VECTORS, TENSORS, AND MATRICES

I Main Topics

A Vector length and direction

B Vector Products

C Tensor notation vs. matrix notation

II Vector Products

A Vector length: $|\mathbf{A}| = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}$

B A vector **A** can be defined by its length |**A**| and the direction of a unit vector **a** that is in the same direction as **A**. The unit vector **a** has x,y,z components $A_x\mathbf{i}/|\mathbf{A}|, A_y\mathbf{j}/|\mathbf{A}|,\ and\ A_z\mathbf{k}/|\mathbf{A}|$, respectively, where **i**,**j**, and **k** are unit vectors along the x,y, and z axes, respectively. $\mathbf{A} = |\mathbf{A}|\mathbf{a}$.

C Example: If $\mathbf{A} = \mathbf{0i} + \mathbf{3j} + \mathbf{4k}$, then $|\mathbf{A}| = \sqrt{0^2 + 3^2 + 4^2} = \sqrt{25} = 5$, and $\mathbf{a} = \frac{\mathbf{0}}{\mathbf{5}}\mathbf{i} + \frac{\mathbf{3}}{\mathbf{5}}\mathbf{j} + \frac{\mathbf{4}}{\mathbf{5}}\mathbf{k}$.

II Products of Vectors

A Dot product: $\mathbf{A} \bullet \mathbf{B} = M$

1 **A** and **B** are vectors, and M is a scalar corresponding to a length.

2 If unit vectors **a** and **b** parallel vectors **A** and **B**, respectively, and the angle from **a** to **b** (and from **A** to **B**) is θ_{ab}, then, recalling that

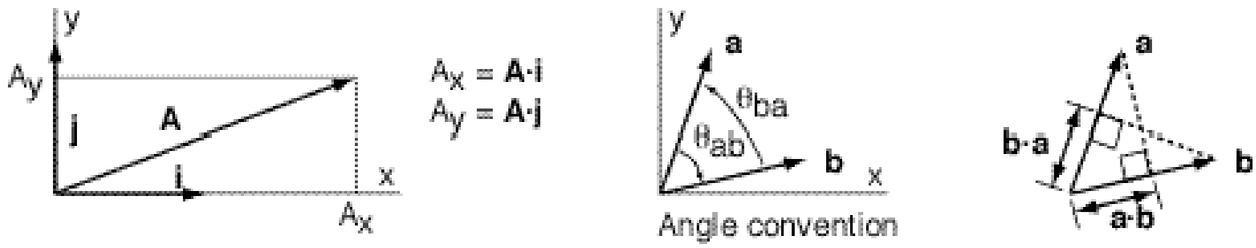
$$\cos\theta_{ab} = \cos(-\theta_{ab}) = \cos(\theta_{ba}) = \cos(\theta) \ldots$$

a $\mathbf{a} \bullet \mathbf{b} = \cos\theta = \mathbf{b} \bullet \mathbf{a}$

b $\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}|\mathbf{a} \bullet |\mathbf{B}|\mathbf{b} = |\mathbf{A}||\mathbf{B}|(\mathbf{a} \bullet \mathbf{b}) = |\mathbf{A}||\mathbf{B}|(\cos\theta)$

c Example: If $\mathbf{A} = 2\mathbf{i} + 0\mathbf{j} + 0\mathbf{K},\ and\ \mathbf{B} = \mathbf{0i} + \mathbf{2j} + \mathbf{0K}, \mathbf{A} \bullet \mathbf{B} = (2)(2)\cos(90^\circ) = 0$

3 If **b** is a unit vector, then $\mathbf{A} \bullet \mathbf{b}$ (or $\mathbf{b} \bullet \mathbf{A}$) is the length of the projection of **A** onto the direction defined by **b.**



4 Dot product tables of Cartesian basis vectors

	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$			$B_x\mathbf{i}$	$B_y\mathbf{j}$	$B_z\mathbf{k}$
$\mathbf{i}\bullet$	1	0	0		$A_x\mathbf{i}\bullet$	A_xB_x	0	0
$\mathbf{j}\bullet$	0	1	0		$A_y\mathbf{j}\bullet$	0	A_yB_y	0
$\mathbf{k}\bullet$	0	0	1		$A_z\mathbf{k}\bullet$	0	0	A_zB_z

5 $\mathbf{A} \bullet \mathbf{B} = \left(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\right)\left(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}\right) = A_xB_x + A_yB_y + A_zB_z$

