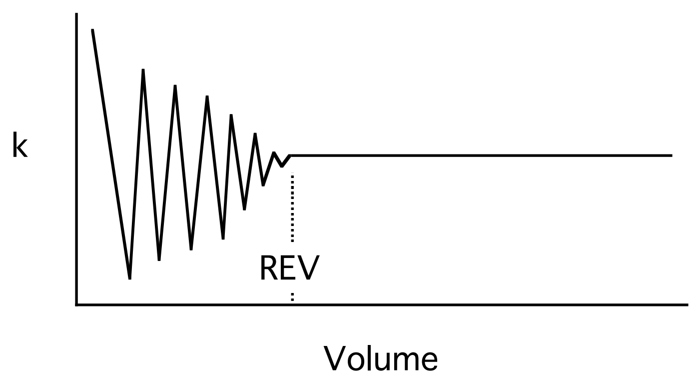
BASIC CONCEPTS OF KINEMATICS AND DEFORMATION

I Main Topics (see chapters 14 and 18 of Means, 1976)
  A Fundamental principles of continuum mechanics
  B Position vectors and coordinate transformation equations
  C Displacement vectors and displacement equations
  D Deformation

II Fundamental principles of continuum mechanics
  A Relates natural world to the realm of mathematics
  B Densities of mass, momentum, and energy exist (no "holes")
  C Number of particles is sufficiently large that the notion of an average bulk material behavior is meaningful
  D Examples of continuous properties
    1 Density $\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}$ So certain derivatives have to exist
    2 Hydraulic conductivity ("permeability")
  E Scale matters (see B, C, and D)



  E Variability
    1 Heterogeneity: material property depends on position
    2 Anisotropy: material property depends on orientation

II Position vectors and coordinate transformation equations
  A **X** = initial (undeformed) position
  B **X'** = final (current, or deformed) position (at time Δt)
  C Coordinate transformation equations
    1 **X'** = f(**X**) Lagrangian: final position set in terms of initial
    2 **X** = g(**X'**) Eulerian: initial position set in terms of final

III Displacement vector (**U**)

A **U** = **X'** - **X**

1 x-component: u_x, u_1, or just u

2 y-component: u_y, u_2, or just v

3 z-component: u_z, u_3, or just w

B **U** = **U**(**X**) Lagrangian: displacement in terms of initial position

C **U** = **U**(**X'**) Eulerian: displacement in terms of final position

IV Deformation: rigid body motion + change in size and/or shape

A Rigid body translation

1 No change in the length of line connecting any points

2 All points displaced by an equal vector (equal amount and direction); no displacement of points relative to one another

3 [X'] = [U] +[X] matrix addition (U is a constant)

B Rigid body rotation

1 No change in the length of line connecting any points

2 All points rotated by an equal amount about a common axis; no angular displacement of points relative to one another

3 [X'] = [a][X] matrix multiplication; <u>rows in [a] are dir. cosines!</u>

C Change in size and /or shape (distortional <u>strain</u>)

1 The lengths of at least some line segments connecting points in a body change (i.e., the relative positions of points changes)

2 **U** is not a constant throughout the body (i.e., <u>**U** varies</u>)

3 Change in linear dimension

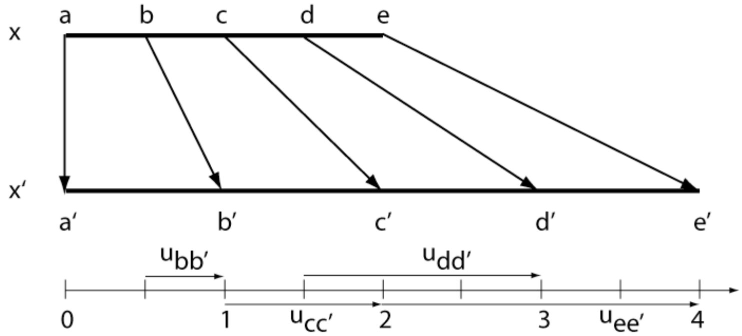
a Extension (or elongation) $\varepsilon = \frac{\Delta L}{L_o} = \frac{L_1 - L_o}{L_o}$ <u>dimensionless</u>!

