HOMOGENEOUS FINITE STRAIN: DISPLACEMENT & DEFORMATION GRADIENTS

- I Main Topics
  - A Position, displacement, and differences in position of two points
  - B Chain rule for a function of two variables
  - C Homogenous deformation
  - D Examples

II Position, displacement, and differences in position of two points\* See diagram on next page

(Capital letters for "total vectors", lower case for components)

- A Initial position vectors: Pt. 1  $(\vec{X}_1 = \vec{x}_1 + \vec{y}_1)$  Pt. 2  $(\vec{X}_2 = \vec{x}_2 + \vec{y}_2)$
- B Final position vectors: Pt. 1  $(\vec{X}_1 = \vec{x}_1 + \vec{y}_1)$  Pt. 2  $(\vec{X}_2 = \vec{x}_2 + \vec{y}_2)$
- C Displacement vectors
  - 1 In terms of positions: Pt. 1  $(\vec{U_1} = \vec{X_1} \vec{X_1})$  Pt. 2  $(\vec{U_2} = \vec{X_2} \vec{X_2})$
  - 2 In terms of components: Pt. 1  $(\vec{U_1} = \vec{u_{1x}} + \vec{u_{1y}})$  Pt. 2  $(\vec{U_2} = \vec{u_{2x}} + \vec{u_{2y}})$
- D Difference in positions of Points 1 and 2
  - 1 Difference in initial positions  $\vec{dX} = \vec{X_2} \vec{X_1}$
  - 2 Difference in final positions  $\vec{dX'} = \vec{X'_2} \vec{X'_1}$



- II Chain rule for a function of two variables
  - A Chain rule for a function of two variables
    - 1 Chain rule of differential calculus, for a function x' = x'(x,y):  $dx' = \frac{\partial x'}{\partial x} dx + \frac{\partial x'}{\partial y} dy$  (where dx is the distance between 2 nearby pts)

The total change in variable x' equals the rate that x' changes with respect to x, multiplied by the change in x, plus the rate that x' changes with respect to y, multiplied by the change in y. Similarly,  $\frac{\partial y'}{\partial y'}$ 

2 
$$dy' = \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial y}dy$$

In matrix form, equations (1) and (2) yield

 $3a \begin{bmatrix} dx' \\ dy' \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \quad \text{OR} \quad 3b \quad [dX'] = [F][dX]$ 

Similarly

6 a 
$$\begin{bmatrix} du \\ dv \end{bmatrix} = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$
  
6 a 
$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$
 OR 6b 
$$[dU] = [J_u][dX]$$

At a particular point the derivatives have a unique (or constant) value, so the equations are linear in dx and dy in the neighborhood of the point (i.e., dx' and dy' depend on dx and dy raised to the first power. If the derivatives do not vary with x or y, then they have the same value everywhere, and the equations are linear in dx and dy no matter how large dx and dy are. This is the condition of homogenous strain. In that case, since  $dx = x_2-x_1$ , if point 1 is at the origin and point 2 is at (x,y), then dx = x. Similarly, dy = y, dx' = x', dy' = y', du = u, and dv = v. These replacements can be made in all the equations above, and the solutions are valid no matter how large x and y are. This leads to the equations in the section III.

B Chain rule for a function of three variables

$$dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy + \frac{\partial x'}{\partial z}dz \quad \text{(etc.)}$$

	Ho A	omogenous (uniform) deformation Equations of homogeneous 2-D strain						
		1	Lagrangian (final positions (x',y') in terms of initial positions (x,y))					
	a $x' = ax + by$ (a and b are constants; compare with II.A.1)							ts; compare with II.A.1)
	<ul> <li>b y' = cx + dy (c and d are constants; compare with II.A.2)</li> <li>2 Eulerian (initial positions (x,y) in terms of final positions (x',y'))</li> <li>a x = Ax' + By' (see derivation below)</li> </ul>							ts; compare with II.A.2)
								s of final positions (x',y'))
				i	y = (x'	– ax)/b	and	y = (y' - cx)/d
				ii	(x' - a:	x)/b = (y' - cx)	/d	
				iii	d(x' - a	ax) = b(y' - cx)		
				iv	cbx – a	adx = by'-dx'		
				v	x = (by	y'-dx')/(cb-ad)		This yields $x = x(x',y')$
b $y = Cx' + Dy'$ (see derivation below)							)	
				i	x = (y'	− dy)/c	and	x = (x' - by)/a
				ii	(y' – d	y)/c = (x' - by)	/a	
				iii	a(y' – (	dy) = c(x' - by)		
				iv	cby – a	ady = cx'-ay'		
				v y	= (cx'-a	ıy')/(cb-ad)		This yields $y = y(x',y')$
	В	Straight parallel lines in an initial state $(X=x,y)$ remain straight and parallel in the final (current) state $(X'=x',y')$ (see appendix)						
	С	C Parallelograms deform into parallelograms in 2-D;						
		ра	parallelepipeds deform into parallelepipeds in 3-D					

