

21. Stresses Around a Hole (I)

- I Main Topics
 - A Introduction to stress fields and stress concentrations
 - B An axisymmetric problem
 - B Stresses in a polar (cylindrical) reference frame
 - C Equations of equilibrium
 - D Solution of boundary value problem for a pressurized hole

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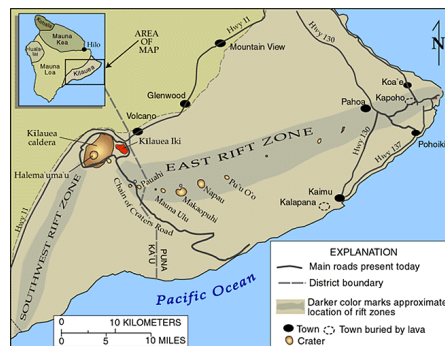
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21. Stresses Around a Hole (I)



<http://www.pacificautoglass.com>



http://hvo.wr.usgs.gov/kilauea/Kilauea_map.html

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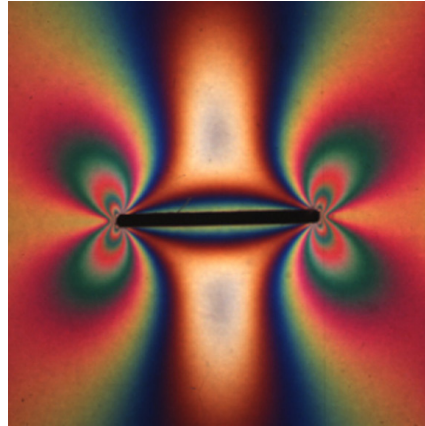
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21. Stresses Around a Hole (I)

II Introduction to stress fields and stress concentrations

A Importance

- 1 Stress (and strain and displacement) vary in space
- 2 Fields extend deformation concepts at a point
- 3 Stress concentrations can be huge and have a large effect



<http://pangea.stanford.edu/research/geomech/Faculty/crack.html>

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21. Stresses Around a Hole (I)

II Introduction to stress fields and stress concentrations (cont.)

B Common causes of stress concentrations

- 1 A force acts on a small area (e.g., beneath a nail being hammered)



http://0.tqn.com/d/homerepair/1/0/l/7/-/-/nail_set.jpg

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21. Stresses Around a Hole (I)

- II Introduction to stress fields and stress concentrations (cont.)
- B Common causes of stress concentrations
 - 2 Geometric effects (e.g., corners on doors and windows)



<http://www.basementsystems.com/foundation-repair/images/Door-Crack.jpg>

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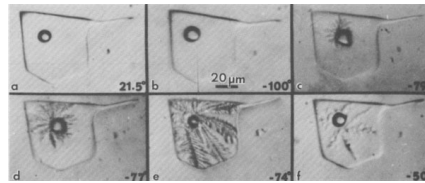
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21. Stresses Around a Hole (I)

- II Introduction to stress fields and stress concentrations (cont.)
- B Common causes of stress concentrations
 - 3 Material heterogeneities (e.g., mineral heterogeneities and voids)

Cracks near a fluid inclusion in halite



http://www.minsocam.org/msa/collectors_corner/arc/img/halite41

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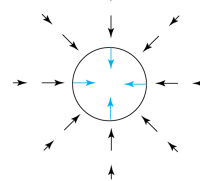
21. Stresses Around a Hole (I)

III An axisymmetric problem

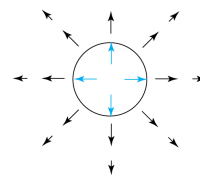
The geometry dictates use of a polar (cylindrical) reference frame.

