19. Principal Stresses

I Main Topics

A Cauchy's formula

B Principal stresses (eigenvectors and eigenvalues)

C Example

19. Principal Stresses

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19. Principal Stresses

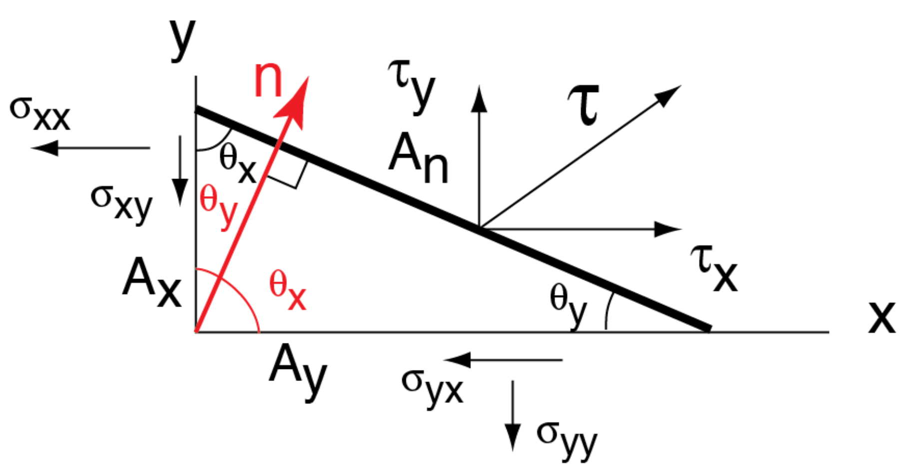
II Cauchy's formula

A Relates traction (stress *vector*) components to stress *tensor* components in the same reference frame

B 2D and 3D treatments analogous

C $\tau_i = \sigma_{ij}\, n_j = n_j \sigma_{ij} = n_j \sigma_{ji}$

Note: all stress components shown are positive



19. Principal Stresses

II Cauchy's formula (cont.)

C $\tau_i = n_j \sigma_{ji}$

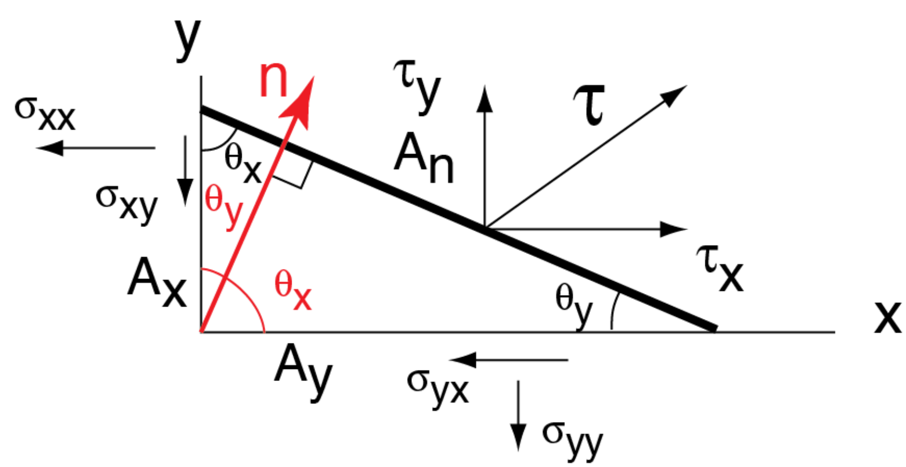
1 Meaning of terms

a τ_i = traction component

b n_j = direction cosine of angle between n-direction and j-direction

c σ_{ji} = stress component

d τ_i and σ_{ji} act *in* the same direction



$n_j = \cos\theta_{nj} = a_{nj}$

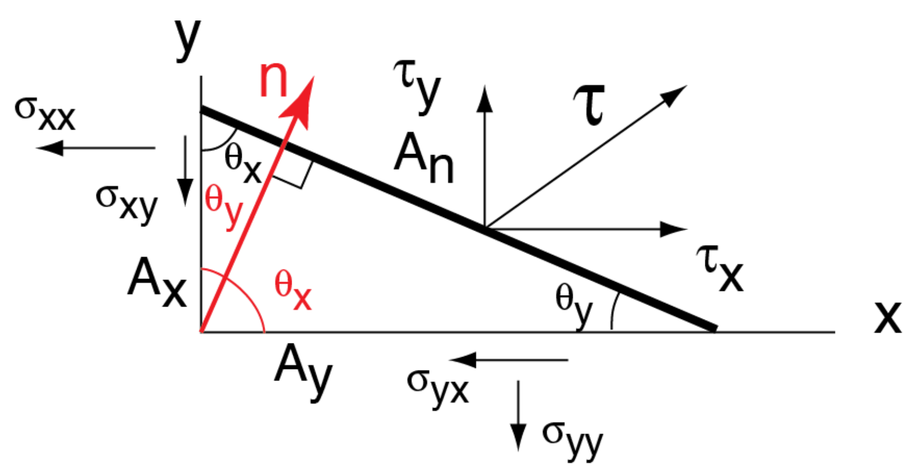
19. Principal Stresses

II Cauchy's formula (cont.)

D Expansion (2D) of $\tau_i = n_j \sigma_{ji}$

1 $\tau_x = n_x \sigma_{xx} + n_y \sigma_{yx}$

2 $\tau_y = n_x \sigma_{xy} + n_y \sigma_{yy}$



$n_j = \cos\theta_{nj} = a_{nj}$

19. Principal Stresses

II Cauchy's formula (cont.)