- D Circles deform into ellipses in 2-D (<u>see appendix</u>); Spheres deform into ellipsoids in 3-D
- E Considered a useful concept in geology for "small" regions over large regions deformation invariably is inhomogeneous (non-uniform)
- F Describing the shape, orientation, and rotation of the strain ellipse of ellipsoid is a useful way to describe the deformation. The rotation is the angle between the axes of the strain ellipse and their counterparts before any deformation occurred.

- G Deformed and undeformed <u>positions</u> are described by linear coordinate transformation equations (2-D examples below in matrix form)
  - 1 x' = ax + by y' = cx + dy (See III.A.1.a and II.A.1.b) 2  $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ , where  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  = deformation gradient matrix [F] 3  $\begin{bmatrix} X' \end{bmatrix} = \begin{bmatrix} F \end{bmatrix} \begin{bmatrix} X \end{bmatrix}$
- H The <u>displacements</u> also are described by linear displacement equations (2-D examples below)

1 
$$u = x' - x = (ax + by) - x = (a - 1)x + by$$
  
 $v = y' - y = (cx + dy) - y = cx + (d - 1)y$  (Compare with II.A.4, II.A.5)

2 
$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
, where

 $\begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \text{displacement vector gradient matrix } [J_u]$ 

- 3  $[U] = [J_u][X]$ 
  - a  $[J_u]$  is the Jacobian matrix for displacements (u)
  - b A Jacobian matrix is a matrix of all the first-order partial derivatives of a vector function. It represents the linear approximation of the function near a point of interest.
  - c For homogeneous strain, the partial derivatives are constants.

4 
$$J_u = F - I$$
, where  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 

IV Examples









No rotation



No rotation



F Pure shear strain (biaxial strain, no dilation) Undeformed (dashed black) and homogenously deformed (solid red) objects

No rotation

## 10/8/08



G Simple shear strain parallel to the x-axis (no dilation) Undeformed (dashed black) and homogenously deformed (solid red) objects

There is a rotation



There is a rotation.

- I General comments on the deformation gradient and displacement gradient tensors
  - 1 The tensors that describe deformation of a body depend on derivatives of positions or displacements.
  - 2 For cases of no deformation (case A) and rigid body translation (case B) the respective tensors are identical, even though the equations they are derived from differ. The constant displacement terms ( $c_x$  and  $c_y$ ) drop out in forming the tensors. The deformation gradient and displacement gradient tensors therefore provide no information on the presence or absence of a rigid body translation.
  - 3 For cases of no deformation (case A) and rigid body <u>rotation</u> (case C) the respective tensors are different. The deformation gradient and displacement gradient tensors therefore do provide information on the presence or absence of a rigid body rotation; this information is in the off-diagonal terms, which have equal magnitude and opposite sign.
  - 4 The meaning of the components of the tensors are straightforward in terms of derivatives, but we still need to understand the meaning of the components in terms of the measures of normal and shear strain.
  - 5 If the initial positions are not known, then they must be inferred (or guessed at): this situation is common in geology and introduces uncertainty in strain calculations.

