22. Stresses Around a Hole (II)

I Main Topics

A General solution for a plane strain case

B Boundary conditions

C Solution that honors boundary conditions

D Significance of solution

E Superposition

F Stress concentrations

21. Stresses Around a Hole (I)

Ship Rock, New Mexico

http://jencarta.com/images/aerial/ShipRock.jpg

Hydraulic Fracture

From Wu et al., 2007

22. Stresses Around a Hole (II)

II General solution for a plane strain case

Start with the governing equation

$$0 = \frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2}$$

Consider a power series solution for u_r and its derivatives

$$u_r = \ldots \quad C_{-3}r^{-3} + \quad C_{-2}r^{-2} + \quad C_{-1}r^{-1} + \; C_0r^0 + \quad C_1r^1 + \quad C_2r^2 + \quad C_3r^3 + \ldots$$

$$\frac{du_r}{dr} = \ldots - 3C_{-3}r^{-4} - 2C_{-2}r^{-3} - 1C_{-1}r^{-2} + 0C_0r^{-1} + 1C_1r^0 + 2C_2r^1 + 3C_3r^2 + \ldots$$

$$\frac{d^2u_r}{dr^2} = \ldots \; 12C_{-3}r^{-5} + 6C_{-2}r^{-4} + 2C_{-1}r^{-3} + 0C_0r^{-2} + 0C_1r^{-1} + 2C_2r^0 + 6C_2r^1 \ldots$$

22. Stresses Around a Hole (II)

II General Solution for a plane strain case

Now substitute the series solutions into the governing equation $0 = \frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{1}{r^2}u_r$

$$\begin{aligned} 0 = \; & \ldots \; 12C_{-3}r^{-5} + 6C_{-2}r^{-4} + 2C_{-1}r^{-3} + 0C_0r^{-2} + 0C_1r^{-1} + 2C_2r^0 + 6C_3r^1 \ldots \\ & + \frac{1}{r}\left(\ldots - 3C_{-3}r^{-4} - 2C_{-2}r^{-3} - 1C_{-1}r^{-2} + 0C_0r^{-1} + 1C_1r^0 + 2C_2r^1 + 3C_3r^2 + \ldots\right) \\ & - \frac{1}{r^2}\left(\ldots C_{-3}r^{-3} + C_{-2}r^{-2} + C_{-1}r^{-1} + C_0r^0 + C_1r^1 + C_2r^2 + C_3r^3 + \ldots\right) \end{aligned}$$

$$\begin{aligned} 0 = \; & \ldots \; 12C_{-3}r^{-5} + 6C_{-2}r^{-4} + 2C_{-1}r^{-3} + 0C_0r^{-2} + 0C_1r^{-1} + 2C_2r^0 + 6C_3r^1 + \ldots \\ & + \left(\ldots - 3C_{-3}r^{-5} - 2C_{-2}r^{-4} - 1C_{-1}r^{-3} + 0C_0r^{-2} + 1C_1r^{-1} + 2C_2r^0 + 3C_3r^1 + \ldots\right) \\ & - \left(\ldots \; C_{-3}r^{-5} + C_{-2}r^{-4} + C_{-1}r^{-3} + C_0r^{-2} + C_1r^{-1} + C_2r^0 + C_3r^1 + \ldots\right) \end{aligned}$$

22. Stresses Around a Hole (II)

II General Solution for a plane strain case

$$
\begin{aligned}
0 = \; & \ldots \; 12C_{-3}r^{-5} + 6C_{-2}r^{-4} + 2C_{-1}r^{-3} + 0C_0r^{-2} + 0C_1r^{-1} + 2C_2r^0 + 6C_3r^1 + \ldots \\
& + \left(\ldots - 3C_{-3}r^{-5} - 2C_{-2}r^{-4} - 1C_{-1}r^{-3} + 0C_0r^{-2} + 1C_1r^{-1} + 2C_2r^0 + 3C_3r^1 + \ldots\right) \\
& + \left(\ldots - 1C_{-3}r^{-5} - 1C_{-2}r^{-4} - 1C_{-1}r^{-3} - 1C_0r^{-2} - 1C_1r^{-1} - 1C_2r^0 - 1C_3r^1 - \ldots\right)
\end{aligned}
$$