E Derivation:

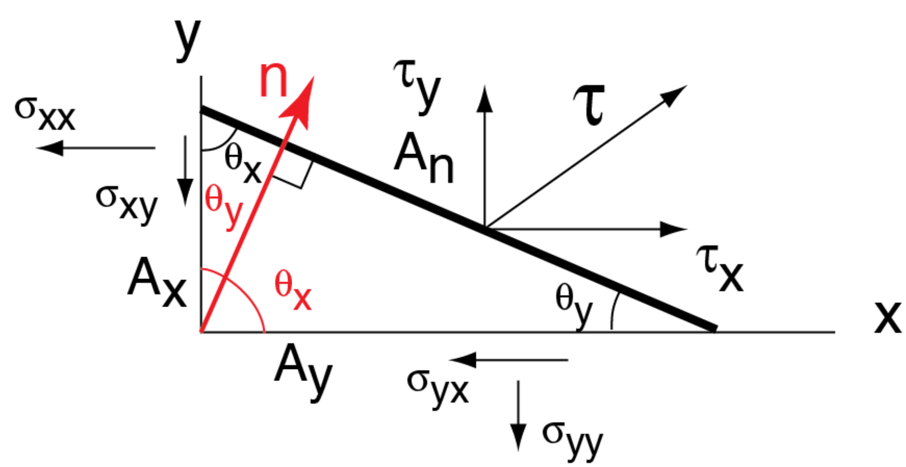
Contributions to τ_x

Note that all contributions must act in x-direction

1 $$\tau_x = w^{(1)}\sigma_{xx} + w^{(2)}\sigma_{yx}$$

2 $$\frac{F_x}{A_n} = \left(\frac{A_x}{A_n}\right)\frac{F_x^{(1)}}{A_x} + \left(\frac{A_y}{A_n}\right)\frac{F_x^{(2)}}{A_y}$$

3 $$\tau_x = n_x\sigma_{xx} + n_y\sigma_{yx}$$



$n_x = \cos\theta_{nx} = a_{nx}$

$n_y = \cos\theta_{ny} = a_{ny}$

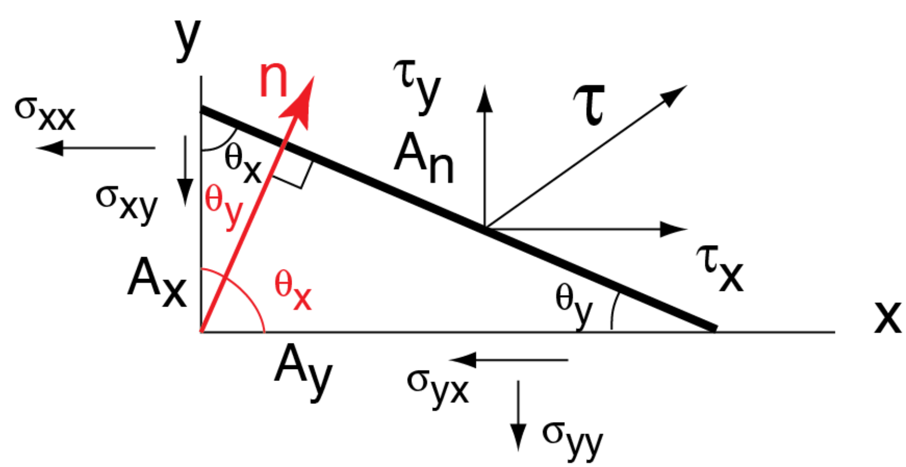
19. Principal Stresses

II Cauchy's formula (cont.)

E Derivation:

Contributions to τ_y

Note that all contributions must act in y-direction



1 $\tau_y = w^{(3)}\sigma_{xy} + w^{(4)}\sigma_{yy}$

2 $\frac{F_y}{A_n} = \left(\frac{A_x}{A_n}\right)\frac{F_y^{(3)}}{A_x} + \left(\frac{A_y}{A_n}\right)\frac{F_y^{(4)}}{A_y}$

3 $\tau_y = n_x\sigma_{xy} + n_y\sigma_{yy}$

$n_x = \cos\theta_{nx} = a_{nx}$

$n_y = \cos\theta_{ny} = a_{ny}$

19. Principal Stresses

II Cauchy's formula (cont.)

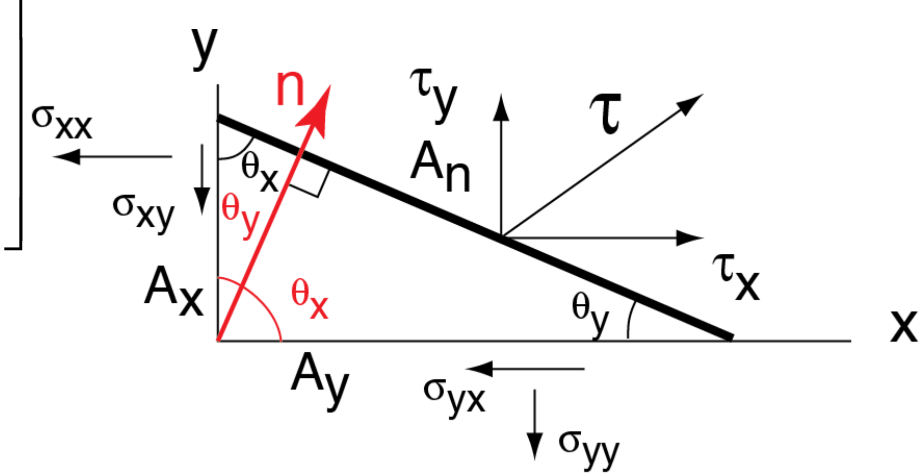
F Alternative forms

1 $\tau_i = n_j \sigma_{ji}$

2 $\tau_i = \sigma_{ji} n_j$

3 $\tau_i = \sigma_{ij} n_j$

4
$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{xy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$



