18. Tensor Transformation of Stresses

I Main Topics

A Objective

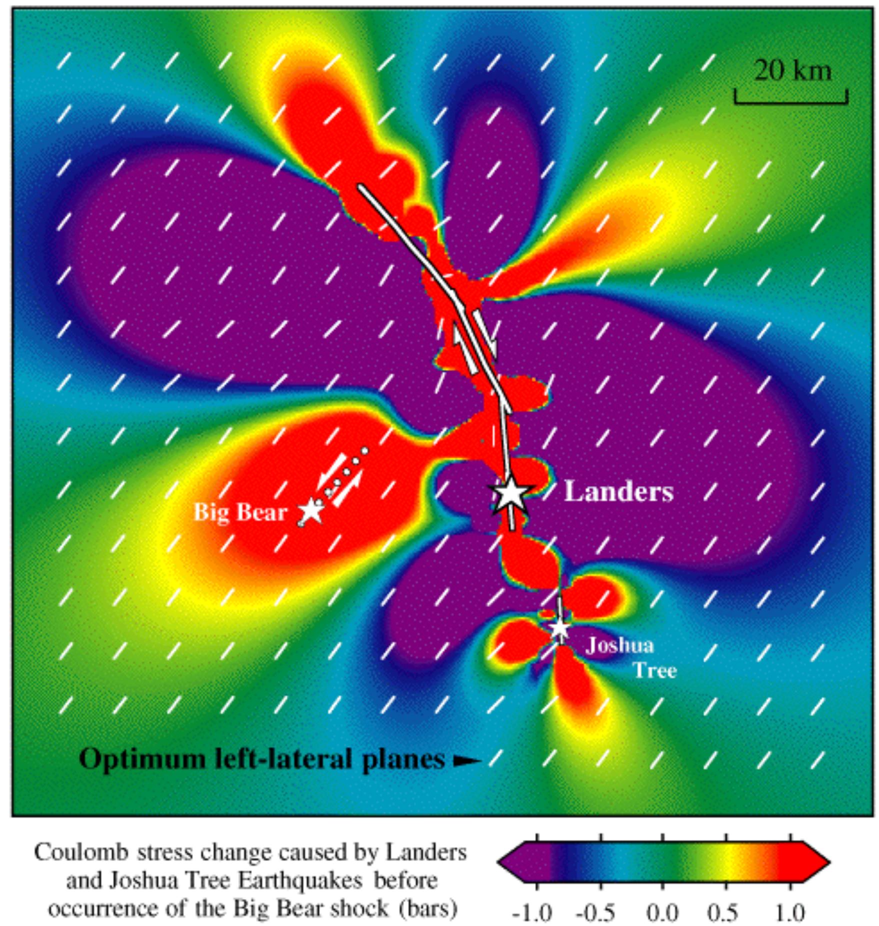
B Approach

C Derivation

D Example

18. Mohr Circle for Tractions

- From King et al., 1994 (Fig. 11)
- Coulomb stress change caused by the Landers rupture. The left-lateral ML=6.5 Big Bear rupture occurred along dotted line 3 hr 26 min after the Landers main shock. The Coulomb stress increase at the future Big Bear epicenter is 2.2-2.9 bars.

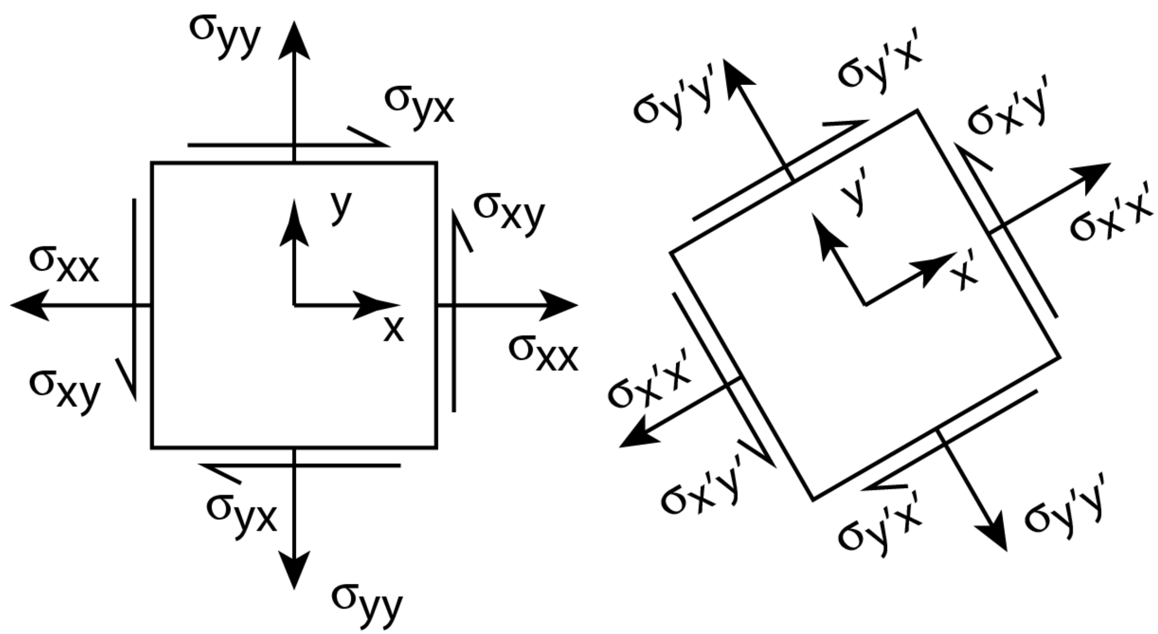


http://earthquake.usgs.gov/research/modeling/papers/landers.php

18. Tensor Transformation of Stresses

II Objective

	Lecture 16	Lecture 18
Transformation	Stresses to tractions	Stresses to stresses
Number of arbitrary planes	1 plane	2 perpendicular planes
Stresses accounted for	Normal stresses only	Normal and shear stresses