## Appendix

A Homogeneous deformation of parallel lines to parallel lines

Consider a pair of parallel lines in the undeformed state 1ay = mx + B 1b  $y = mx + B^*$ 

The x,y coordinates in terms of the x',y' coordinates are:

$$2 x = (dx'-by')/(ad-bc)$$

$$3 y = (ay'-cx')/(ad-bc)$$

Substituting (2) and (3) into (1) yields

$$4a \frac{ay' - cx'}{ad - bc} = m \frac{dx' - by'}{ad - bc} + B \qquad 4b \qquad \frac{ay' - cx'}{ad - bc} = m \frac{dx' - by'}{ad - bc} + B^*$$

Move the x' and y' terms to opposite sides of the equations

$$5a \frac{(a+mb)y'}{ad-bc} = \frac{(c+md)x'}{ad-bc} + B \qquad 5b \qquad \frac{(a+mb)y'}{ad-bc} = \frac{(c+md)x'}{ad-bc} + B^*$$

Multiplying both sides of (5) by  $\frac{ad-bc}{a+mb}$  yields equations for two lines

6a 
$$y' = \frac{(c+md)}{(a+mb)}x' + B\frac{ad-bc}{(a+mb)}$$
 6b  $y' = \frac{(c+md)}{(a+mb)}x' + B*\frac{ad-bc}{(a+mb)}$ 

Parallel lines y = mx + B and  $y = mx + B^*$  transform to two parallel lines.

B Homogeneous deformation of a unit circle to an ellipse

$$1 \quad x^{2} + y^{2} = 1 \quad \text{or} \qquad x^{2} + y^{2} - 1 = 0$$

$$2 \quad \left[\frac{by' - dx'}{cb - ad}\right]^{2} + \left[\frac{cx' - ay'}{cb - ad}\right]^{2} - 1 = 0$$

$$3 \quad x'^{2} \left[\frac{d^{2} + c^{2}}{cb - ad}\right]^{2} - x'y' \left[\frac{2bd + 2ac}{cb - ad}\right]^{2} + y'^{2} \left[\frac{a^{2} + b^{2}}{cb - ad}\right]^{2} - 1 = 0$$

Equation (3) has the form of an ellipse oblique to the x,y axes

4 
$$C_1 x^2 + C_2 xy + C_3 y^2 + C_4 x + C_5 y + C_6 = 0$$
 (C<sub>1</sub>, C<sub>3</sub> > 0)

Since (3) and (4) have the same form, the circle transforms to an ellipse.