5 Matlab

a t = s'*n

b t = s*n

6 Note that the stress matrix (tensor) transforms the normal vector to the plane (**n**) to the traction vector acting on the plane (**τ**)

$$n_j = \cos\theta_{nj} = a_{nj}$$

19. Principal Stresses

III Principal stresses (eigenvectors and eigenvalues)

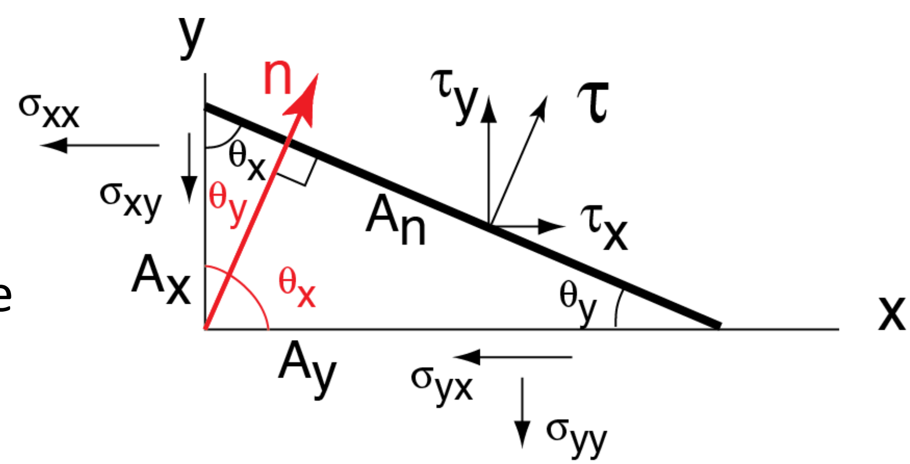
A Now we seek (a) the orientation of the unit normal (given by n_x and n_y) to any special plane where the associated traction vector is perpendicular (normal) to that plane, and (b) the magnitude (λ) of that traction vector.

B These traction vectors have no shear component and hence correspond to the principal stresses.

C The orientations of the special traction vectors are called eigenvectors, and the magnitudes of these special traction vectors are called eigenvalues.

D An eigenvector points in the same direction as the normal to the plane, so the transformation of the normal vector to the traction vector by Cauchy's formula does not involve a rotation.

Note that the traction vector below parallels the normal vector to the plane



$$n_j = \cos\theta_{nj} = a_{nj}$$

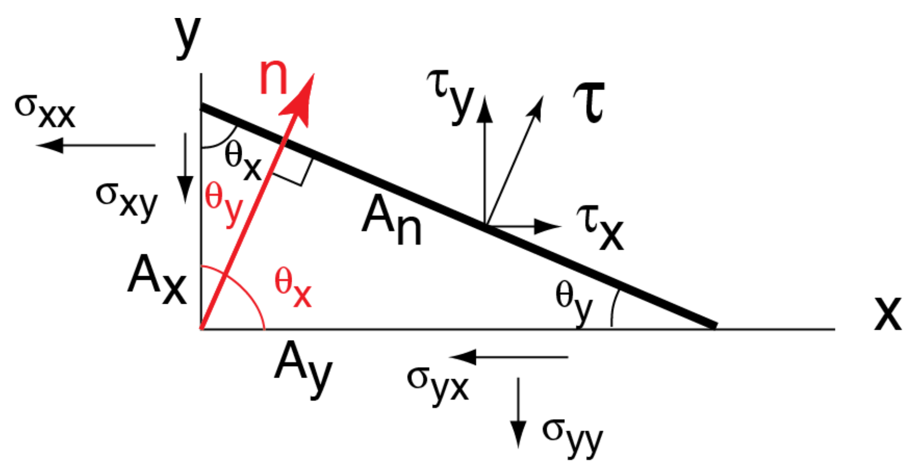
19. Principal Stresses

III Principal stresses (eigenvectors and eigenvalues)

E The x- and y- components of such a principal traction vector are obtained by projecting the vector onto the x- and y- axes:

$$\begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} = \left|\vec{\tau}\right| \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

Since the magnitude of the eigenvector is a scalar, both the normal to the plane and the eigenvector point in the same direction.