Displacements are purely radial by symmetry

Axisymmetric hole with suction



Axisymmetric hole with pressure



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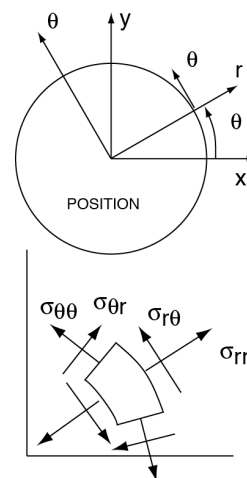
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21. Stresses Around a Hole (I)

IV Stresses in a polar (cylindrical) reference frame (on-in convention)

A Coordinate transformations

- 1 $x = r \cos\theta = r \cos\theta_{xr}$
- 2 $y = r \sin\theta = r \cos\theta_{yr}$
- 3 $r = (x^2 + y^2)^{1/2}$
- 4 $\theta = \tan^{-1}(y/x)$
- 5 $a_{rx} = \cos\theta_{rx} = \cos\theta$
- 6 $a_{ry} = \cos\theta_{ry} = \sin\theta$
- 7 $a_{\theta x} = \cos\theta_{\theta x} = -\sin\theta$
- 8 $a_{\theta y} = \cos\theta_{\theta y} = \cos\theta$



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21. Stresses Around a Hole (I)

B Stress transformations

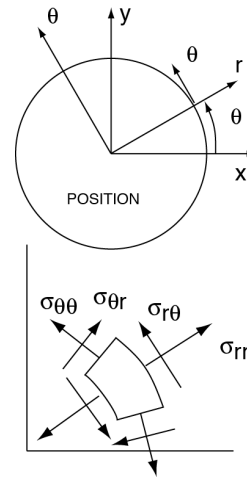
$$\sigma_{i'j'} = a_{i'k} a_{j'l} \sigma_{kl}$$

$$1 \quad \sigma_{rr} = a_{rx} a_{rx} \sigma_{xx} + a_{rx} a_{ry} \sigma_{xy} + a_{ry} a_{rx} \sigma_{yx} + a_{ry} a_{ry} \sigma_{yy}$$

$$2 \quad \sigma_{r\theta} = a_{rx} a_{\theta x} \sigma_{xx} + a_{rx} a_{\theta y} \sigma_{xy} + a_{ry} a_{\theta x} \sigma_{yx} + a_{ry} a_{\theta y} \sigma_{yy}$$

$$3 \quad \sigma_{\theta r} = a_{\theta x} a_{rx} \sigma_{xx} + a_{\theta x} a_{ry} \sigma_{xy} + a_{\theta y} a_{rx} \sigma_{yx} + a_{\theta y} a_{ry} \sigma_{yy}$$

$$4 \quad \sigma_{\theta\theta} = a_{\theta x} a_{\theta x} \sigma_{xx} + a_{\theta x} a_{\theta y} \sigma_{xy} + a_{\theta y} a_{\theta x} \sigma_{yx} + a_{\theta y} a_{\theta y} \sigma_{yy}$$



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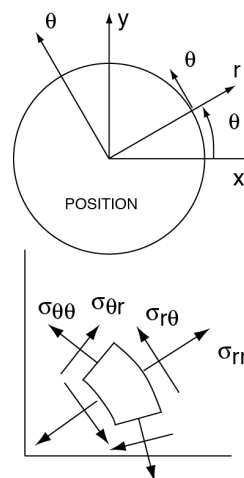
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21. Stresses Around a Hole (I)

V Equations of equilibrium (force balance)

A In polar (r, θ) reference frame for the **axisymmetric** problem here, $\sigma_{r\theta} = \sigma_{\theta r} = 0$



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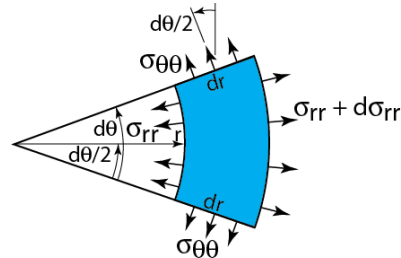
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21. Stresses Around a Hole (I)

V Equations of equilibrium
(force balance)

