SCALARS, VECTORS, AND TENSORS

I Main Topics
- A Why deal with tensors?
- B Order of scalars, vectors, and tensors
- C Linear transformation of scalars and vectors (and tensors)

II Why deal with tensors?
- A They broaden how we can look at the world; geologists who don't become acquainted with them are handi-capped.
- B They can be extremely useful

III "Order" of scalars, vectors, and tensors
- A Scalars (magnitudes)
  - 1 Numbers with no associated direction (zero-order tensors)
  - 2 No subscripts in notation
  - 3 Examples: Time, mass, length volume
  - 4 Matrix representation: 1x1 matrix $[x]$
- B Vectors (magnitude and a direction)
  - 1 Quantities with one associated direction (first-order tensors)
  - 2 One subscript in notation (e.g., u_x)
  - 3 Examples: Displacement, velocity, acceleration
  - 4 Matrix representation: 1xn row matrix, or nx1 column matrix, with n components
    - a Two-dimensional vector (2 components):

      $[x \quad y]$ or $[x_1 \quad x_2]$ 1 row, 2 columns

      x = component in x-direction, y = component in y-direction

      x_1 = component in x-direction, x_2 = component in y-direction b Three-dimensional vector (3 components):

$[x \quad y \quad z]$ or $[x_1 \quad x_2 \quad x_3]$ 1 row, 3 columns

5 Don't confuse the dimensionality of a tensor with its order

C Tensors (magnitude and two directions)

1 Quantities with two associated directions (second-order tensors)

2 Two subscripts in notation (e.g., σ_{ij})

3 Examples: Stress, strain, permeability

4 Matrix representation: nxn matrix, with n^2 components

a Two-dimensional tensor (4 components):

$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}$ or $\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ 2 rows, 2 columns

b Three-dimensional vector (9 components):

$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$ or $\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$ 3 rows, 3 columns

5 An n-dimension tensor consists of n rows of n-dimension vectors

IV Linear transformation of scalars, vectors, and tensors

A "Transformations" refers to how components change when the coordinate system changes in which the quantities are measured. "Linear" means the transformation depends on the length of the components, not, for example, on the square of the component lengths. Transformations are used to when we change reference frames in order to present physical quantities from a different (clearer) perspective.

B Scalars

1 Scalar quantities don't change in response to a transformation of coordinates; they are invariant

2 Examples: The mass, volume, and density of a body are independent of the reference frame orientation.

C Vectors

1 Vector components change with a transformation of coordinates (see Figures 6.1-6.3 for examples of a rotation)

2 Example: the components of a position vector change from a coordinate system where x = east, y = north, z = up to a new system where x' = north, y' =west, z' = down (see Fig. 2.1)

3 Transformation rule for vectors (See Figs. 6.1-6.3)

a Expanded form (explicit but can be tedious to write)

i
$$\begin{aligned} x_{1'} &= a_{1'1}x_1 + a_{1'2}x_2 \\ x_{2'} &= a_{2'1}x_1 + a_{2'2}x_2 \end{aligned} \qquad (2\text{-D})$$

$$\begin{aligned} x_{1'} &= a_{1'1}x_1 + a_{1'2}x_2 + a_{1'3}x_3 \\ x_{2'} &= a_{2'1}x_1 + a_{2'2}x_2 + a_{2'3}x_3 \\ x_{3'} &= a_{3'1}x_1 + a_{3'2}x_2 + a_{3'3}x_3 \end{aligned} \qquad (3\text{-D})$$

$a_{j'i}$ is the component of a unit vector in the i-direction that points in the j'-direction (i.e., a direction cosine).

ii Every component in the unprimed reference frame can contribute to each component in the primed reference frame. The direction cosines are weighting factors that specify how much each unprimed component contributes to a primed component.

b Summation notation $x_{j'} = \sum_{i=1}^{n} a_{j'i} \; x_i$

The resultant vector in the x'_j direction is the sum of the components that act in that direction

c Matrix form $[x'] = [a][x]$ (or in Matlab X=a*x)

where [a] is a rotation matrix containing the direction cosines of the angles between (unit vectors along) the x,y,z axes and the x',y',z' axes.

