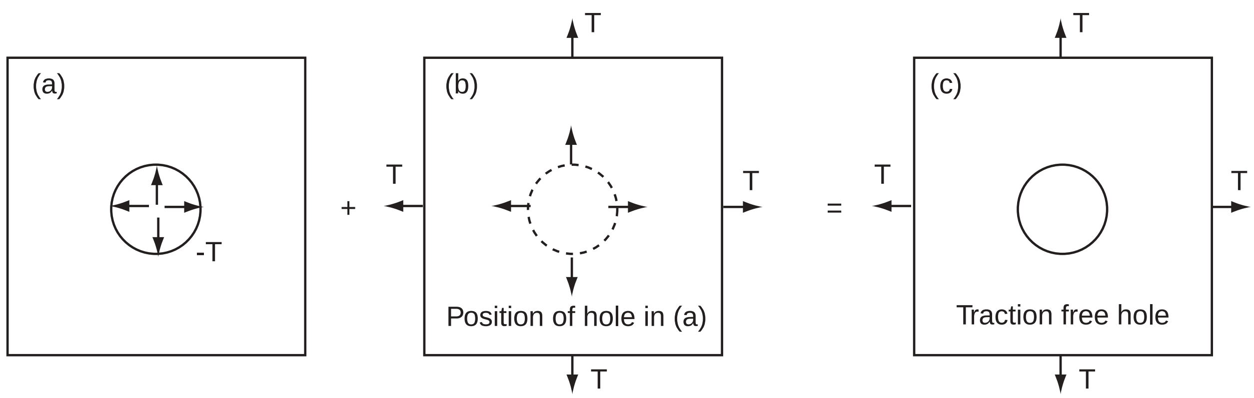
A For a ***pressure*** in a hole with no remote load at r = ∞, <u>the radial normal stress is a principal stress</u> because $\sigma_{r\theta} = 0$, and σ_{rr} is <u>the most compressive stress</u>.

B For a ***pressure*** in a hole with no remote load at r = ∞, <u>the circumferential normal stress is a principal stress</u> because $\sigma_{r\theta} = 0$, and $\sigma_{\theta\theta}$ is <u>the most tensile stress. A high pressure could cause radial cracking</u> (e.g., <u>radial dikes around a magma chamber</u>).

C Superposition of a biaxial tension

The solution for a pressurized hole in an infinite plate can be used to build other solutions, such as the solution for an unpressurized hole in a plate loaded in biaxial (isotropic) tension T.

SUPERPOSITION



Pressure in a hole + Biaxial Tension (no hole) =

$$\sigma_{rr} = -T(a/r)^2 \quad + \quad \sigma_{rr} = T \quad = \quad \sigma_{rr} = T - T(a/r)^2$$

$$\sigma_{\theta\theta} = -(-T)(a/r)^2 \qquad \sigma_{\theta\theta} = T \qquad \sigma_{\theta\theta} = T + T(a/r)^2$$

$$\boxed{\sigma_{\theta\theta}\big|_{r=a} = 2T}$$

Stress concentration!

This is one of the most important solutions in the field of rock mechanics. The presence of a hole causes a doubling of the normal stress that exists far from the hole. This is an example of a stress concentration. Note that the magnitude of the hoop stress (circumferential stress) at the hole is independent of the size of the hole, so a tiny cylindrical hole causes the same stress concentration as a large one. A tiny hole near the wall of a larger hole might be expected to have an even larger stress concentration (why?).

Stress concentrations explain myriad phenomena, such as why paper doesn't explode when pulled upon hard, why paper tears along lines of tiny holes, why cracks in riveted steel plates start from the rivet holes, why cracks in drying mudflats originate from the where grass stems have poked through the mud, etc. Consider how many holes (i.e., potential stress concentrations) exist in a porous sandstone, and one begins to understand why “strong” rocks can fail under “low” stresses.