6 For unit vectors $\mathbf{e}_r$ and $\mathbf{e}_s$ along axes of a Cartesian frame

a $\mathbf{e}_r \bullet \mathbf{e}_s = 1 \;\; if \;\; r = s$

b $\mathbf{e}_r \bullet \mathbf{e}_s = 0 \;\; if \;\; r \neq s$

7 In Matlab, $\mathrm{C} = \mathbf{A} \bullet \mathbf{B}$ is performed as C=A(:)'*B(:) *or* C=sum(A.*B)

8 Uses in geology for dot products: all kinds of projections

B Cross product: $\mathbf{A} \times \mathbf{B} = \mathbf{C}$

1 **C** is a <u>vector</u> perpendicular to both **A** and **B**, so **C** is perpendicular to the plane containing **A** and **B**. **C** points in the direction of our thumb if the other fingers on your right hand first point in the direction of **A** and then curl to point in the direction of **B.** (i.e., **A**, **B**, and **C** form a right-handed set). As a result, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

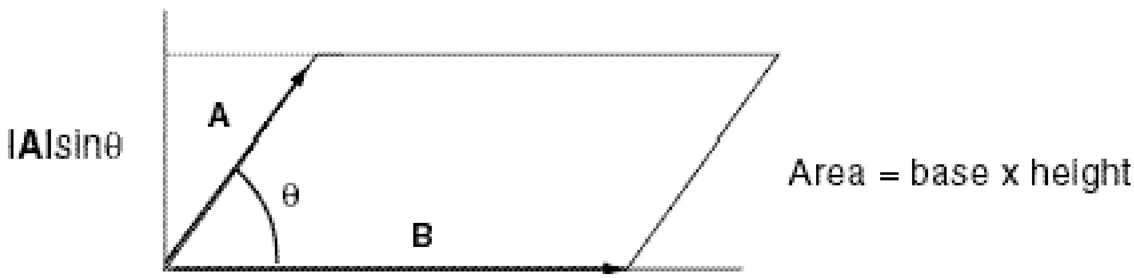
2 If unit vectors **a** and **b** parallel vectors **A** and **B**, respectively, and the angle between **a** and **b** (and between **A** and **B**) is θ, then ...

a $\mathbf{a} \times \mathbf{b} = \sin\theta\ \mathbf{n}$, where **n** is a unit vector normal to the **a,b** plane

b $\mathbf{A} \times \mathbf{B} = |\mathbf{A}|\mathbf{a} \times |\mathbf{B}|\mathbf{b} = |\mathbf{A}||\mathbf{B}|(\mathbf{a} \times \mathbf{b}) = |\mathbf{A}||\mathbf{B}|(\sin\theta)\mathbf{n}$

c Example: If $\mathbf{A} = 2\mathbf{i} + 0\mathbf{j} + 0\mathbf{k},\ and\ \mathbf{B} = \mathbf{0i} + \mathbf{2j} + \mathbf{0k},\ \mathbf{A} \bullet \mathbf{B} = (2)(2)\sin(90^\circ)\mathbf{k} = 4\mathbf{k}$

3 The length (magnitude) of **C** is the <u>area</u> of the parallelogram defined by vectors **A** and **B**, where **A** and **B** are along adjacent side of the parallelogram. In the figure below, **AxB** points into the page, and **BxA** points out of the page.



4 Cross product tables of Cartesian basis vectors

	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$			$B_x\mathbf{i}$	$B_y\mathbf{j}$	$B_z\mathbf{k}$
$\mathbf{i}\times$	0	$\mathbf{k}$	$-\mathbf{j}$		$A_x\mathbf{i}\times$	$\mathbf{0}$	$A_xB_y\mathbf{k}$	$-A_xB_z\mathbf{j}$
$\mathbf{j}\times$	$-\mathbf{k}$	0	$\mathbf{i}$		$A_y\mathbf{j}\times$	$-A_yB_x\mathbf{k}$	$\mathbf{0}$	$A_yB_z\mathbf{i}$
$\mathbf{k}\times$	$\mathbf{j}$	$-\mathbf{i}$	0		$A_z\mathbf{k}\times$	$A_zB_x\mathbf{j}$	$-A_zB_y\mathbf{i}$	$\mathbf{0}$

5
$$\mathbf{A} \bullet \mathbf{B} = \left(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\right)\left(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}\right)$$
$$= \left(A_yB_z - A_zB_y\right)\mathbf{i} - \left(A_xB_z - A_zB_x\right)\mathbf{j} + \left(A_xB_y - A_yB_x\right)\mathbf{k}$$