In expanded matrix form $\begin{bmatrix} x_{1'} \\ x_{2'} \end{bmatrix} = \begin{bmatrix} a_{1'1} & a_{1'2} \\ a_{2'1} & a_{2'2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (2-D)

In expanded matrix form $\begin{bmatrix} x_{1'} \\ x_{2'} \\ x_{3'} \end{bmatrix} = \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (3-D)

To multiply two matrices [A] and [B], the row elements in [A] multiply the column elements in B. So if [A] is an **nxm matrix** (**n = # of rows, m = # of columns**), [B] must be an **mxp** matrix (m = # of rows, p = # of columns). <u>The number of columns in the first matrix must match the number of rows in the second matrix.</u> The solution [C] will be an **nxp** matrix, (n = # of rows, p = # of columns). The elements of [C] are given by:

$$C_{ij} = \sum_{k=1}^{k=m} A_{ik} \; B_{kj}$$

Examples

A 1x2 matrix times a 2x1 matrix gives a 1x1 matrix

$$\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}(1)(3)+(2)(4)\end{bmatrix} = \begin{bmatrix}11\end{bmatrix}$$

A 2x1 matrix times a 1x2 matrix gives a 2x2 matrix

$$\begin{bmatrix}3\\4\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix} = \begin{bmatrix}(3)(1) & (3)(2)\\(4)(1) & (4)(2)\end{bmatrix} = \begin{bmatrix}3 & 6\\4 & 8\end{bmatrix}$$

A 2x2 matrix times a 2x1 matrix gives a 2x1 matrix

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}(1)(5)+(2)(6)\\(3)(5)+(4)(6)\end{bmatrix} = \begin{bmatrix}17\\39\end{bmatrix}$$

A 2x2 matrix times a 2x2 matrix gives a 2x2 matrix

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}(1)(1)+(2)(0) & (1)(0)+(2)(1)\\(3)(1)+(4)(0) & (3)(0)+(4)(1)\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$$

A 3x2 matrix times a 2x2 matrix gives a 3x2 matrix

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}7 & 8\\9 & 10\end{bmatrix} = \begin{bmatrix}(1)(7)+(2)(9) & (1)(8)+(2)(10)\\(3)(7)+(4)(9) & (3)(8)+(4)(10)\\(5)(7)+(6)(9) & (5)(8)+(6)(10)\end{bmatrix} = \begin{bmatrix}25 & 28\\57 & 64\\89 & 100\end{bmatrix}$$

A 3x2 matrix times a 2x3 matrix gives a 3x3 matrix

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}7 & 8 & 9\\10 & 11 & 12\end{bmatrix} = \begin{bmatrix}(1)(7)+(2)(10) & (1)(8)+(2)(11) & (1)(9)+(2)(12)\\(3)(7)+(4)(10) & (3)(8)+(4)(11) & (3)(9)+(4)(12)\\(5)(7)+(6)(10) & (5)(8)+(6)(11) & (5)(9)+(6)(12)\end{bmatrix} = \begin{bmatrix}27 & 30 & 3\\61 & 68 & 7\\95 & 106 & 1\end{bmatrix}$$

d Tensor notation $x_{j'} = a_{j'i}\ x_i$; double subscripts imply summation

D Second-order tensors (to be covered later)

1 Components change with a transformation of coordinates
2 Transformation rule for tensors is somewhat analogous to that for vectors except there are two rotations involved.

3 In tensor notation: $\sigma'_{i'j'} = a_{i'k} a_{j'l} \sigma_{kl} = a_{i'k} \sigma_{kl} a_{j'l}$; i,j,k,l = (1,2,3)

Consider any two vectors u and v that we wish to transform into a new coordinate system

$u_{i'} = a_{i'k} u_k$ 3x1 matrix = (3x3 matrix)(3x1 matrix) column matrix

$v_{j'} = a_{j'l} v_l$ 3x1 matrix = (3x3 matrix)(3x1 matrix) column matrix

$v_{j'} = v_l a_{j'l}$ 1x3 matrix = (1x3 matrix)(3x3 matrix) row matrix

Now suppose each component of $u_{i'}$ is to be multiplied by each component of $v_{j'}$ to yield a total of 9 components. To do this, we need to multiply a 3x1 (column) matrix by a 1x3 (row) matrix.

$$u_{i'} v_{j'} = \left(a_{i'k} u_k\right)\left(v_l a_{j'l}\right)$$

In this tensor notation we can rearrange the order of the terms

$$u_{i'} v_{j'} = a_{i'k} a_{j'l} u_k v_l$$

Now we let the products $u_{i'} v_{j'}$ be called $A_{i'j'}$, and $u_k u_l$ be called A_{kl}

$$A_{i'j'} = a_{i'k} a_{j'l} A_{kl}$$

Quantities that transform like $A_{i'j'}$, and A_{kl} are known as second-order tensors.