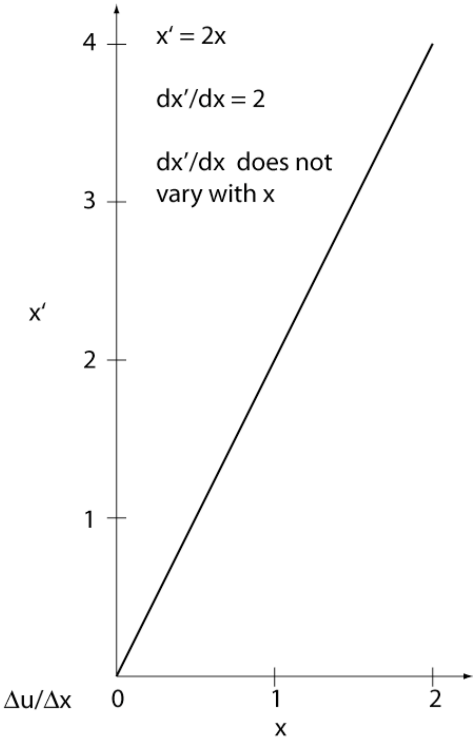
b Stretch $S = \frac{L_1}{L_o} = \frac{L_o}{L_o} + \frac{L_1 - L_o}{L_o} = 1 + \varepsilon$ <u>dimensionless</u>!

c Quadratic elongation $\lambda = \left(\frac{L_1}{L_o}\right)^2 = S^2 = (1+\varepsilon)^2$ <u>dimensionless</u>!

4 Change in right angles (change in angle between originally orthogonal lines): $\gamma = \tan\psi$ Note: for small angular changes, $\tan\psi \rightarrow \psi$

Example of homogeneous strain in one dimension





$x' = 2x$ Lagrangian coordinate transformation equation

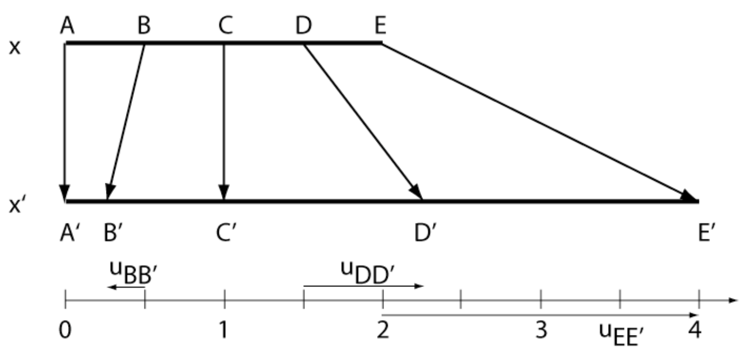
$x = x'/2$ Eulerian coordinate transformation equation

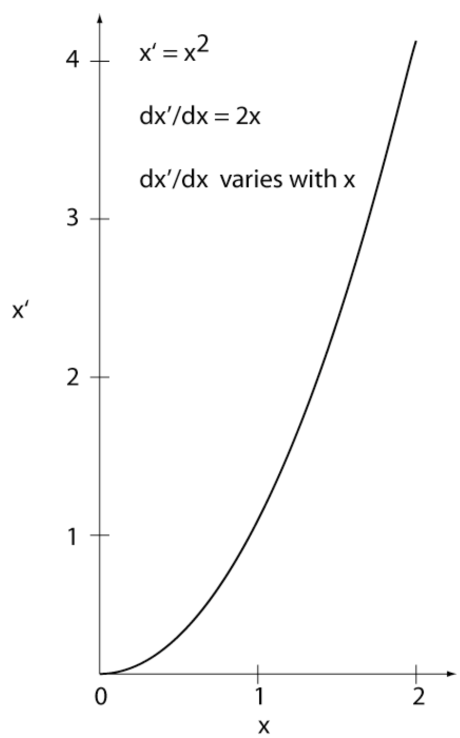
$u = x$ Lagrangian displacement equation
$u = u(x) = x' - x = 2x - x = x$

$u = x'/2$ Eulerian displacement equation
$u = u(x) = x' - x = x' - '/2x = x'/2$

$\varepsilon = \Delta L/L_0 = \{ (x_2' - x_1') - (x_2 - x_1) \} / (x_2 - x_1)$
$= \{ (x_2' - x_2) - (x_1' - x_1) \} / (x_2 - x_1) = \{u_2 - u_1\} / (x_2 - x_1) = \Delta u/\Delta x$

Example of inhomogeneous strain in one dimension





$x' = x^2$ Lagrangian coordinate transformation equation

$x = x'^{1/2}$ Eulerian coordinate transformation equation

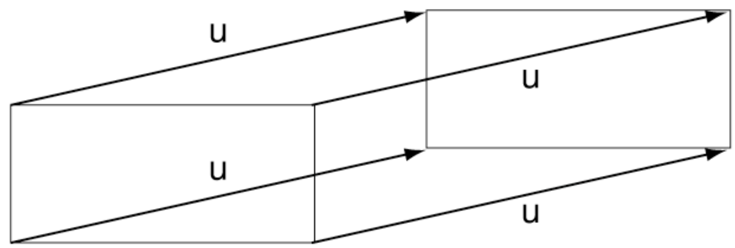
$u = x^2 - x$ Lagrangian displacement equation
$u = u(x) = x' - x = x^2 - x = (x)(x-1)$