B Force balance in radial
direction



$$\begin{aligned}
 1 \quad 0 &= F_r \\
 &= (-\sigma_{rr})(rd\theta)(dz) \\
 &\quad + (\sigma_{rr} + d\sigma_{rr})(r + dr)(d\theta)(dz) \\
 &\quad - (2\sigma_{\theta\theta})(dr)\left(\sin\frac{d\theta}{2}\right)(dz)
 \end{aligned}$$

The dz term can be divided out, yielding:

$$\begin{aligned}
 2 \quad 0 &= (-\sigma_{rr})(rd\theta) \\
 &\quad + (\sigma_{rr} + d\sigma_{rr})(r + dr)(d\theta) \\
 &\quad - (2\sigma_{\theta\theta})(dr)\left(\sin\frac{d\theta}{2}\right)
 \end{aligned}$$

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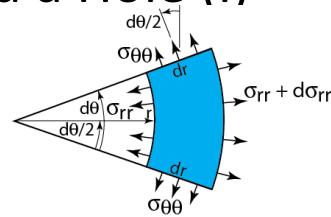
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21. Stresses Around a Hole (I)

V Equations of equilibrium
(force balance)

B Force balance in radial
direction



$$2 \quad F_r = \underline{(-\sigma_{rr})(rd\theta)} + \underline{(\sigma_{rr} + d\sigma_{rr})(r + dr)(d\theta)} - (2\sigma_{\theta\theta})(dr)\left(\sin\frac{d\theta}{2}\right) = 0$$

Simplifying eq. (2) by cancelling the underlined terms yields

$$3 \quad (\sigma_{rr})(dr)(d\theta) + (d\sigma_{rr})(r + dr)(d\theta) - \underline{(2\sigma_{\theta\theta})(dr)\left(\sin\frac{d\theta}{2}\right)} = 0$$

Now as $d\theta \rightarrow 0$, $\sin(d\theta/2) \rightarrow d\theta/2$, so $\underline{2\sin(d\theta/2)} \rightarrow d\theta$

$$4 \quad (\sigma_{rr})(dr)(d\theta) + (d\sigma_{rr})(r + dr)(d\theta) - \underline{(\sigma_{\theta\theta})(dr)(d\theta)} = 0$$

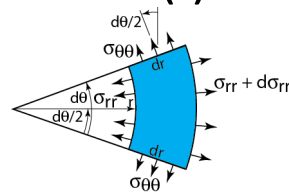
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21. Stresses Around a Hole (I)

C Force balance in radial direction



$$4 \quad (\sigma_{rr})(dr)(d\theta) + (d\sigma_{rr})(r+dr)(d\theta) - (\sigma_{\theta\theta})(dr)(d\theta) = 0$$

Now as $dr \rightarrow 0$ and $d\theta \rightarrow 0$, so $(d\sigma_{rr} dr d\theta) \rightarrow 0$, hence

$$5 \quad (\sigma_{rr})(dr)(d\theta) + (d\sigma_{rr})(r)(d\theta) - (\sigma_{\theta\theta})(dr)(d\theta) = 0$$

Now divide through by $(r)(dr)(d\theta)$

$$6 \quad \frac{d\sigma_{rr}}{dr} + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} = 0$$

This equation equilibrium in the radial direction is the governing equation of for an axisymmetric problem.