$n_j = \cos\theta_{nj} = a_{nj}$

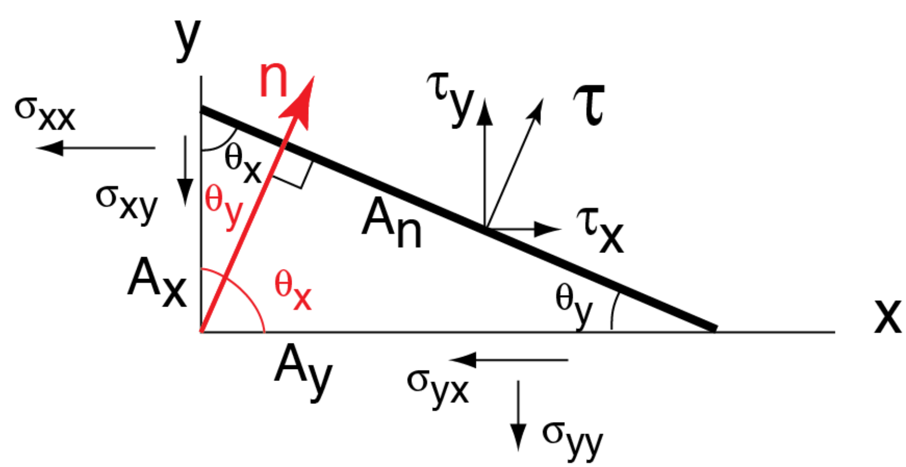
19. Principal Stresses

III Principal stresses (eigenvectors and eigenvalues)

F $$\begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} = \left|\vec{\tau}\right| \begin{bmatrix} n_x \\ n_y \end{bmatrix} \qquad Let\ \lambda = \left|\vec{\tau}\right|$$

G $$\begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

H $$\begin{bmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \lambda \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$



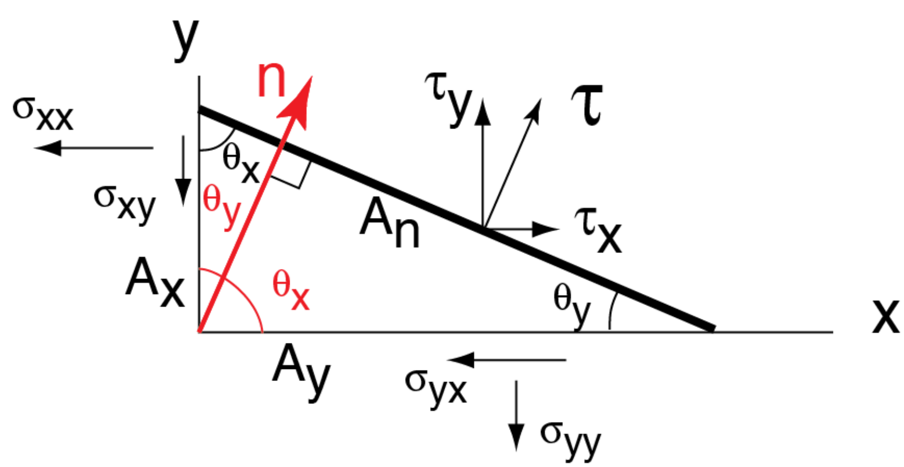
The form of (H) is [A][X]=λ[X], and [σ] is symmetric

19. Principal Stresses

III Principal stresses (eigenvectors and eigenvalues)

From previous notes

Subtract the right side from both sides



I
$$\begin{bmatrix} \sigma_{xx} - T & \sigma_{yx} - 0 \\ \sigma_{xy} - 0 & \sigma_{yy} - T \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

J $[\sigma - IT][n] = [0]$, where $[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now, a brief interlude to show how to solve analytically for the eigenvalues in 2D

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

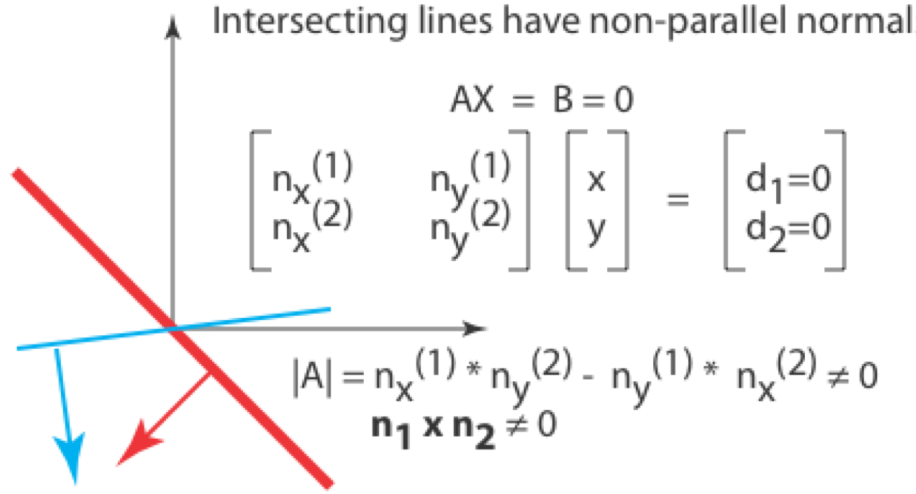
From previous notes

III Determinant (cont.)

