TENSOR TRANSFORMATION OF STRESSES

Transformation of stresses between planes of arbitrary orientation

In the 2-D example of lecture 16, the normal and shear stresses (tractions) were found on one arbitrarily oriented plane in the n,s reference frame. Here we consider two perpendicular (but otherwise arbitrarily oriented) planes, one perpendicular to the x'-axis, and the other perpendicular to the y'-axis and seek to find the normal and shear stress acting on them. In lecture 16 we considered the case where no shear stresses acted on the planes perpendicular to the x- and y-axes. We address here the more general case where shear stresses might exist, but still apply the strategy of lecture 16.

The key concept is that the total value of each stress component in one reference frame is the sum of the weighted contributions from all the components in another frame (see Figures 18.1 – 18.4). We start with a two-dimensional example:

$$\sigma_{x'x'} = w^{(1)}\sigma_{xx} + w^{(2)}\sigma_{xy} + w^{(3)}\sigma_{yx} + w^{(4)}\sigma_{yy}$$

All four stress components are needed to completely describe stresses at a point in the x,y reference must be known. The w terms are the weighting factors. In order for the equation to be dimensionally consistent, all the weighting factors must be dimensionless. Figures 18.1 – 18.4 show that the weighting factors describe as dimensionless ratios how a force in one direction projects into another direction (here the x' direction), and how the ratio of an area with a normal in one direction projects to an area with a normal in another direction (here the x' direction). Algebraically, we can write this as

$$\frac{F_{x'}}{A_{x'}} = \left(\frac{A_x}{A_{x'}}\frac{F_{x'}}{F_x}\right)\frac{F_x}{A_x} + \left(\frac{A_x}{A_{x'}}\frac{F_{x'}}{F_y}\right)\frac{F_y}{A_x} + \left(\frac{A_y}{A_{x'}}\frac{F_{x'}}{F_x}\right)\frac{F_x}{A_y} + \left(\frac{A_y}{A_{x'}}\frac{F_{x'}}{F_y}\right)\frac{F_y}{A_y}$$

Each term in parentheses weights the following x,y stress component to give a weighted contribution to the stress term in the x',y' reference frame. Each ratio in parentheses equals the direction cosine between the reference frame axes of the numerator and denominator, so:

$$\sigma_{x'x'} = a_{x'x}a_{x'x}\sigma_{xx} + a_{x'x}a_{x'y}\sigma_{xy} + a_{x'y}a_{x'x}\sigma_{yx} + a_{x'y}a_{x'y}\sigma_{yy}$$

Similarly, each of the other three 2-D stress components in the x',y' reference frame also involve all four stress components in the x,y reference frame:

$$\frac{F_{y'}}{A_{x'}} = \left(\frac{A_x}{A_{x'}}\frac{F_{y'}}{F_x}\right)\frac{F_x}{A_x} + \left(\frac{A_y}{A_{x'}}\frac{F_{y'}}{F_x}\right)\frac{F_y}{A_x} + \left(\frac{A_x}{A_{x'}}\frac{F_{y'}}{F_y}\right)\frac{F_x}{A_y} + \left(\frac{A_y}{A_{x'}}\frac{F_{y'}}{F_y}\right)\frac{F_y}{A_y}$$

$$\sigma_{x'y'} = a_{x'x}\,a_{y'x}\,\sigma_{xx} + a_{x'x}\,a_{y'y}\,\sigma_{xy} + a_{x'y}\,a_{y'x}\,\sigma_{yx} + a_{x'y}\,a_{y'y}\,\sigma_{yy}$$

$$\frac{F_{x'}}{A_{y'}} = \left(\frac{A_x}{A_{y'}}\frac{F_{x'}}{F_x}\right)\frac{F_x}{A_x} + \left(\frac{A_x}{A_{y'}}\frac{F_{x'}}{F_y}\right)\frac{F_y}{A_x} + \left(\frac{A_y}{A_{y'}}\frac{F_{x'}}{F_x}\right)\frac{F_x}{A_y} + \left(\frac{A_y}{A_{y'}}\frac{F_{x'}}{F_y}\right)\frac{F_y}{A_y}$$