$u = x' - x'^{1/2}$ Eulerian displacement equation
$u = u(x) = x' - x = x' - x'^{1/2} = (x'^{1/2})(x'^{1/2} - 1)$

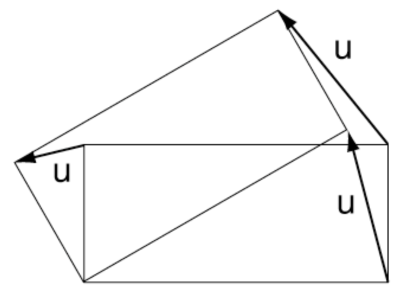
$\varepsilon = \lim_{\Delta x \to 0} \Delta L/L_0 = \lim_{\Delta x \to 0} \Delta u/\Delta x = du/dx = 2x - 1$

Rigid Body Motion

Rigid Body Translation

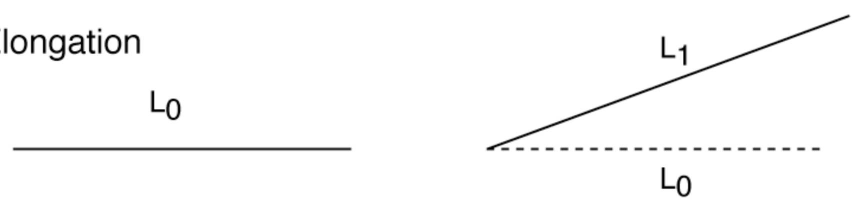


Rigid Body Rotation



Basic Measures of Strain

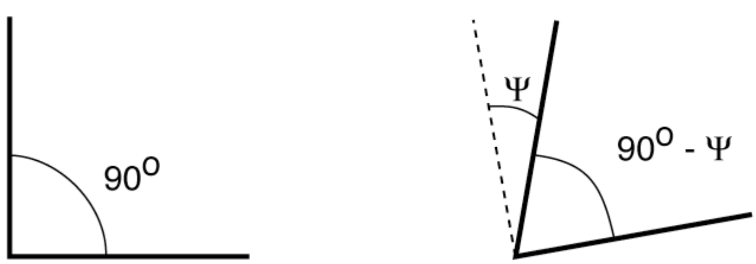
Elongation



$\varepsilon = (L_1 - L_0) / L_0$

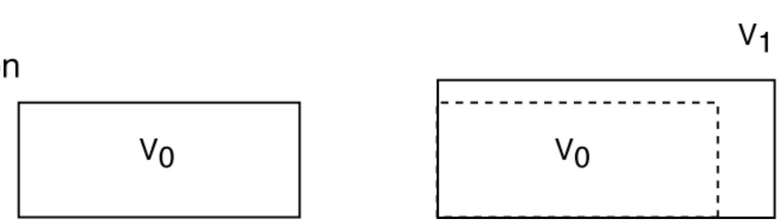
$S = L_1 / L_0$

Shear Strain



$\gamma = \tan \Psi$

Dilation



$\Delta = (V_1 - V_0) / V_0$

D Change in volume (dilational strain)

1 Dilation $\Delta = \frac{\Delta V}{V_o} = \frac{V_1 - V_o}{V_o}$ dimensionless!

2 Example 1: Consider a rectangular box with sides of length a_o, b_o, c_o, the sides lying along the 1,2,3 axes. Its volume is $a_o b_o c_o$, or

$$V_0 = \begin{vmatrix} a_o & 0 & 0 \\ 0 & b_o & 0 \\ 0 & 0 & c_o \end{vmatrix} = a_o b_o c_o$$

So V_0 is a determinant.

Suppose the box is stretched along the 1,2,3 axes such that its new dimensions are a_1, b_1, c_1. Its new volume V_1 is

$$V_1 = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{vmatrix} = \begin{vmatrix} a_0(1+\varepsilon_1) & 0 & 0 \\ 0 & b_0(1+\varepsilon_2) & 0 \\ 0 & 0 & c_0(1+\varepsilon_3) \end{vmatrix} = a_1 b_1 c_1$$

$$V_1 = a_0 b_0 c_0 \begin{vmatrix} (1+\varepsilon_1) & 0 & 0 \\ 0 & (1+\varepsilon_2) & 0 \\ 0 & 0 & (1+\varepsilon_3) \end{vmatrix} = a_0 b_0 c_0 \begin{vmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{vmatrix} = a_0 b_0 c_0 S_1 S_2 S_3$$

$$\Delta = \frac{V_1 - V_0}{V_0} = \frac{a_0 b_0 c_0 S_1 S_2 S_3 - a_0 b_0 c_0}{a_0 b_0 c_0} = S_1 S_2 S_3 - 1 \approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

(The expression at the right side applies for small strains [<~1%])