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```
function GG303 lec14b(a,b,c,d)
% Plots undeformed postions (X) of points on a square and a circle
% and deformed positions (X')
% and the displacements (U) relating the undeformed and deformed
positions
% given the coefficients of the 2-D coordinate transformation
equations
\% for homogeneous 2-D plane strain (uz = 0).
 [x'] = [a b] [x] 
% [y'] [c d] [y]
% Malvern (1969, p. 156) and Means (1976, p. 198) call the a,b,c,d
% coordinate transformation matrix the deformation gradient matrix
(F).
% Ramsay & Huber (1983) call this the "strain" matrix (p. 71).
 {X'} = [F][X] 
\% Fij = dX'i/dxj
% U = X' - X
 \sup (ax + by) - x = (a-1)x + by = ex + fy 
uy = (cx + dy) - y = cx + (d-1)y = qx + hy
% [ux] = [e f] [x]
% [uy] [g h] [y]
[U] = [Ju][X] = [F-I][X]
% Malvern (1969, p. 124) and Means (1976, p. 197) call
% the e,f,q,h matrix the displacement gradient matrix (Ju).
% Ramsay & Huber (1983) call this
% the "displacement vector gradient" matrix (p. 71).
% Juij = dui/dxj
% Examples
% GG303 lec14b(1,0,0,1)
% GG303 lec14b(1,0,0,1)
% GG303 lec14b(cos(60*pi/180),-
sin(60*pi/180), sin(60*pi/180), cos(60*pi/180))
% GG303 lec14b(1,0,0,0.5)
% GG303 lec14b(2,0,0,2)
% GG303_lec14b(2,0,0,0.5)
% GG303 lec14b(1,2,0,1)
% GG303 lec14b(2,1,0,-0.5)
% Define initial positions and the coordinate transformation and
displacement matrices
x = [0 \ 0 \ 1 \ 1 \ 0];
                       % x-coordinates of the square, clockwise
from origin
                       % y-coordinates of the square, clockwise
y = [0 \ 1 \ 1 \ 0 \ 0];
from origin
                        % 2x5 matrix
X = [x;y]
% Find the center of the square before deformation
x c = (x(1) + x(3))/2;
y_c = (y(1) + y(3))/2;
X_c = [x_c; y_c];
theta = \overline{0}:pi/360:2*pi;
xc = 0.5 + (0.5 + cos(theta));
yc = 0.5+(0.5*sin(theta));
Xc = [xc;yc];
                        % Position of undeformed circle center
                        % 2x2 matrix
F = [a b; c d]
```

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```
Ju = F - eye(size(F)) % 2x2 matrix
                         % 2x2 matrix. See Malvern, p. 159
C = F' * F
Bp = (inv(F))'*inv(F) % 2x2 matrix inv(B). See Malvern, p. 159
% Calculate deformed positions (X') and displacements (U)
Xp = F * X
                        % Xp = X' = deformed positions
                        % xp is the first row of Xp
xp = Xp(1,:);
yp = Xp(2,:);
                        % yp is the second row of Xp
Xcp = F*Xc;
                        % XCp = XC' = deformed positions
xcp = Xcv;% xcp = xcv = deformed positionxcp = Xcp(1,:);% xcp is the first row of XCpycp = Xcp(2,:);% ycp is the second row of XCpU = Ju*X% U = displacementsux = U(1,:);% ux is the first row of U
ux = U(1,:);
uy = U(2,:);
                        % uy is the second row of U
% Find the center of the deformed square
X cp = F*X c;
% Calculate the volume of the undeformed (V) and deformed (Vp)
squares
V = abs(det([x(4)-x(1),y(4)-y(1); x(2)-x(1),y(2)-y(1)]))
Vp= abs(det([xp(4)-xp(1),yp(4)-yp(1); xp(2)-xp(1),yp(2)-yp(1)]))
dilation = (Vp-V)./V
% Plot the undeformed square and the deformed "square"
figure(1)
clf
plot(x,y,'--k');
                                % Plots the undeformed square in
black
hold on
plot(xp,yp,'r');
                  % Plots the deformed square in red
hold on
% Plot the undeformed circle and the deformed "circle"
plot(xc,yc,'--k'); % Plots the undeformed circle in
black
hold on
plot(xcp,ycp,'r');
                        % Plots the deformed circle in red
% Find the principal axes using the Greens deformation tensor C
[vC,dC] = eiq(C)
plot([X_cp(1),X_cp(1)+vC(1,1)],[X_cp(2),X_cp(2)+vC(1,2)],'r');
plot([X_cp(1),X_cp(1)+vC(2,1)],[X_cp(2),X_cp(2)+vC(2,2)],'r');
[vBp,dBp] = eig(Bp)
plot([X c(1),X c(1)+vBp(1,1)],[X c(2),X c(2)+vBp(1,2)],'--k');
plot([X_c(1),X_c(1)+vBp(2,1)],[X_c(2),X_c(2)+vBp(2,2)],'--k');
% Now plot the displacement vectors with "nice-looking" heads
% The arrow heads produced by the "stock" version of quiver
(below) are too big
%quiver(x,y,ux,uy,0); % Plots the displacement vectors with
no scaling
% So first plot lines with no arrowheads connecting the vector
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```

```
tails and heads
for i=1:4
    line([x(i),xp(i)],[y(i),yp(i)])
end
% and then use quiver to draw arrows of UNIT length with heads
where I want heads
dx = zeros(size(x));
dy = zeros(size(y));
scalefactor = sqrt( (x-xp).^2 + (y-yp).^2);
k = find(scalefactor);
                                            % Finds nonzero
scalefactors
dx(k) = 0.25*(xp(k) - x(k))./scalefactor(k);
dy(k) = 0.25*(yp(k) - y(k))./scalefactor(k);
quiver (xp-dx,yp-dy,dx,dy,0);
axis('equal')
% Now label the plots
xlabel('x')
ylabel('y')
title('Undeformed (dashed black) and homogenously deformed (solid
red) objects'
```

## References

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- \*Ramsey, J.G., and Huber, M.I., 1983, The techniques of modern structural geology, Volume 1: Strain analysis: Academic Press, San Diego, 307 p.
- \* These are particularly good for undergraduate students