$$\sigma_{y'x'} = a_{y'x}\,a_{x'x}\,\sigma_{xx} + a_{y'x}\,a_{x'y}\,\sigma_{xy} + a_{y'y}\,a_{x'x}\,\sigma_{yx} + a_{y'y}\,a_{x'y}\,\sigma_{yy}$$

$$\frac{F_{y'}}{A_{y'}} = \left(\frac{A_x}{A_{y'}}\frac{F_{y'}}{F_x}\right)\frac{F_x}{A_x} + \left(\frac{A_x}{A_{y'}}\frac{F_{y'}}{F_y}\right)\frac{F_y}{A_x} + \left(\frac{A_y}{A_{y'}}\frac{F_{y'}}{F_x}\right)\frac{F_x}{A_y} + \left(\frac{A_y}{A_{y'}}\frac{F_{y'}}{F_y}\right)\frac{F_y}{A_y}$$

$$\sigma_{y'y'} = a_{y'x}\,a_{y'x}\,\sigma_{xx} + a_{y'x}\,a_{y'y}\,\sigma_{xy} + a_{y'y}\,a_{y'x}\,\sigma_{yx} + a_{y'y}\,a_{y'y}\,\sigma_{yy}$$

The shorthand tensor notation for the four equations above is

$\sigma_{i'j'} = a_{i'k}a_{j'l}\;\sigma_{kl}$;   i,j,k,l = (x,y) or (1,2)

<u>The exact same notation applies for 3-D!</u>

$\sigma_{i'j'} = a_{i'k}a_{j'l}\;\sigma_{kl}$;   i,j,k,l = (x,y,z) or (1,2,3)

The same expressions result from multiplying the following matrices.

In 2-D

$$\begin{bmatrix} \sigma_{1'1'} & \sigma_{1'2'} \\ \sigma_{2'1'} & \sigma_{2'2'} \end{bmatrix} = \begin{bmatrix} a_{1'1} & a_{1'2} \\ a_{2'1} & a_{2'2} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} a_{1'1} & a_{2'1} \\ a_{1'2} & a_{2'2} \end{bmatrix}$$

In 3-D

$$\begin{bmatrix} \sigma_{1'1'} & \sigma_{1'2'} & \sigma_{1'3'} \\ \sigma_{2'1'} & \sigma_{2'2'} & \sigma_{2'3'} \\ \sigma_{3'1'} & \sigma_{3'2'} & \sigma_{3'3'} \end{bmatrix} = \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{1'1} & a_{2'1} & a_{3'1} \\ a_{1'2} & a_{2'2} & a_{3'2} \\ a_{1'3} & a_{2'3} & a_{3'3} \end{bmatrix}$$

In general

$\left[\sigma_{i'j'}\right] = [a]\;[\sigma]\;[a]^T$ <u>Works for either</u> 2-D or 3-D!

Advantages of Mohr circle approach over tensor/matrix approach

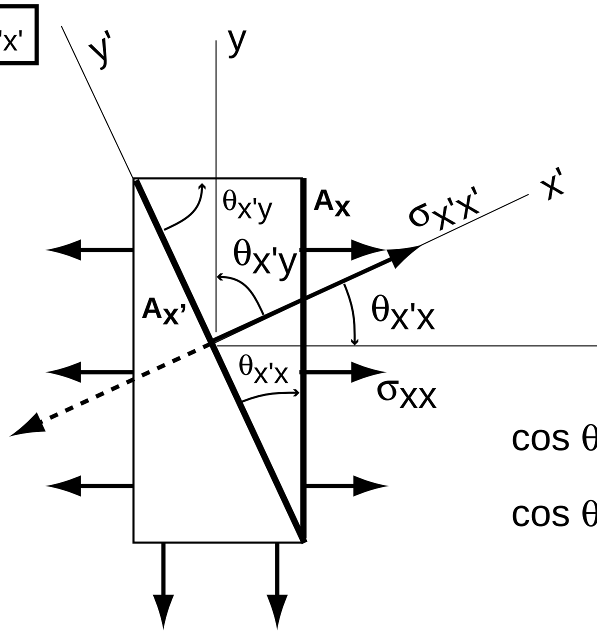
1. Gives a geometric meaning to stress relationships.
2. Can do stress rotation problems in your head.

