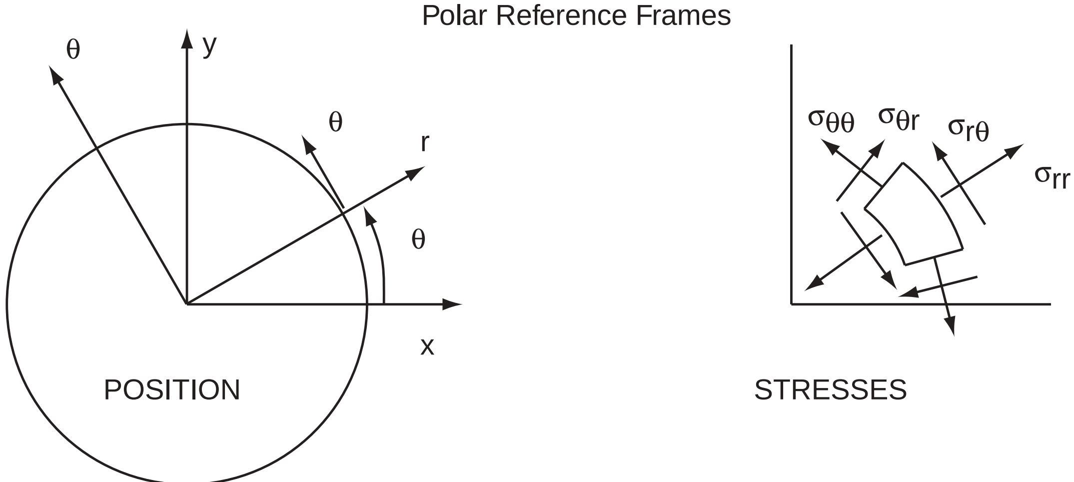
STRESSES AROUND A HOLE (I)

I Main Topics
  A Introduction to stress fields and stress concentrations
  B Stresses in a polar (cylindrical) reference frame
  C Equations of equilibrium
  D Solution of boundary value problem for a pressurized hole

II Introduction to stress fields and stress concentrations
  A Importance
    1 Stress (and strain and displacement) vary in space
    2 Stress concentrations can be huge and have a large effect
  B Common causes of stress concentrations
    1 A force acts on a small area (e.g., beneath a nail being hammered)
    2 Geometric effects (e.g., corners on doors and windows)
    3 Material heterogeneities (e.g., mineral heterogeneities and voids)

III Stresses in a polar (cylindrical) reference frame (on-in convention)

Polar Reference Frames



$$\sigma_{i'j'} = a_{i'k}\, a_{j'l}\, \sigma_{kl}$$

$x = r\cos\theta \qquad y = r\sin\theta$

$r = (x^2 + y^2)^{1/2} \qquad \theta = \mathrm{atan}(y/x)$

$a_{rx} = \cos\theta_{rx} = \cos\theta$
$a_{ry} = \cos\theta_{ry} = \sin\theta$
$a_{\theta x} = \cos\theta_{\theta x} = -\sin\theta$
$a_{\theta y} = \cos\theta_{\theta y} = \cos\theta$

$$\sigma_{rr} = a_{rx}\, a_{rx}\, \sigma_{xx} + a_{rx}\, a_{ry}\, \sigma_{xy} + a_{ry}\, a_{rx}\, \sigma_{yx} + a_{ry}\, a_{ry}\, \sigma_{yy}$$

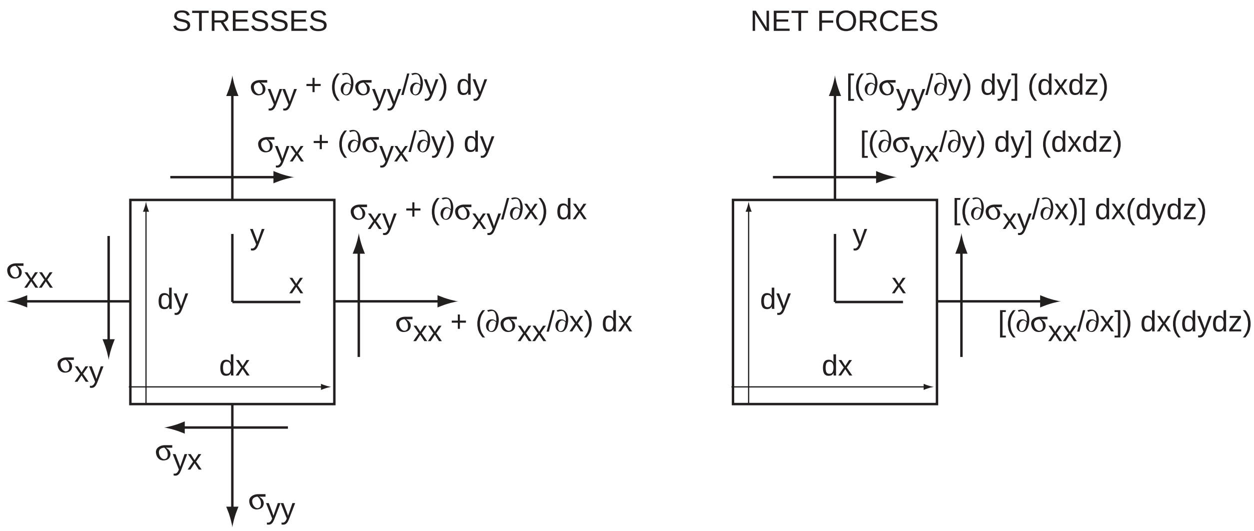
$$\sigma_{r\theta} = a_{rx}\, a_{\theta x}\, \sigma_{xx} + a_{rx}\, a_{\theta y}\, \sigma_{xy} + a_{ry}\, a_{\theta x}\, \sigma_{yx} + a_{ry}\, a_{\theta y}\, \sigma_{yy}$$