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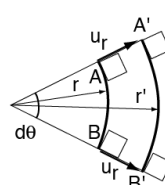
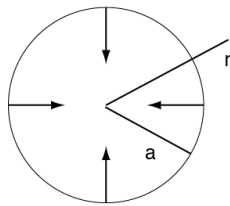
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21. Stresses Around a Hole (I)

VI Solution of boundary value problem for a pressurized hole

Axisymmetric Displacements and Strains



$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r} \\ e_{\theta\theta} &= \frac{\text{change in arc length}}{\text{original arc length}} \\ &= \frac{(A'B' - AB)}{AB} \\ &= \frac{(r'd\theta - rd\theta)}{rd\theta} \\ &= \frac{(r' - r)d\theta}{rd\theta} \\ &= \frac{(r' - r)}{r} \\ e_{\theta\theta} &= u_r/r \end{aligned}$$

- A Homogeneous isotropic material
- B Uniform positive traction on wall of hole
- C Uniform radial displacement
- D Radial shear stresses = 0 because radial shear strains = 0

$\epsilon_{r\theta} = 0$
The right angles at A and B between radial lines and circumferential arcs do not change as $A \rightarrow A'$ and $B \rightarrow B'$: $\epsilon_{r\theta} = 0$

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21. Stresses Around a Hole (I)

- V Solution of boundary value problem for a pressurized hole
 - F One General Solution Method
 - 1 Replace the stresses by strains using Hooke's law
 - 2 Replace the strains by displacement derivatives to yield a governing equation in terms of displacements.
 - 3 Solve differential governing equation for displacements
 - 4 Take the derivatives of the displacements to find the strains
 - 5 Solve for the stresses in terms of the strains using Hooke's law (not hard, but somewhat lengthy)

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

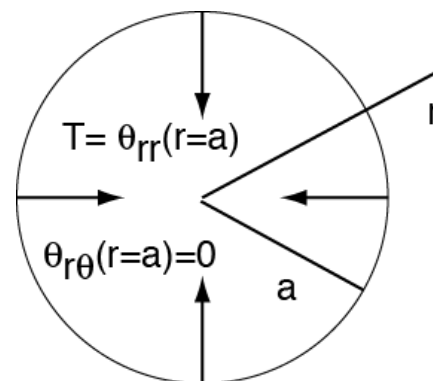
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21. Stresses Around a Hole (I)

- V Solution of boundary value problem for a pressurized hole
 - F One General Solution Method
 - 6 The general solution will contain constants. Their values are found in terms of the stresses or displacements on the boundaries of our body (i.e., the wall of the hole and any external boundary), that is in terms of the **boundary conditions** for our problem.



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21. Stresses Around a Hole (I)

V Solution of boundary value problem for a pressurized hole

G Strain-displacement relationships

Cartesian coordinates	Polar coordinates
$\epsilon_{xx} = \frac{\partial u}{\partial x}$	$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$
$\epsilon_{yy} = \frac{\partial v}{\partial y}$	$\epsilon_{\theta\theta} = \frac{u_r}{r}$
$\epsilon_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$	$\epsilon_{r\theta} = 0$

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21. Stresses Around a Hole (I)

V Solution of boundary value problem for a pressurized hole

H Strain-stress relationships: Plane Stress ($\sigma_{zz} = 0$)

Cartesian coordinates	Polar coordinates
$\epsilon_{xx} = \frac{1}{E} [\sigma_{xx} - \nu\sigma_{yy}]$	$\epsilon_{rr} = \frac{1}{E} [\sigma_{rr} - \nu\sigma_{\theta\theta}]$
$\epsilon_{yy} = \frac{1}{E} [\sigma_{yy} - \nu\sigma_{xx}]$	$\epsilon_{\theta\theta} = \frac{1}{E} [\sigma_{\theta\theta} - \nu\sigma_{rr}]$
$\epsilon_{xy} = \frac{1}{2G} \sigma_{xy}$	$\epsilon_{r\theta} = \frac{1}{2G} \sigma_{r\theta}$

From specializing the 3D relationships

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21. Stresses Around a Hole (I)

V Solution of boundary value problem for a pressurized hole

J Stress-strain relationships: Plane Stress ($\sigma_{zz} = 0$)