Advantages of tensor or matrix approach over Mohr circle approach

1. The physical underpinning behind how stresses transform is explicit; it is not obvious with a Mohr circle construction. First, the notion that all members of a stress tensor are involved in the transformation is more straightforward than with a Mohr circle. Second, the two rotation terms $a_{i'k}$ and $a_{j'l}$ reflect (a) the rotation of the area that the stress components act on, and (b) the rotation of the direction that the components act in, so the tendency to incorrectly consider stress as a force is reduced; the tensor quality of stresses is more apparent.
2. The double angle expressions in Mohr circle, which can be difficult to remember and work with, are not present here.
3. Easier to address with a computer or a calculator.
4. Can apply just as easy to 3-D as 2-D; far more useful for 3-D problems than Mohr diagrams.
5. Powerful methods of linear algebra exist for finding the magnitudes of the principal stresses ("eigenvalues") and the direction of the principal stresses ("eigenvectors), so the underlying nature of the stress field is easier to identify; this is especially important in 3-D.
6. The shear stress convention with tensors is logical.

Fig. 18.1

Contribution of σ_{xx} to $\sigma_{x'x'}$



$\cos \theta_{x'x} = a_{x'x}$

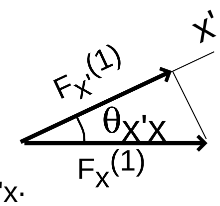
$\cos \theta_{x'y} = a_{x'y}$

What does σ_{xx} on face A_x of area A_x contribute to $\sigma_{x'x'}$ on face $A_{x'}$ of area $A_{x'}$?

Start with the definition of stress: $\sigma_{x'x'}{}^{(1)} = F_{x'}{}^{(1)} / A_{x'}$.

The unknown quantities $F_{x'}{}^{(1)}$ and $A_{x'}$ must be found from the known quantities σ_{xx} and θ.

To do this we first find the force $F_x{}^{(1)}$ associated with σ_{xx}:



Force = (stress)(area)

$F_x{}^{(1)} = \sigma_{xx} A_x$

The component of $F_x{}^{(1)}$ that acts along the x'-direction is $F_x{}^{(1)} \cos \theta_{x'x}$.

$F_{x'}{}^{(1)} = F_x{}^{(1)} \cos \theta_{x'x}$

As can be seen from the diagram atop the page $A_x = A_{x'} \cos \theta_{x'x}$, so

$A_{x'} = A_x / \cos \theta_{x'x}$

So the contribution of σ_{xx} to $\sigma_{x'x'}$ is:

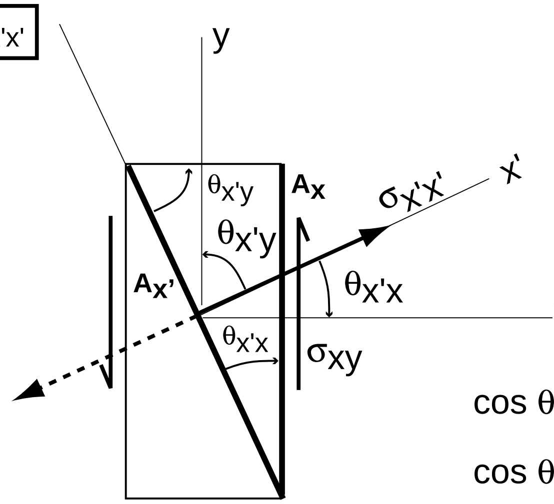
$\sigma_{x'x'}{}^{(1)} = F_{x'}{}^{(1)} / A_{x'} = F_x{}^{(1)} \cos \theta_{x'x} / (A_x / \cos \theta_{x'x}) = (F_x{}^{(1)} / A_x) \cos \theta_{x'x} \cos \theta_{x'x}$