Now collect terms of the same powers

$$0 = \; \ldots \; 8C_{-3}r^{-5} + 3C_{-2}r^{-4} + 0C_{-1}r^{-3} - 1C_0r^{-2} + 0C_1r^{-1} + 3C_2r^0 + 8C_3r^1 + \ldots$$

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22. Stresses Around a Hole (II)

II General Solution for a plane strain case

$$0 = \ldots 8C_{-3}r^{-5} + 3C_{-2}r^{-4} + 0C_{-1}r^{-3} - 1C_0r^{-2} + 0C_1r^{-1} + 3C_2r^0 + 8C_3r^1 + \ldots$$

For this to hold for *all* values of *r*, the product of each leading coefficient and constant must equal 0 because the powers of *r* are linearly independent. *All* coefficients *except* C_{-1} and C_1 thus must be zero.

$$u_r = \ldots C_{-3}r^{-3} + C_{-2}r^{-2} + C_{-1}r^{-1} + C_0r^0 + C_1r^1 + C_2r^2 + C_3r^3 + \ldots$$

$$u_r = C_{-1}r^{-1} + C_1r^1$$

- General solution for radial displacements
- Solve for constants via boundary conditions

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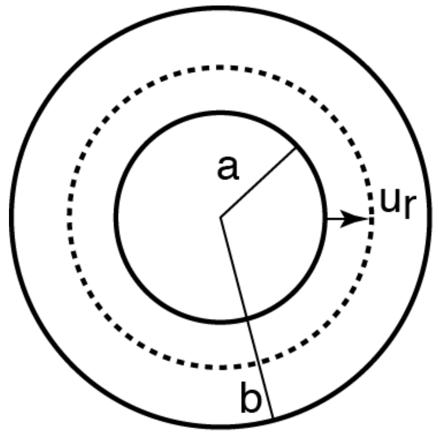
22. Stresses Around a Hole (II)

III Boundary conditions

A Two boundary conditions must be specified to solve our problem because our general solution has two unknown coefficients:

$$u_r = C_{-1}r^{-1} + C_1 r^1$$

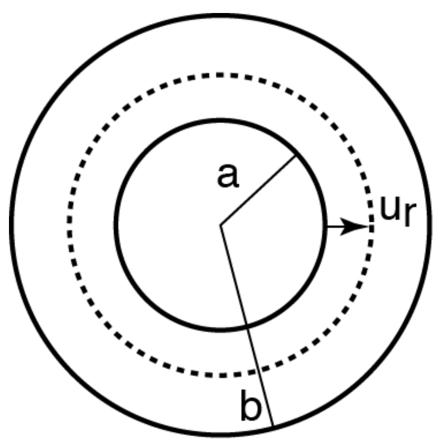


22. Stresses Around a Hole (II)

III Boundary conditions: Radial displacements

B $u_r = u_0$ at the wall of the hole: $u_r|_{r=a} = u_0$

C $u_r = 0$ at infinity: $u_r|_{r=b=\infty} = 0$



22. Stresses Around a Hole (II)

IV Solution that honors boundary conditions

Gov. eq.: $u_r = C_{-1}r^{-1} + C_1r^1$

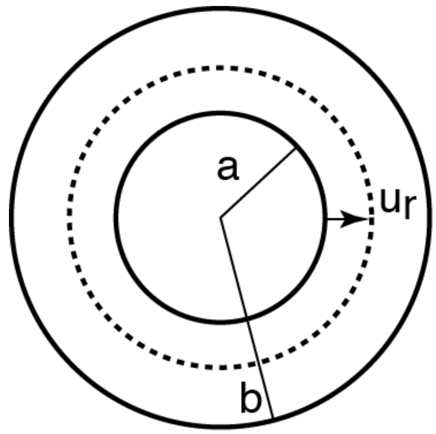
BC 1: $u_r|_{r=a} = u_0$

BC 2: $u_r|_{r=b=\infty} = 0$

As $r \rightarrow \infty$, $u_r \rightarrow C_1r$

BC 2 requires $C_1 = 0$, so

$u_r = C_{-1}r^{-1}$



22. Stresses Around a Hole (II)

IV Solution that honors boundary conditions

Gov. eq.: $u_r = C_{-1}r^{-1} + C_1r^1$

BC 1: $u_r|_{r=a} = u_0$

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As $r \rightarrow \infty$, $u_r \rightarrow C_1r$