4 Transformation by matrix multiplication (watch the order of the matrices)

$$\left[A_{i'j'}\right] = \left[a_{i'k}\right]\left[A_{kl}\right]\left[a_{i'k}\right]^T$$

$$\begin{bmatrix} \sigma_{1'1'} & \sigma_{1'2'} & \sigma_{1'3'} \\ \sigma_{2'1'} & \sigma_{2'2'} & \sigma_{2'3'} \\ \sigma_{3'1'} & \sigma_{3'2'} & \sigma_{3'3'} \end{bmatrix} = \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{1'1} & a_{2'1} & a_{3'1} \\ a_{1'2} & a_{2'2} & a_{3'2} \\ a_{1'3} & a_{2'3} & a_{3'3} \end{bmatrix}$$

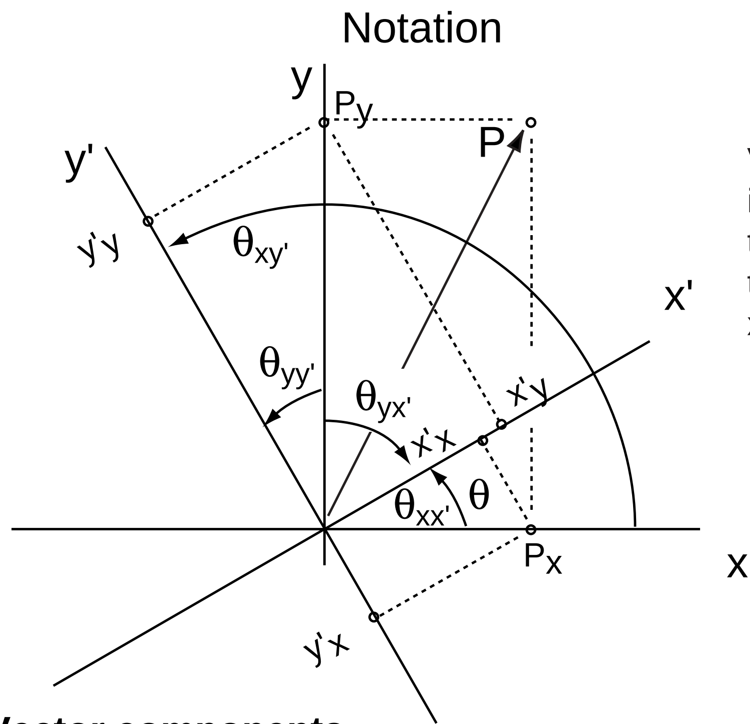


Fig. 6.1

Vector P can be considered in terms of its components in the x,y reference frame, or in terms of components in the x',y' reference frame.

Vector components

The x component of P (i.e., P_x) has components in the x' and y' directions:
x'_x = x' component of x y'_x = y' component of x

The y component of P (i.e., P_y) has components in the x' and y' directions:
x'_y = x' component of y y'_y = y' component of y