$\sigma_{x'x'}{}^{(1)} = a_{x'x}\, a_{x'x}\, \sigma_{xx}$ ← Contribution of σ_{xx} to $\sigma_{x'x'}$

$$\frac{F_{x'}{}^{(1)}}{A_{x'}} = \frac{A_x}{A_{x'}} \frac{F_{x'}}{F_x} \frac{F_x{}^{(1)}}{A_x}$$ ← How the components project

Fig. 18.2

Contribution of σ_{xy} to $\sigma_{x'x'}$



$\cos \theta_{x'x} = a_{x'x}$

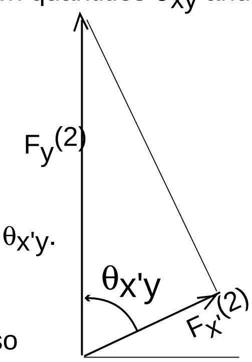
$\cos \theta_{x'y} = a_{x'y}$

What does σ_{xy} on face A_x of area A_x contribute to $\sigma_{x'x'}$ on face $A_{x'}$ of area $A_{x'}$?

Start with the definition of stress: $\sigma_{x'x'}^{(2)} = F_{x'}^{(2)} / A_{x'}$.

The unknown quantities $F_{x'}^{(1)}$ and $A_{x'}$ must be found from the known quantities σ_{xy} and θ.

To do this we first find the force $F_y^{(2)}$ associated with σ_{xy}:



Force = (stress)(area)

$F_y^{(2)} = \sigma_{xy} A_x$

The component of $F_y^{(2)}$ that acts along the x'-direction is $F_y^{(2)} \cos \theta_{x'y}$.

$F_{x'}^{(2)} = F_y^{(2)} \cos \theta_{x'y}$

As can be seen from the diagram atop the page $A_x = A_{x'} \cos \theta_{x'x}$, so

$A_{x'} = A_x / \cos \theta_{x'x}$

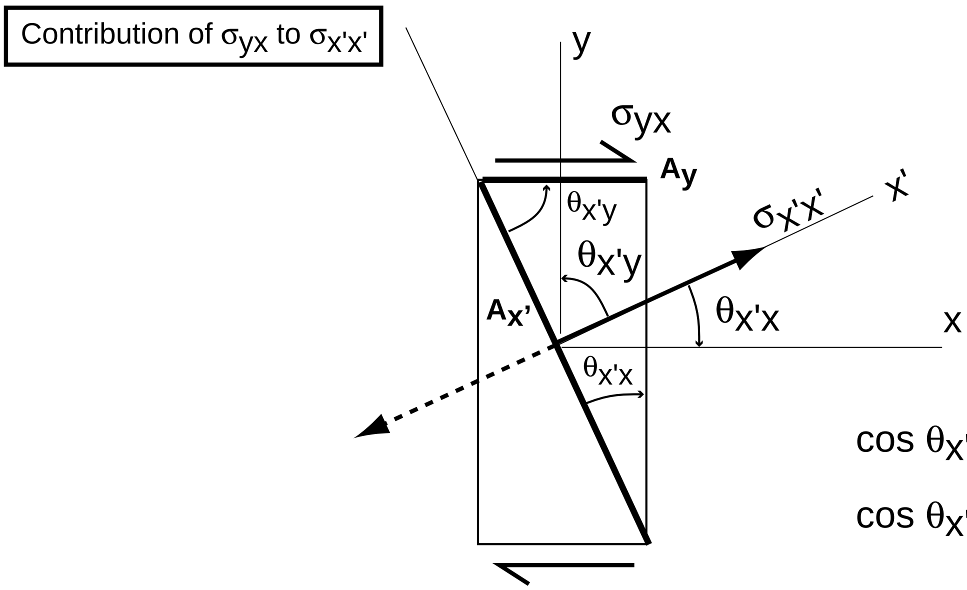
So the contribution of σ_{xx} to $\sigma_{x'x'}$ is:

$\sigma_{x'x'}^{(2)} = F_{x'}^{(2)} / A_{x'} = F_y \cos \theta_{x'y} / (A_x / \cos \theta_{x'x}) = (F_y / A_x) \cos \theta_{x'x} \cos \theta_{x'y}$

$\sigma_{x'x'}^{(2)} = a_{x'x} a_{x'y} \sigma_{xy}$ ← Contribution of σ_{xy} to $\sigma_{x'x'}$

$$\frac{F_{x'}^{(2)}}{A_{x'}} = \frac{A_x}{A_{x'}} \frac{F_{x'}}{F_y} \frac{F_y^{(2)}}{A_x}$$ ← How the components project

Fig. 18.3



$\cos\theta_{x'x} = a_{x'x}$

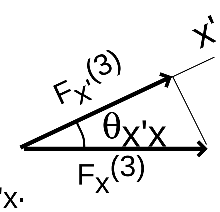
$\cos\theta_{x'y} = a_{x'y}$

What does σ_{yx} on face A_y of area A_y contribute to $\sigma_{x'x'}$ on face $A_{x'}$ of area $A_{x'}$?

Start with the definition of stress: $\sigma_{x'x'}{}^{(3)} = F_{x'}{}^{(3)}/A_{x'}$.

The unknown quantities $F_{x'}{}^{(3)}$ and $A_{x'}$ must be found from the known quantities σ_{xx} and θ.

To do this we first find the force $F_x{}^{(3)}$ associated with σ_{yx}:



Force = (stress)(area)

$F_x{}^{(3)} = \sigma_{yx} A_x$

The component of $F_x{}^{(3)}$ that acts along the x'-direction is $F_x{}^{(3)} \cos\theta_{x'x}$.

$F_{x'}{}^{(3)} = F_x{}^{(3)} \cos\theta_{x'x}$

As can be seen from the diagram atop the page $A_y = A_{x'} \cos\theta_{x'y}$, so

$A_{x'} = A_y/\cos\theta_{x'y}$

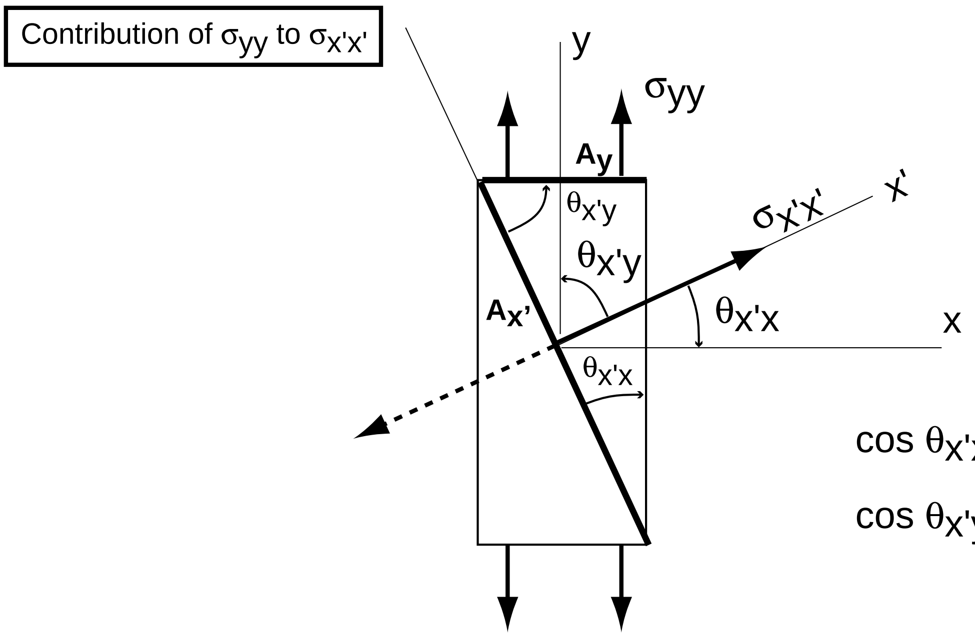
So the contribution of σ_{yx} to $\sigma_{x'x'}$ is:

$\sigma_{x'x'}{}^{(3)} = F_{x'}{}^{(3)} / A_{x'} = F_x{}^{(3)} \cos\theta_{x'x} / (A_y/\cos\theta_{x'y}) = (F_x{}^{(3)} / A_y) \cos\theta_{x'x} \cos\theta_{x'x}$

$\sigma_{x'x'}{}^{(3)} = a_{x'y}\, a_{x'x}\, \sigma_{yx}$ ← Contribution of σ_{xx} to $\sigma_{x'x'}$

$$\frac{F_{x'}{}^{(3)}}{A_{y'}} = \frac{A_x}{A_{y'}}\ \frac{F_{x'}}{F_x}\ \frac{F_x{}^{(3)}}{A_x}$$ ← How the components project

Fig. 18.4



$\cos \theta_{x'x} = a_{x'x}$

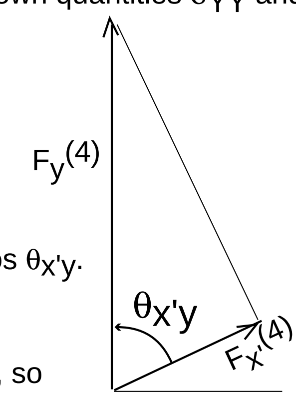
$\cos \theta_{x'y} = a_{x'y}$

What does σ_{yy} on face A_y of area A_y contribute to $\sigma_{x'x'}$ on face $A_{x'}$ of area $A_{x'}$?

Start with the definition of stress: $\sigma_{x'x'}^{(4)} = F_{x'}^{(4)} / A_{x'}$.

The unknown quantities $F_{x'}^{(4)}$ and $A_{x'}$ must be found from the known quantities σ_{YY} and θ.

To do this we first find the force $F_y^{(4)}$ associated with σ_{yy}:



Force = (stress)(area)

$F_y^{(4)} = \sigma_{yy} A_y$

The component of $F_y^{(4)}$ that acts along the x'-direction is $F_y^{(4)} \cos \theta_{x'y}$.

$F_{x'}^{(4)} = F_y^{(4)} \cos \theta_{x'y}$

As can be seen from the diagram atop the page $A_y = A_{x'} \cos \theta_{x'y}$, so

$A_{x'} = A_y / \cos \theta_{x'y}$

So the contribution of $\sigma_{yy}^{(4)}$ to $\sigma_{x'x'}^{(4)}$ is:

$\sigma_{x'x'}^{(4)} = F_{x'}^{(4)} / A_{x'} = F_y^{(4)} \cos \theta_{x'y} / (A_y / \cos \theta_{x'y}) = (F_y^{(4)} / A_y) \cos \theta_{x'y} \cos \theta_{x'y}$

$$\sigma_{x'x'}^{(4)} = a_{x'y}\, a_{x'y}\, \sigma_{yy}^{(4)}$$ ← Contribution of σ_{yy} to $\sigma_{x'x'}$

$$\frac{F_{x'}^{(4)}}{A_{x'}} = \frac{A_y}{A_{x'}} \frac{F_{x'}}{F_y} \frac{F_y^{(4)}}{A_y}$$ ← How the components project