$$\sigma_{\theta r} = a_{\theta x}\, a_{rx}\, \sigma_{xx} + a_{\theta x}\, a_{ry}\, \sigma_{xy} + a_{\theta y}\, a_{rx}\, \sigma_{yx} + a_{\theta y}\, a_{ry}\, \sigma_{yy}$$

$$\sigma_{\theta\theta} = a_{\theta x}\, a_{\theta x}\, \sigma_{xx} + a_{\theta x}\, a_{\theta y}\, \sigma_{xy} + a_{\theta y}\, a_{\theta x}\, \sigma_{yx} + a_{\theta y}\, a_{\theta y}\, \sigma_{yy}$$

IV Equations of equilibrium (force balance)

A In Cartesian (x,y) reference frame



1 Force = (stress)(area)

2 Force balance in the x-direction

a $\sum F_x = 0 = [-\sigma_{xx}](dydz) + [-\sigma_{yx}](dxdz) + [\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx](dydz) + [\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy](dxdz)$

b $\sum F_x = 0 = [\frac{\partial \sigma_{xx}}{\partial x} dx](dydz) + [\frac{\partial \sigma_{yx}}{\partial y} dy](dxdz)$

c $\sum F_x = 0 = [\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}](dxdydz)$

d $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0$

The rate of σ_{xx} increase is balanced by the rate of σ_{yx} decrease

3 Force balance in the y-direction

a $\sum F_y = 0 = [-\sigma yy](dxdz) + [-\sigma_{xy}](dydz) + [\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y} dy](dxdz) + [\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial y} dx](dydz)$

b $\sum F_y = 0 = [\frac{\partial \sigma_{yy}}{\partial y} dy](dxdz) + [\frac{\partial \sigma_{xy}}{\partial x} dx](dydz)$

c $\sum F_x = 0 = [\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x}](dxdydz)$

d $\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$

The rate of σ_{yy} increase is balanced by the rate of σ_{xy} decrease

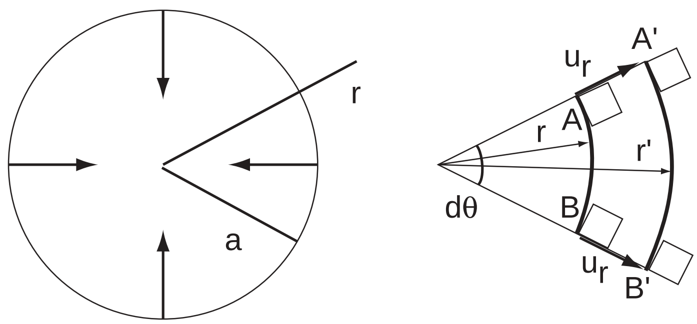
B In polar (r,θ) reference frame (apply chain rule; see supplement)

1 $\sum F_r = 0 = \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}$

2 $\sum F_\theta = 0 = \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r}$

V Solution of boundary value problem for a pressurized hole

Axisymmetric Displacements and Strains



$e_{rr} = \partial u_r/\partial r$

$\varepsilon_{\theta\theta}$ = change in arc length / original arc length

= (A'B' - AB)/AB

= (r'dθ - rdθ)/rdθ

= (r'- r)dθ/rdθ

= (r'- r)/r

$\varepsilon_{\theta\theta} = u_r/r$

$\varepsilon_{r\theta} = 0$

The right angles at A and B between radial lines and circumferential arcs do not change as A→ A' and B→B': $\varepsilon_{r\theta} = 0$

A Governing equation for an axisymmetric problem

1 Displacements are purely radial for a uniform suction (pressure) in hole; the displacements are toward (away from) the hole

2 For constant values of r, stresses and displacements in a polar reference frame do not change with θ, so $\partial(u_r, \theta_{rr}, \text{ etc.})/\partial\theta = 0$

3 By symmetry the shear stress $\sigma_{r\theta} = 0$

4 Equation B2 of section IV is identically equal to zero

5 Equation B1 becomes: $\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$, and then $\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$

This is our ***governing equation***. It is a differential equation.