Cartesian coordinates	Polar coordinates
$\sigma_{xx} = \frac{E}{(1-\nu^2)} [\epsilon_{xx} + \nu\epsilon_{yy}]$	$\sigma_{rr} = \frac{E}{(1-\nu^2)} [\epsilon_{rr} + \nu\epsilon_{\theta\theta}]$
$\sigma_{yy} = \frac{E}{(1-\nu^2)} [\epsilon_{yy} + \nu\epsilon_{xx}]$	$\sigma_{\theta\theta} = \frac{E}{(1-\nu^2)} [\epsilon_{\theta\theta} + \nu\epsilon_{rr}]$
$\sigma_{xy} = 2G\epsilon_{xy}$	$\sigma_{r\theta} = 2G\epsilon_{r\theta}$

From specializing the 3D relationships

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21. Stresses Around a Hole (I)

V Solution of boundary value problem for a pressurized hole

L Axisymmetric Governing Equation

Plane Stress ($\sigma_{zz} = 0$)

In terms of stress	In terms of displacement
$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$	$\frac{d^2u_r}{dr^2} + \frac{1}{r} \frac{du_r}{dr} - \frac{u_r}{r^2} = 0$

See Appendix for the algebra to convert the governing equation from a function of stresses to radial displacement

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Appendix

- Equations of equilibrium in Cartesian form
- Plane strain relationships ($\epsilon_{zz} = 0$)
- Conversion of axisymmetric governing equation to a function of radial displacement

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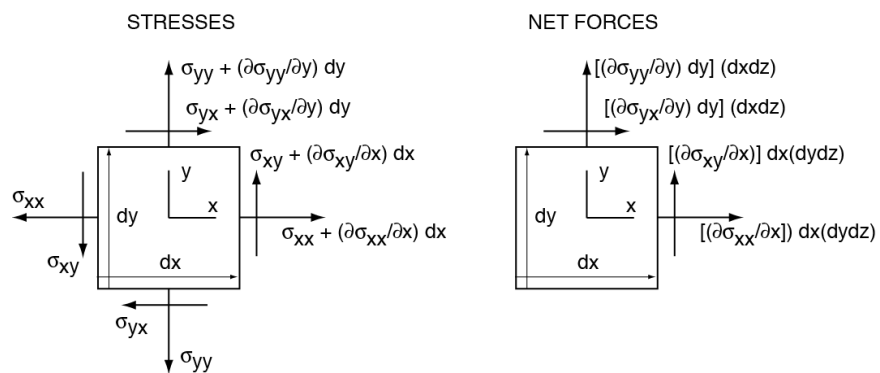
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21. Stresses Around a Hole (I)

Equations of equilibrium (force balance)

A In Cartesian (x,y) reference frame



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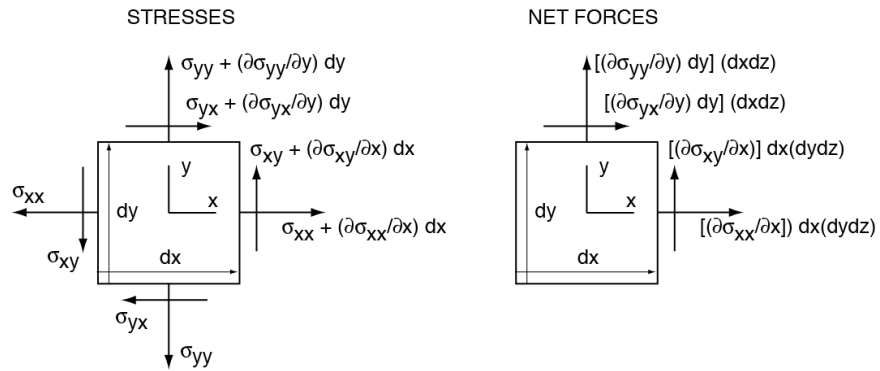
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21. Stresses Around a Hole (I)

IV Equations of equilibrium (force balance)

A In Cartesian (x,y) reference frame



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21. Stresses Around a Hole (I)

IV Equations of equilibrium (force balance) (cont.)