Angles

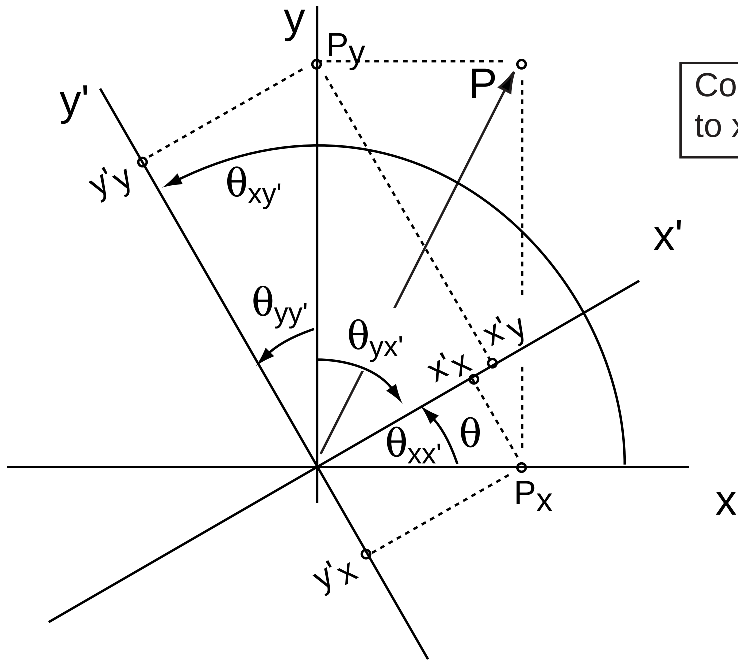
$\theta_{xx'}$ is the angle from the x axis to the x' axis; $\theta_{xx'} = -\theta_{x'x}$
$\theta_{xy'}$ is the angle from the x axis to the y' axis; $\theta_{xy'} = -\theta_{y'x}$
$\theta_{yx'}$ is the angle from the y axis to the x' axis; $\theta_{yx'} = -\theta_{x'y}$
$\theta_{yy'}$ is the angle from the y axis to the y' axis; $\theta_{yy'} = -\theta_{y'y}$
Switching the order of the angle subscripts changes the sign of the angle

Direction cosines

$a_{xx'} = \cos\theta_{xx'} = \cos(-\theta_{xx'}) = \cos\theta_{x'x} = a_{xx'} = \cos\theta$
$a_{xy'} = \cos\theta_{xy'} = \cos(-\theta_{xy'}) = \cos\theta_{y'x} = a_{yx'} = -\sin\theta$
$a_{yx'} = \cos\theta_{yx'} = \cos(-\theta_{yx'}) = \cos\theta_{x'y} = a_{xy'} = \sin\theta$
$a_{yy'} = \cos\theta_{yy'} = \cos(-\theta_{yy'}) = \cos\theta_{y'y} = a_{yy'} = \cos\theta$
Switching the order of the cosine subscripts does not change the sign of the cosine.
This makes using direction cosines somewhat immune to sign errors.

Transformation of 2-D Vectors (I)

Fig. 6.2



Conversion from x,y components to x',y' components

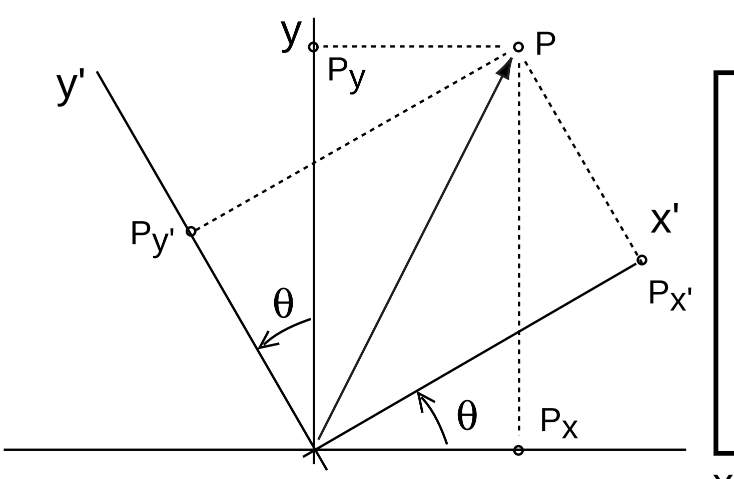
Note that $\theta_{x'x} = \theta$

P_x has components in the x' and y' directions:
$x'_x = (a_{x'x})\,(x) = (\cos\theta)(x)$ $y'_x = (a_{y'x})\,(x) = (-\sin\theta)(x)$

P_y has components in the x' and y' directions:
$x'_y = (a_{x'y})\,(y) = (\sin\theta)(y)$ $y'_y = (a_{y'y})\,(x) = (\cos\theta)(y)$

The x' and y' components of P are then:

$x' = x'_x + x'_y$ $x' = a_{x'x}\,x + a_{x'y}\,y$ $x' = \cos(\theta)\,x + \sin(\theta)\,y$
$y' = y'_x + y'_y$ $y' = a_{y'x}\,x + a_{y'y}\,y$ $y' = -\sin(\theta)\,x + \cos(\theta)\,y$