D Geometric meanings of the real matrix equation AX = B = 0

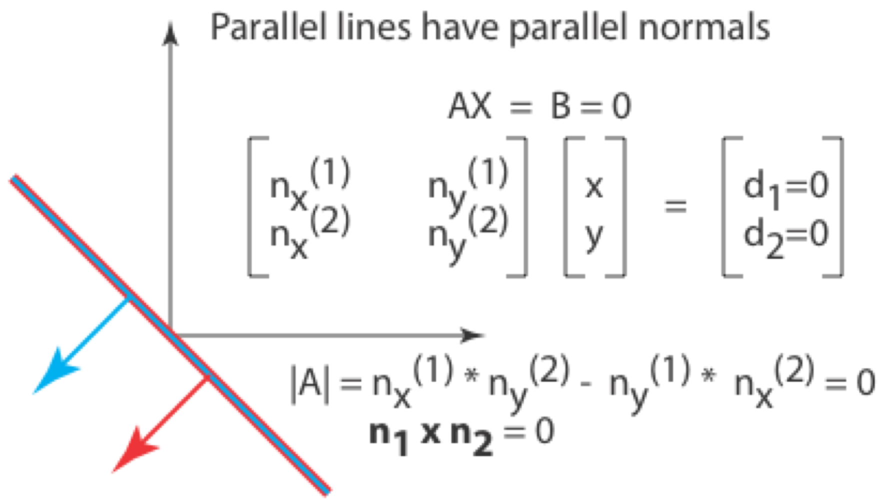
1 |A| ≠ 0 ;

- a $[A]^{-1}$ exists
- b Describes two lines (or 3 planes) that intersect at the origin
- c X has a unique solution



2 |A| = 0 ;

- a $[A]^{-1}$ does not exist
- b Describes two co-linear lines that that pass through the origin (or three planes that intersect a line or plane through the origin)
- c X has no unique solution



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

From previous notes

III Eigenvalue problems, eigenvectors and eigenvalues (cont.)

E Alternative form of an eigenvalue equation

→ 1 $[A][X]=\lambda[X]$

Subtracting $\lambda[IX] = \lambda[X]$ from both sides yields:

→ 2 $[A-I\lambda][X]=0$ (same form as $[\mathcal{A}][X]=0$)

F Solution conditions and connections with determinants

1 Unique trivial solution of $[X] = 0$ if and only if $|A-I\lambda| \neq 0$

→ 2 Eigenvector solutions ($[X] \neq 0$) if and only if $|A-I\lambda|=0$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

From previous notes

III Eigenvalue problems, eigenvectors and eigenvalues (cont.)

→ G Characteristic equation: $|A-I\lambda|=0$

1 Eigenvalues of a ***symmetric*** 2x2 matrix $A=\begin{bmatrix} a & b \\ b & d \end{bmatrix}$

a $$\lambda_1,\lambda_2=\frac{(a+d)\pm\sqrt{(a+d)^2-4\left(ad-b^2\right)}}{2}$$

b $$\lambda_1,\lambda_2=\frac{(a+d)\pm\sqrt{(a+2ad+d)^2-4ad+4b^2}}{2}$$

c $$\lambda_1,\lambda_2=\frac{(a+d)\pm\sqrt{(a-2ad+d)^2+4b^2}}{2}$$

→ d $$\lambda_1,\lambda_2=\frac{(a+d)\pm\sqrt{(a-d)^2+4b^2}}{2}$$

Radical term cannot be negative. Eigenvalues are real.

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

From previous notes

VI Solutions for symmetric matrices (cont.)

→ B Any distinct eigenvectors ($\mathbf{X_1}$, $\mathbf{X_2}$) of a ***symmetric*** nxn matrix are perpendicular ($\mathbf{X_1} \bullet \mathbf{X_2} = 0$)

1a $AX_1 = \lambda_1 X_1$ 1b $AX_2 = \lambda_2 X_2$

AX_1 parallels X_1, AX_2 parallels X_2 (property of eigenvectors)

Dotting AX_1 by X_2 and AX_2 by X_1 can test whether X_1 and X_2 are orthogonal.

2a $\mathbf{X}_2 \bullet AX_1 = \mathbf{X}_2 \bullet \lambda_1 \mathbf{X}_1 = \lambda_1 (\mathbf{X}_2 \bullet \mathbf{X}_1)$

2b $\mathbf{X}_1 \bullet AX_2 = \mathbf{X}_1 \bullet \lambda_2 \mathbf{X}_2 = \lambda_2 (\mathbf{X}_1 \bullet \mathbf{X}_2)$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

From previous notes

If $A=A^T$, then the left sides of (2a) and (2b) are equal:

3 $\mathbf{X}_2 \bullet A\mathbf{X}_1 = A\mathbf{X}_1 \bullet \mathbf{X}_2 = [A\mathbf{X}_1]^T[\mathbf{X}_2] = [[\mathbf{X}_1]^T[A]^T][\mathbf{X}_2]$

$= [\mathbf{X}_1]^T[A]\,[\mathbf{X}_2] = [\mathbf{X}_1]^T[[A]\,[\mathbf{X}_2]] = \mathbf{X}_1 \bullet A\mathbf{X}_2$

Since the left sides of (2a) and (2b) are equal, the right sides must be equal too. Hence,

4 $\lambda_1 (\mathbf{X}_2 \bullet \mathbf{X}_1) = \lambda_2 (\mathbf{X}_1 \bullet \mathbf{X}_2)$