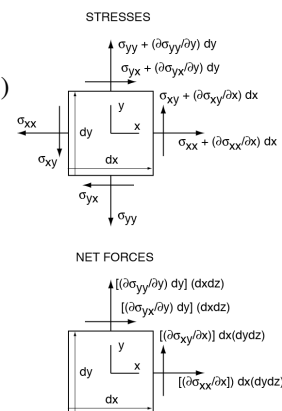
$$1 \sum F_x = 0 = [-\sigma_{xx}](dydz) + [-\sigma_{yx}](dx dz) + [\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx](dydz) + [\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy](dx dz)$$

$$2 \sum F_y = 0 = [\frac{\partial \sigma_{xx}}{\partial x} dx](dydz) + [-\frac{\partial \sigma_{yx}}{\partial y} dy](dx dz)$$

$$3 \sum F_z = 0 = [\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}](dx dy dz)$$

$$4 \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0$$

The rate of σ_{xx} increase is balanced by the rate of σ_{yx} decrease



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21. Stresses Around a Hole (I)

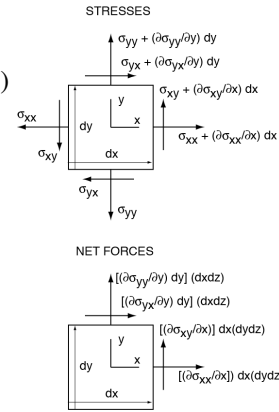
IV Equations of equilibrium (force balance) (cont.)

$$1 \quad \sum F_y = 0 = [-\sigma_{yy}](dxdz) + [-\sigma_{xy}](dydz) + [\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y} dy](dxdz) + [\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x} dx](dydz)$$

$$2 \quad \sum F_x = 0 = [\frac{\partial \sigma_{yy}}{\partial y} dy](dxdz) + [\frac{\partial \sigma_{xy}}{\partial x} dx](dydz)$$

$$3 \quad \sum F_y = 0 = [\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x}](dxdydz)$$

$$4 \quad \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \quad \text{The rate of } \sigma_{yy} \text{ increase is balanced by the rate of } \sigma_{xy} \text{ decrease}$$



21. Stresses Around a Hole (I)

V Solution of boundary value problem for a pressurized hole

K Stress-strain relationships: Plane Strain ($\epsilon_{zz} = 0$)

Cartesian coordinates	Polar coordinates
$\sigma_{xx} = \frac{E}{(1+\nu)} \left[\epsilon_{xx} + \left(\frac{\nu}{1-2\nu} \right) (\epsilon_{xx} + \epsilon_{yy}) \right]$	$\sigma_{rr} = \frac{E}{(1+\nu)} \left[\epsilon_{rr} + \left(\frac{\nu}{1-2\nu} \right) (\epsilon_{rr} + \epsilon_{\theta\theta}) \right]$
$\sigma_{yy} = \frac{E}{(1+\nu)} \left[\epsilon_{yy} + \left(\frac{\nu}{1-2\nu} \right) (\epsilon_{yy} + \epsilon_{xx}) \right]$	$\sigma_{\theta\theta} = \frac{E}{(1+\nu)} \left[\epsilon_{\theta\theta} - \left(\frac{\nu}{1-2\nu} \right) (\epsilon_{\theta\theta} + \epsilon_{rr}) \right]$
$\sigma_{xy} = 2G\epsilon_{xy}$	$\epsilon_{r\theta} = 2G\epsilon_{r\theta}$

From specializing the 3D relationships

21. Stresses Around a Hole (I)

V Solution of boundary value problem for a pressurized hole

I Strain-stress relationships: Plane Strain ($\epsilon_{zz} = 0$)