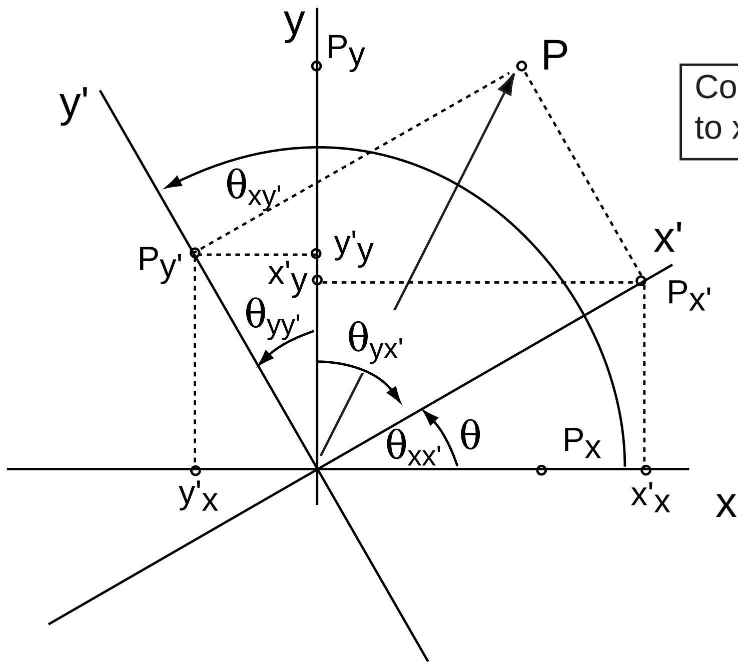
In matrix notation these two equations are written:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation matrix with direction cosines

Transformation of 2-D Vectors (II)

Fig. 6.3



Conversion from x',y' components to x,y components

Note that $\theta_{x'x} = \theta$

$P_{x'}$ (i.e., x') has components in the x and y directions:

$x'_x = (a_{xx'})\ (x') = (\cos\theta)(x')$ $\quad$ $x'_y = (a_{yx'})\ (x') = (\sin\theta)(x')$

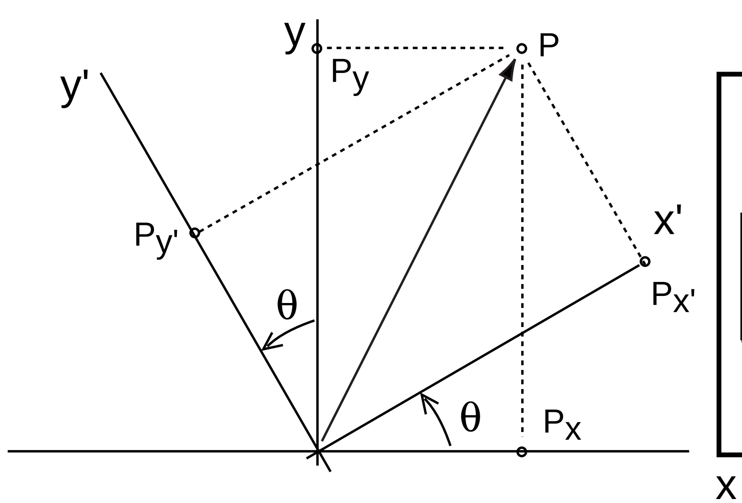
$P_{y'}$ (i.e., y') has components in the x and y directions:

$y'_x = (a_{xy'})\ (y') = (-\sin\theta)(y')$ $\quad$ $y'_y = (a_{yy'})\ (x) = (\cos\theta)(y')$

The x and y components of P are then:

$x = x'_x + y'_x$ $\quad$ $x = a_{xx'}\,x' + a_{xy'}\,y'$ $\quad$ $x = \cos(\theta)\,x' - \sin(\theta)\,y'$

$y = x'_y + y'_y$ $\quad$ $y = a_{yx'}\,x' + a_{yy'}\,y'$ $\quad$ $y = \sin(\theta)\,x' + \cos(\theta)\,y'$



In matrix notation these two equations are written:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Rotation matrix with direction cosines