BC 2 requires $C_1 = 0$, so

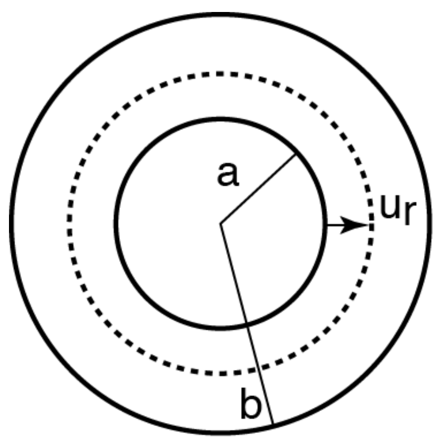
$u_r = C_{-1}r^{-1}$

By BC 1, $u_r|_{r=a} = u_0 = C_{-1}a^{-1}$

So, $C_{-1} = a\, u_0$

$u_r = (a/r)\, u_0$

Solution satisfying boundary conditions

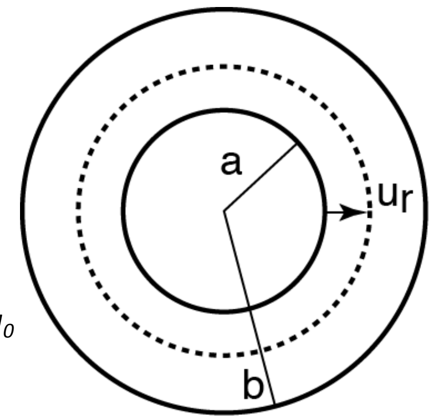


22. Stresses Around a Hole (II)

IV Solution that honors boundary conditions (cont.)

$u_r = (a/r)\, u_0$

- The hole radius *a* provides a scale
- The displacements decay with distance *r* from the hole (as suspected)
- The displacements scale with u_0
- Problems with different boundary conditions have different solutions



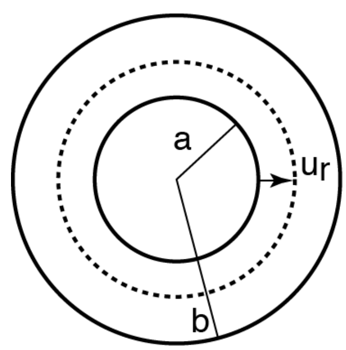
22. Stresses Around a Hole (II)

IV Solution that honors boundary conditions (cont.)

Strains

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{\partial\left(u_0 a r^{-1}\right)}{\partial r} = u_0 a \frac{\partial\left(r^{-1}\right)}{\partial r}$$

$$= \frac{-u_0 a}{r^2} = -u_0 a r^{-2}$$



$$\varepsilon_{\theta\theta} = \frac{u_r}{r} = \frac{u_0 a r^{-1}}{r} = \frac{u_0 a}{r^2} = u_0 a r^{-2}$$

$$\varepsilon_{r\theta} = 0$$

22. Stresses Around a Hole (II)

IV Solution that honors boundary conditions (cont.)

<u>Stresses</u> (in terms of u_0)

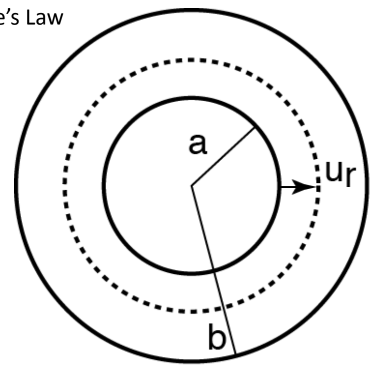
Hooke's Law

Strains

$$\sigma_{rr} = \frac{E}{(1+\nu)}\left[\varepsilon_{rr} + \frac{\nu}{(1-2\nu)}(\varepsilon_{rr} + \varepsilon_{\theta\theta})\right]$$

$$= \frac{E}{(1+\nu)}\left[\frac{-u_0 a}{r^2} + \frac{\nu}{(1-2\nu)}\left(\frac{-u_0 a}{r^2} + \frac{u_0 a}{r^2}\right)\right]$$

$$= \frac{E}{(1+\nu)}\left[\frac{-u_0 a}{r^2}\right]$$



22. Stresses Around a Hole (II)

IV Solution that honors boundary conditions (cont.)