Cartesian coordinates	Polar coordinates
$\epsilon_{xx} = \frac{1-\nu^2}{E} \left[\sigma_{xx} - \left(\frac{\nu}{1-\nu} \right) \sigma_{yy} \right]$	$\epsilon_{rr} = \frac{1-\nu^2}{E} \left[\sigma_{rr} - \left(\frac{\nu}{1-\nu} \right) \sigma_{\theta\theta} \right]$
$\epsilon_{yy} = \frac{1-\nu^2}{E} \left[\sigma_{yy} - \left(\frac{\nu}{1-\nu} \right) \sigma_{xx} \right]$	$\epsilon_{\theta\theta} = \frac{1-\nu^2}{E} \left[\sigma_{\theta\theta} - \left(\frac{\nu}{1-\nu} \right) \sigma_{rr} \right]$
$\epsilon_{xy} = \frac{1}{2G} \sigma_{xy}$	$\epsilon_{r\theta} = \frac{1}{2G} \sigma_{r\theta}$

From specializing the 3D relationships

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Conversion of axisymmetric governing equation to a function of radial displacement

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\sigma_{rr} = \frac{E}{(1-\nu^2)} [\epsilon_{rr} + \nu\epsilon_{\theta\theta}] = \frac{E}{(1-\nu^2)} \left[\frac{du_r}{dr} + \nu \frac{u_r}{r} \right]$$

$$\sigma_{\theta\theta} = \frac{E}{(1-\nu^2)} [\epsilon_{\theta\theta} + \nu\epsilon_{rr}] = \frac{E}{(1-\nu^2)} \left[\frac{u_r}{r} + \nu \frac{du_r}{dr} \right]$$

$$\frac{d\sigma_{rr}}{dr} = \frac{E}{(1-\nu^2)} \left[\frac{d^2 u_r}{dr^2} + \nu \left(u_r \frac{-1}{r^2} + \frac{1}{r} \frac{du_r}{dr} \right) \right]$$

$$\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \frac{E}{(1-\nu^2)} \frac{1}{r} \left[\frac{du_r}{dr} + \nu \frac{u_r}{r} - \frac{u_r}{r} - \nu \frac{du_r}{dr} \right]$$

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \frac{E}{(1-\nu^2)} \left[\frac{d^2 u_r}{dr^2} + \nu \left(u_r \frac{-1}{r^2} + \frac{1}{r} \frac{du_r}{dr} \right) + \frac{1}{r} \left(\frac{du_r}{dr} + \nu \frac{u_r}{r} - \frac{u_r}{r} - \nu \frac{du_r}{dr} \right) \right]$$

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \frac{E}{(1-\nu^2)} \left[\frac{d^2 u_r}{dr^2} + \frac{1}{r} \frac{du_r}{dr} - \frac{1}{r} \frac{u_r}{r} + \nu \left(\frac{-u_r}{r^2} + \frac{1}{r} \frac{du_r}{dr} \right) + \frac{1}{r} \left(\nu \frac{u_r}{r} - \nu \frac{du_r}{dr} \right) \right]$$

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \frac{E}{(1-\nu^2)} \left[\frac{d^2 u_r}{dr^2} + \frac{1}{r} \frac{du_r}{dr} - \frac{1}{r} \frac{u_r}{r} \right] = 0$$

$$\frac{d^2 u_r}{dr^2} + \frac{1}{r} \frac{du_r}{dr} - \frac{u_r}{r^2} = 0$$

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21. Stresses Around a Hole (I)

V Solution of boundary value problem for a pressurized hole

L Axisymmetric Governing Equations

Plane Stress ($\sigma_{zz} = 0$)

Plane Strain ($\epsilon_{zz} = 0$)

In terms of stress	In terms of displacement	In terms of stress	In terms of displacement
$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$	$\frac{d^2 u_r}{dr^2} + \frac{1}{r} \frac{du_r}{dr} - \frac{u_r}{r^2} = 0$	$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$	$\frac{d^2 u_r}{dr^2} + \frac{1}{r} \frac{du_r}{dr} - \frac{u_r}{r^2} = 0$

Governing equations are the same for plane stress and plane strain