B One General Solution Method

1 First solve for the stresses in terms of the strains using Hooke's law, and then solve for the strains in terms of the displacements. This leaves a differential equation to solve in terms of the displacements.

Second, after solving for the displacements, take the derivatives of the displacements to find the strains. Third, solve for the stresses in terms of the strains using Hooke's law. The solution is not hard, but it is somewhat lengthy.

2 The general solution will contain constants. Their values are found in terms of the stresses or displacements on the boundaries of our body (i.e., the wall of the hole and any external boundary), that is in terms of the ***boundary conditions*** for our problem.

C Strain-displacement relationships

Cartesian coordinates

a $\varepsilon_{xx} = \frac{\partial u}{\partial x}$

b $\varepsilon_{yy} = \frac{\partial v}{\partial y}$

c $\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

Polar coordinates

$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$

$\varepsilon_{\theta\theta} = \frac{u_r}{r}$

$\varepsilon_{r\theta} = 0$

D Strain- stress relationships

1 Plane stress (σ_{zz} = 0)

Cartesian coordinates

a $\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - v\sigma_{yy}\right]$

b $\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - v\sigma_{xx}\right]$

c $\varepsilon_{xy} = \frac{1}{2G}\sigma_{xy}$

Polar coordinates

$\varepsilon_{rr} = \frac{1}{E}\left[\sigma_{rr} - v\sigma_{\theta\theta}\right]$

$\varepsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - v\sigma_{rr}\right]$

$\varepsilon_{xy} = \frac{1}{2G}\sigma_{xy}$

2 Plane strain (ε_{zz} = 0)

Cartesian coordinates

a $\varepsilon_{xx} = \frac{1-v^2}{E}\left[\sigma_{xx} - \left(\frac{v}{1-v}\right)\sigma_{yy}\right]$

b $\varepsilon_{yy} = \frac{1-v^2}{E}\left[\sigma_{yy} - \left(\frac{v}{1-v}\right)\sigma_{xx}\right]$

c $e_{r\theta} = \frac{1}{2G}\sigma_{r\theta}$

Polar coordinates

$\varepsilon_{rr} = \frac{1-v^2}{E}\left[\sigma_{rr} - \left(\frac{v}{1-v}\right)\sigma_{\theta\theta}\right]$

$\varepsilon_{\theta\theta} = \frac{1-v^2}{E}\left[\sigma_{\theta\theta} - \left(\frac{v}{1-v}\right)\sigma_{rr}\right]$

$e_{r\theta} = \frac{1}{2G}\sigma_{r\theta}$

E Stress-strain relationships

1 Plane stress (σ_{zz} = 0)

Cartesian coordinates

a $\sigma_{xx} = \frac{E}{(1-\nu^2)}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right]$

b $\sigma_{yy} = \frac{E}{(1-\nu^2)}\left[\varepsilon_{yy} + \nu\varepsilon_{xx}\right]$

c $\sigma_{xy} = 2G\varepsilon_{xy}$

Polar coordinates

$\sigma_{rr} = \frac{E}{(1-\nu^2)}\left[\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right]$

$\sigma_{\theta\theta} = \frac{E}{(1-\nu^2)}\left[\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\right]$

$\sigma_{r\theta} = 2G\varepsilon_{r\theta}$

2 Plane strain (ε_{zz} = 0)

Cartesian coordinates

a $\sigma_{xx} = \frac{E}{(1+\nu)}\left[\varepsilon_{xx} + \frac{\nu}{(1-2\nu)}\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right]$

b $\sigma_{yy} = \frac{E}{(1+\nu)}\left[\varepsilon_{yy} + \frac{\nu}{(1-2\nu)}\left(\varepsilon_{yy} + \varepsilon_{xx}\right)\right]$

c $\sigma_{xy} = 2G\varepsilon_{xy}$

Polar coordinates

$\sigma_{rr} = \frac{E}{(1+\nu)}\left[\varepsilon_{rr} + \frac{\nu}{(1-2\nu)}\left(\varepsilon_{rr} + \varepsilon_{\theta\theta}\right)\right]$

$\sigma_{\theta\theta} = \frac{E}{(1+\nu)}\left[\varepsilon_{\theta\theta} + \frac{\nu}{(1-2\nu)}\left(\varepsilon_{\theta\theta} + \varepsilon_{rr}\right)\right]$

$\sigma_{r\theta} = 2G\varepsilon_{r\theta}$

F Axisymmetric Equilibrium (Governing) Equations

1 Plane stress (σ_{zz} = 0)

In terms of stress

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

In terms of displacement

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0$$

2 Plane strain (ε_{zz} = 0)

In terms of stress

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

In terms of displacement

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0$$