<u>Stresses</u> (in terms of u_0)

Hooke's Law

Strains

$$\sigma_{rr} = \frac{E}{(1+\nu)}\left[\varepsilon_{rr} + \frac{\nu}{(1-2\nu)}(\varepsilon_{rr} + \varepsilon_{\theta\theta})\right]$$

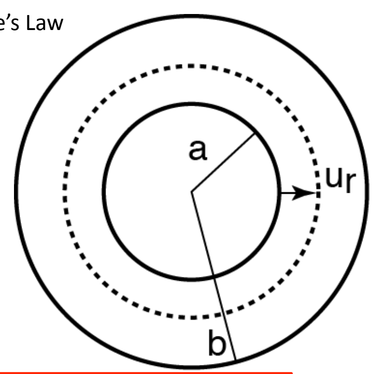
$$= \frac{E}{(1+\nu)}\left[\frac{-u_0 a}{r^2} + \frac{\nu}{(1-2\nu)}\left(\frac{-u_0 a}{r^2} + \frac{u_0 a}{r^2}\right)\right]$$

$$= \frac{E}{(1+\nu)}\left[\frac{-u_0 a}{r^2}\right]$$

$$\sigma_{\theta\theta} = \frac{E}{(1+\nu)}\left[\varepsilon_{\theta\theta} + \frac{\nu}{(1-2\nu)}(\varepsilon_{\theta\theta} + \varepsilon_{rr})\right]$$

$$= \frac{E}{(1+\nu)}\left[\frac{u_0 a}{r^2} + \frac{\nu}{(1-2\nu)}\left(\frac{u_0 a}{r^2} + \frac{-u_0 a}{r^2}\right)\right]$$

$$= \frac{E}{(1+\nu)}\left[\frac{u_0 a}{r^2}\right]$$



Same absolute magnitude, opposite sign; $\sigma_{rr} = -\sigma_{\theta\theta}$

$$\sigma_{r\theta} = 2G\varepsilon_{r\theta} = 0$$

22. Stresses Around a Hole (II)

IV Solution that honors boundary conditions (cont.)

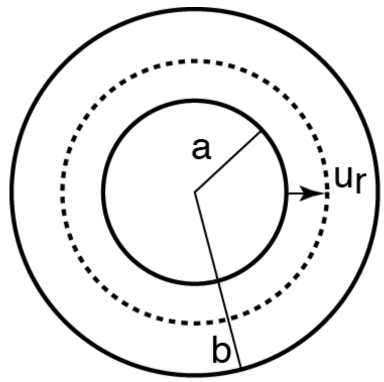
Stresses (in terms of tractions)

First evaluate the radial stress *on the wall of the hole* (*r=a*), which equals traction *T*, and from that solve for u_0

$$\sigma_{rr}\big|_{r=a} = \frac{E}{(1+\nu)}\left[\frac{-u_0 a}{a^2}\right] = \frac{E}{(1+\nu)}\left[\frac{-u_0}{a}\right] = T$$

$$u_0 = -aT \Big/ \frac{E}{(1+\nu)}$$

Now substitute for u_0 in the general expression for σ_{rr} on previous page



$$\sigma_{rr} = \frac{E}{(1+\nu)}\left[\frac{\left(aT \Big/ \frac{E}{(1+\nu)}\right)a}{r^2}\right] = T\left(\frac{a}{r}\right)^2$$

$$\sigma_{\theta\theta} = -\sigma_{rr} = -T\left(\frac{a}{r}\right)^2$$

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22. Stresses Around a Hole (II)

- Solution for in-plane displacements in terms of displacement boundary conditions *does not* depend on the elastic properties

$$u_r = (a/r)u_0$$
$$u_\theta = 0$$

- Solution for in-plane stresses in terms of traction (stress) boundary conditions *does not* depend on the elastic properties

$$\sigma_{\theta\theta} = -\sigma_{rr} = -T\left(\frac{a}{r}\right)^2$$
$$\sigma_{r\theta} = \sigma_{\theta r} = 0$$

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22. Stresses Around a Hole (II)