6 $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

7 For unit vectors $\mathbf{e}_r$ and $\mathbf{e}_s$ along axes of a Cartesian frame

a $\mathbf{e}_p \times \mathbf{e}_q = \mathbf{e}_r \;\; if \;\; p,q = 1,2 \;\; or \;\; 2,3 \;\; or \;\; 3,1$

b $\mathbf{e}_r \times \mathbf{e}_q = -\mathbf{e}_p \;\; if \;\; r,q = 3,2 \;\; or \;\; 2,1 \;\; or \;\; 1,3$

b $\mathbf{e}_p \times \mathbf{e}_q = 0 \;\; if \;\; p = q$

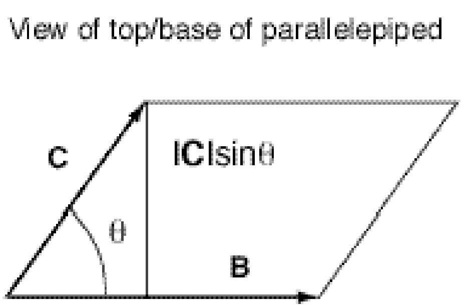
8 In Matlab, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is performed as C=cross(A,B)

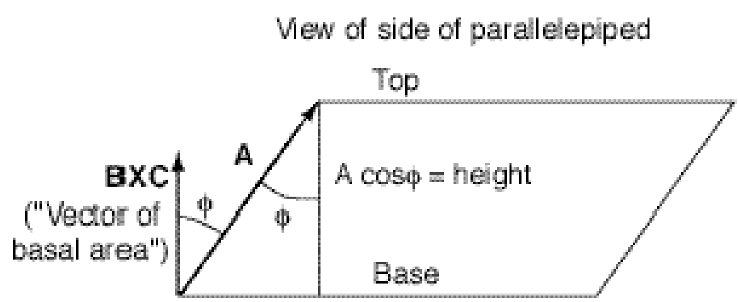
9 Uses in geology for cross products: finding poles to planes in three-point problems; finding fold axes from poles to bedding.

C Scalar triple product: $(\mathbf{A,B,C}) = \mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = V$

1 The vector triple product is a scalar (i.e., a number) that corresponds to a <u>volume</u>.

2 |V| is the volume of a parallelepiped with edges along **A**, **B**, and **C.** (**B**x**C**) gives the area of the base, and the dot product of this with **A** gives the base times the component of **A** normal to the base (i.e., the base times the height). The absolute value of V guarantees that the volume is non-negative.





Basal area = (C sinθ)(B) = |**BxC**|

Volume = (base)(height) = |**BxC**|(A cosφ) = |**A·(BxC)**|

3 $V = \mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \bullet \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{matrix} A_x(B_y C_z - B_z C_y) - \\ A_y(B_x C_z - B_z C_x) + \\ A_z(B_x C_y - B_y C_x) \end{matrix}$

4 The determinant of a 3x3 matrix gives the volume of a parallelepiped.

5 In Matlab, $V = \mathbf{A} \bullet (\mathbf{B} \times \mathbf{C})$ is performed as V = sum(A.*cross(B,C))

6 Use in geology: solutions of equations, estimating volume of ore bodies

7 <u>If at least two of the vectors **A,B,C** are parallel to each other, then **A,B,C** cannot define a parallelepiped, at least two rows of the matrix in (3) are linearly dependent, and the determinant of (3) is zero, and the three planes defined by **A,B,C** will not intersect in a unique point</u>

8 Proof that $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is perpendicular to the plane of **A** and **B**

a If C is <u>not</u> perpendicular to the **AB** plane, then C must be non-perpendicular to both **A** and **C**, i.e., $A \bullet C \neq 0 \;\; and \;\; A \bullet B \neq 0$.

b $|\mathbf{A} \bullet \mathbf{C}| = |\mathbf{A} \bullet (\mathbf{A} \times \mathbf{B})| = |\mathbf{B} \bullet (\mathbf{A} \times \mathbf{A})| = \mathbf{0}$

c $|\mathbf{B} \bullet \mathbf{C}| = |\mathbf{B} \bullet (\mathbf{A} \times \mathbf{B})| = |\mathbf{A} \bullet (\mathbf{B} \times \mathbf{B})| = \mathbf{0}$

d The postulate that C is <u>not</u> perpendicular to the **AB** plane thus is <u>disproved</u>, so C <u>is</u> perpendicular to the **AB** plane.

D Invariants

1 Quantities that do not depend on the orientation of a coordinate system.

2 Examples

a Dot product of two vectors (a length)

b Scalar triple product (a volume)