Now subtract the right side of (4) from the left

5 $(\lambda_1 - \lambda_2)(\mathbf{X}_2 \bullet \mathbf{X}_1) = 0$

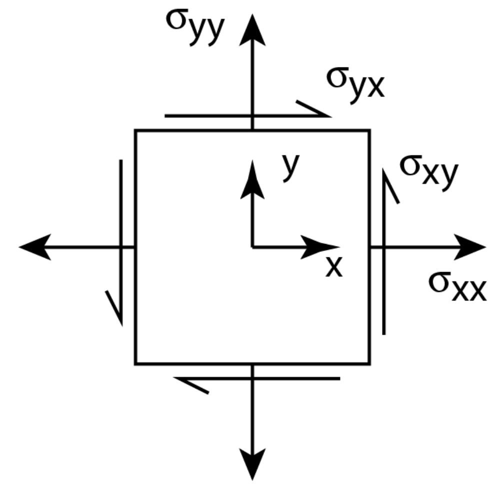
- The eigenvalues generally are different, so $\lambda_1 - \lambda_2 \neq 0$.
- This means for (5) to hold that $\mathbf{X}_2 \bullet \mathbf{X}_1 = 0$.
- Therefore, the eigenvectors ($\mathbf{X}_1$, $\mathbf{X}_2$) of a symmetric 2x2 matrix are perpendicular

End of brief interlude

19. Principal Stresses

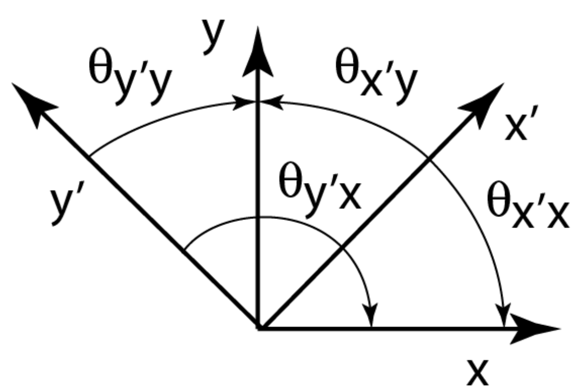
IV Example

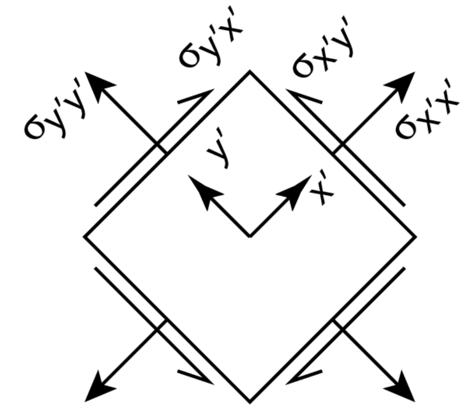
Find the principal stresses



given $\sigma_{ij} = \begin{bmatrix} \sigma_{xx} = -4MPa & \sigma_{xy} = -4MPa \\ \sigma_{yx} = -4MPa & \sigma_{yy} = -4MPa \end{bmatrix}$

$$\theta_{x'x} = -45^\circ, \theta_{x'y} = 45^\circ, \theta_{y'x} = -135^\circ, \theta_{y'y} = -45^\circ$$

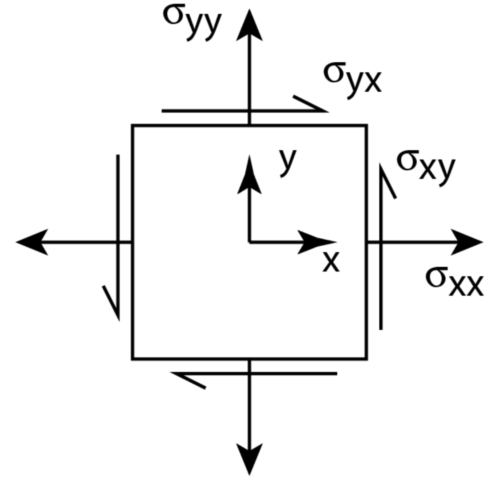




19. Principal Stresses

IV Example

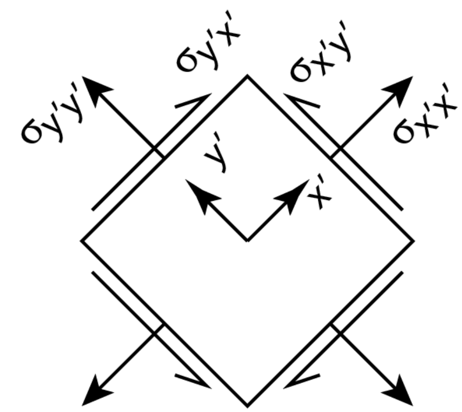
$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} = -4MPa & \sigma_{xy} = -4MPa \\ \sigma_{yx} = -4MPa & \sigma_{yy} = -4MPa \end{bmatrix}$$



First find eigenvalues

$$\lambda_1, \lambda_2 = \frac{\left(\sigma_{xx} + \sigma_{yy}\right) \pm \sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\sigma_{xy}{}^2}}{2}$$

$$\lambda_1, \lambda_2 = -4MPa \pm \frac{\sqrt{64}}{2} MPa = 0MPa, -8MPa$$



19. Principal Stresses

IV Example

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} = -4\,MPa & \sigma_{xy} = -4\,MPa \\ \sigma_{yx} = -4\,MPa & \sigma_{yy} = -4\,MPa \end{bmatrix}$$

$$\lambda_1, \lambda_2 = -4\,MPa \pm \frac{\sqrt{64}}{2}\,MPa = 0\,MPa, -8\,MPa$$

Then solve for eigenvectors (the dimensions of stress are unnecessary below and are dropped)

$$For\ \lambda_1 = 0: \begin{bmatrix} -4-0 & -4 \\ -4 & -4-0 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -4n_x - 4n_y = 0 \Rightarrow n_x = -n_y \\ -4n_x - 4n_y = 0 \Rightarrow n_x = -n_y \end{matrix}$$