- Solution for in-plane stresses in terms of displacement boundary conditions *does* depend on the elastic properties
- Solution for in-plane displacements in terms of traction (stress) boundary conditions *does* depend on the elastic properties

$$\sigma_{rr} = \frac{-E}{(1+\nu)}\left[\frac{u_0 a}{r^2}\right]$$
$$\sigma_{\theta\theta} = \frac{E}{(1+\nu)}\left[\frac{u_0 a}{r^2}\right]$$

$$u_r = \frac{-(1+\nu)}{E}\frac{a}{r}aT$$
$$u_\theta = 0$$

22. Stresses Around a Hole (II)

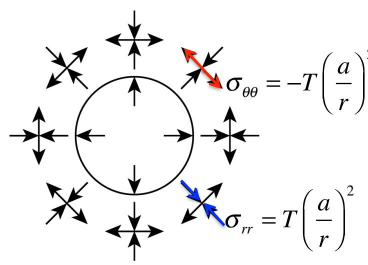
V Significance

For a ***pressure*** in a hole with no remote load at $r = \infty$:

A The radial normal stress σ_{rr} is a principal stress because $\sigma_{r\theta} = 0$; σ_{rr} is the most compressive stress.

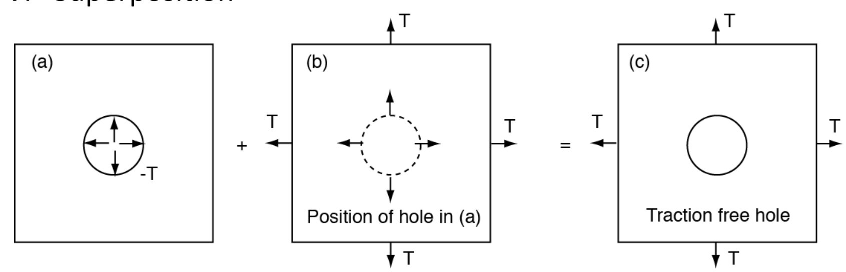
B The circumferential normal stress $\sigma_{\theta\theta}$ is a principal stress because $\sigma_{\theta r} = 0$; $\sigma_{\theta\theta}$ is the most tensile stress.

C A high pressure could cause radial cracking (e.g., radial dikes around a magma chamber).



Stresses Around a Hole (II)

VI Superposition



Pressure in a hole | Biaxial Tension (no hole)

$\sigma_{rr} = -T(a/r)^2$ + $\sigma_{rr} = T$ = $\sigma_{rr} = T - T(a/r)^2$

$\sigma_{\theta\theta} = -(-T)(a/r)^2$ $\quad \sigma_{\theta\theta} = T$ $\quad \sigma_{\theta\theta} = T + T(a/r)^2$

$\sigma_{\theta\theta}\big|_{r=a} = 2T$

Stress concentration!

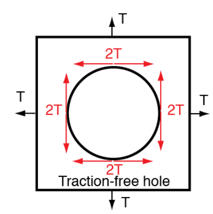
22. Stresses Around a Hole (II)

VII Stress concentrations

A The hole causes a doubling of the normal stress far from the hole (i.e.,a stress concentration)

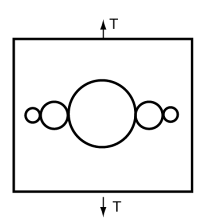
B The magnitude of the circumferential stress at the hole is independent of the size of the hole, so a tiny cylindrical hole causes the same stress concentration as a large one.

C A tiny hole near the wall of a larger hole might be expected to have an even larger stress concentration (why?)



$\sigma_{\theta\theta}\big|_{r=a} = 2T$

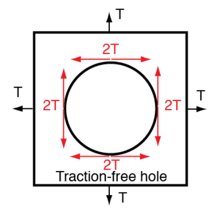
Stress concentration!



22. Stresses Around a Hole (II)

VI Stress concentrations

- D Stress concentrations explain myriad phenomena, such as
  1. Why paper doesn't explode when pulled upon hard
  2. Why paper tears along "the dotted line"
  3. Why cracks in riveted steel plates start from the rivet holes
  4. Why cracks in drying mudflats originate from the where grass stems have poked through the mud etc.
  5. Why "strong" rocks can fail under "low" stresses
  6. Why dikes can propagate through the Earth's crust
  7. Why seismic ruptures grow so large



$\sigma_{\theta\theta}|_{r=a} = 2T$

Stress concentration!