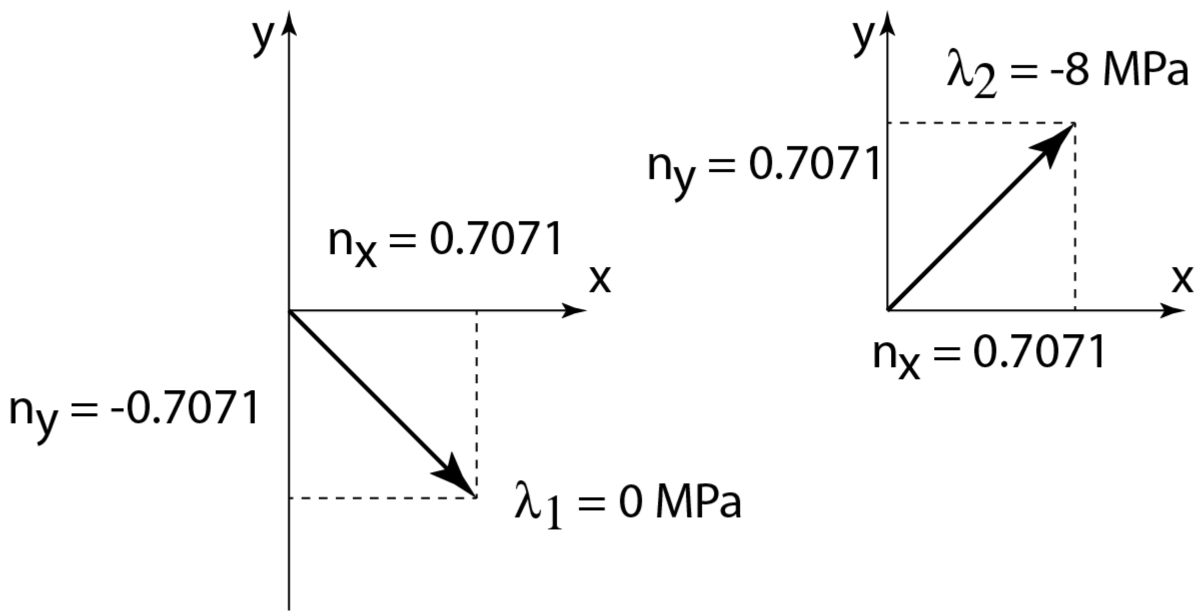
$$For\ \lambda_2 = -8: \begin{bmatrix} -4-(-8) & -4 \\ -4 & -4-(-8) \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 4n_x - 4n_y = 0 \Rightarrow n_x = n_y \\ -4n_x + 4n_y = 0 \Rightarrow n_x = n_y \end{matrix}$$

19. Principal Stresses

IV Example

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} = -4\,MPa & \sigma_{xy} = -4\,MPa \\ \sigma_{yx} = -4\,MPa & \sigma_{yy} = -4\,MPa \end{bmatrix}$$

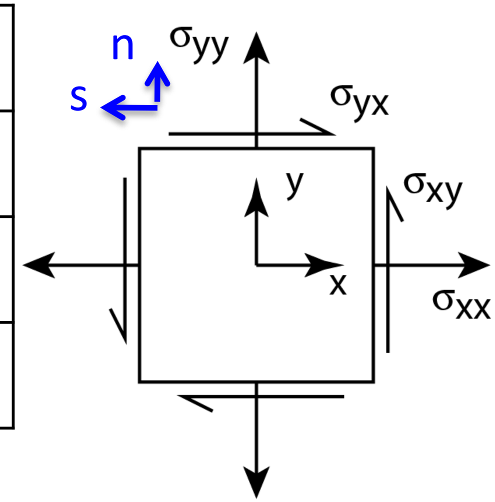
Eigenvalues	Eigenvectors
$\lambda_1 = 0\,MPA$	$n_x = -n_y$
$\lambda_2 = -8\,MPA$	$n_x = n_y$

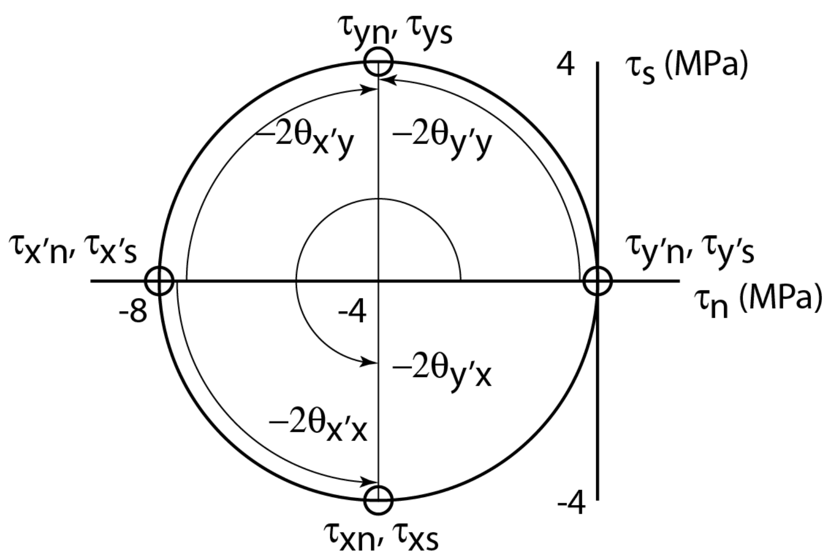


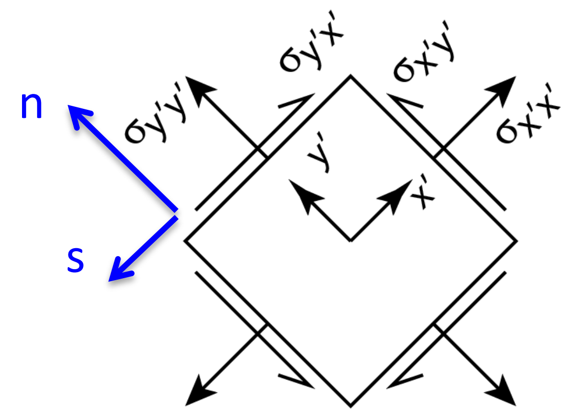
19. Principal Stresses

IV Example (values in MPa)

σ_{xx} = - 4	τ_{xn} = - 4	$\sigma_{x'x'}$ = - 8	$\tau_{x'n}$ = - 8
σ_{xy} = - 4	τ_{xs} = - 4	$\sigma_{x'y'}$ = 0	$\tau_{x's}$ = 0
σ_{yx} = - 4	τ_{ys} = + 4	$\sigma_{y'x'}$ = -0	$\tau_{y's}$ = +0
σ_{yy} = - 4	τ_{yn} = -4	$\sigma_{y'y'}$ = 0	$\tau_{y'n}$ = 0







19. Principal Stresses

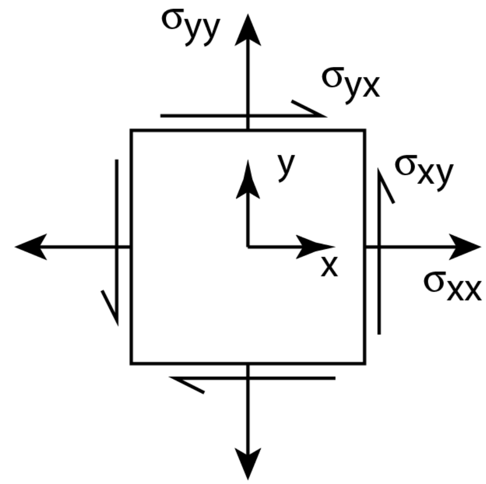
IV Example

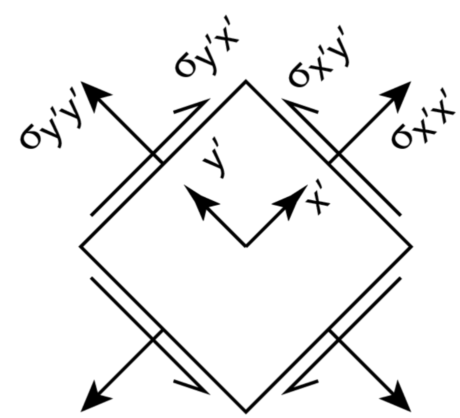
Matrix form

$$\begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \sigma_{y'x'} & \sigma_{y'y'} \end{bmatrix} = \begin{bmatrix} a_{x'x} & a_{x'y} \\ a_{y'x} & a_{y'y} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} a_{x'x} & a_{x'y} \\ a_{y'x} & a_{y'y} \end{bmatrix}^T$$

$$\left[\sigma_{i'j'}\right] = [a]\left[\sigma_{ij}\right][a]^T$$

This expression is valid in 2D and 3D!

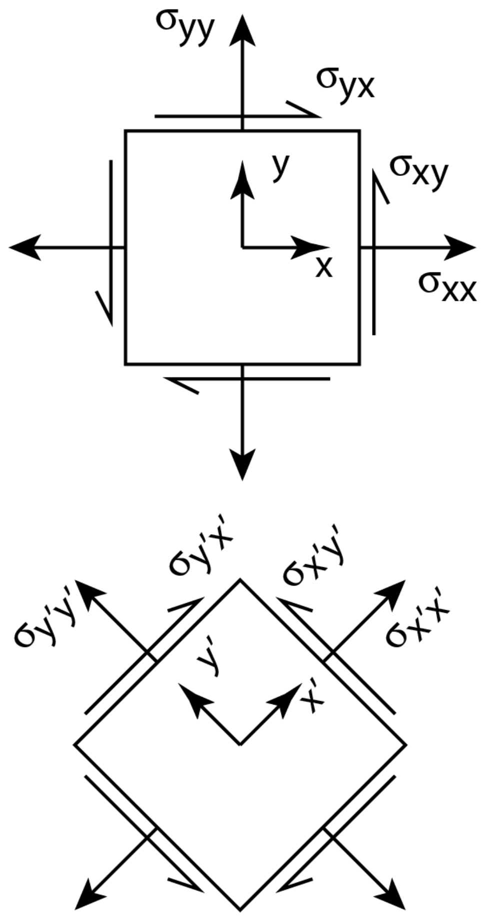




19. Principal Stresses

IV Example

Matrix form/Matlab




```
>> sij = [-4 -4;-4 -4]
sij =
    -4    -4
    -4    -4
>> [vec,val]=eig(sij)
vec =
    0.7071   -0.7071
    0.7071    0.7071
val =
    -8    0
     0    0
```

Eigenvectors (in columns)

Corresponding eigenvalues (in columns)