18. Tensor Transformation of Stresses

III Approach

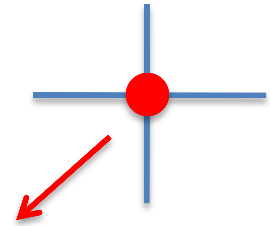
	Vectors	Tensors
Equation	$v_{i'} = a_{i'j} v_j$	$\sigma_{i'j'} = a_{i'k} a_{j'l} \sigma_{kl}$
Number of subscripts in quantity being converted	1	2
Number of direction cosines in equation	1	2

Key concept
Total value of each stress component in one reference frame is the sum of the weighted contributions from all the components in another frame

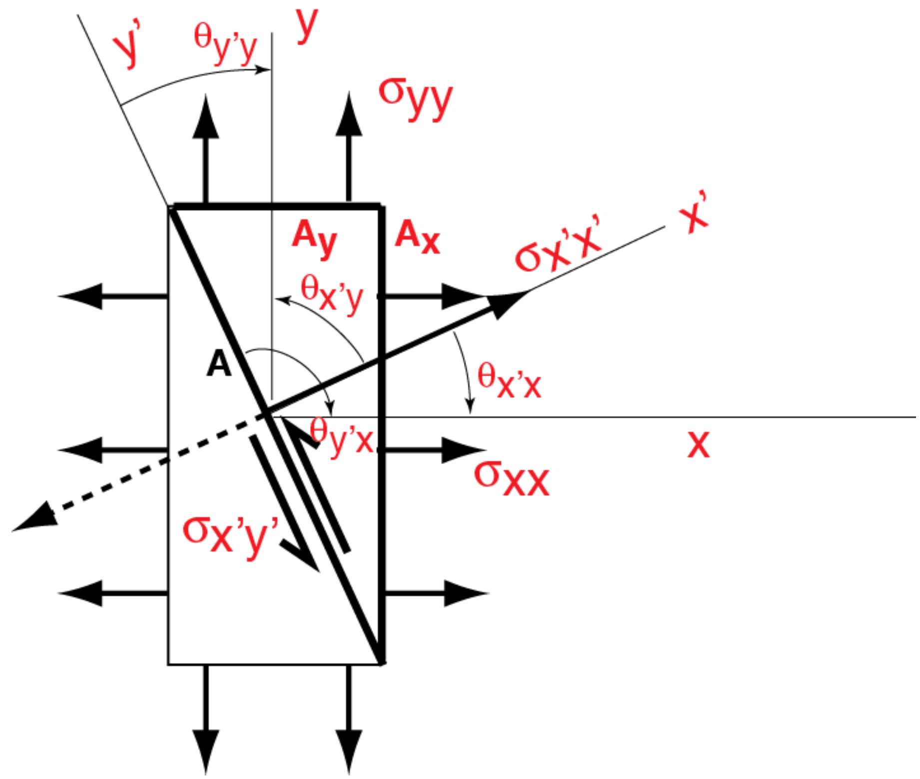
18. Tensor Transformation of Stresses

V Derivation

A Description of terms



Term	Meaning
$A_{x'}$, A_x, A_y	Sides of prism
σ_{xx}/σ_{xy}	Normal/shear stress on A_x
σ_{yy}/σ_{yx}	Normal/shear stress on A_y
$\sigma_{x'x'}$ /$\sigma_{x'y'}$	Normal/shear stress on $A_{x'}$
$\theta_{x'x}$	Angle from x' to x axis
$\theta_{x'y}$	Angle from x' to y axis
$\theta_{y'x}$	Angle from y' to x axis
$\theta_{y'y}$	Angle from y' to y axis



18. Tensor Transformation of Stresses

IV Derivation

B Contribution of σ_{xx} to $\sigma_{x'x'}$

w = dimensionless weighting factor

1 $$\sigma_{x'x'} = w^{(1)}\sigma_{xx} + w^{(2)}\sigma_{xy} + w^{(3)}\sigma_{yx} + w^{(4)}\sigma_{yy}$$

2 $$\frac{F_{x'}}{A_{x'}} = \left(\frac{A_x}{A_{x'}}\frac{F_{x'}^{(1)}}{F_x^{(1)}}\right)\frac{F_x^{(1)}}{A_x} + \left(\frac{A_x}{A_{x'}}\frac{F_{x'}^{(2)}}{F_y^{(2)}}\right)\frac{F_y^{(2)}}{A_x} + \left(\frac{A_y}{A_{x'}}\frac{F_{x'}^{(3)}}{F_x^{(3)}}\right)\frac{F_x^{(3)}}{A_y} + \left(\frac{A_y}{A_{x'}}\frac{F_{x'}^{(4)}}{F_y^{(4)}}\right)\frac{F_y^{(4)}}{A_y}$$

3 $$\sigma_{x'x'} = a_{x'x}a_{x'x}\sigma_{xx} + a_{x'x}a_{x'y}\sigma_{xy} + a_{x'y}a_{x'x}\sigma_{yx} + a_{x'y}a_{x'y}\sigma_{yy}$$

18. Tensor Transformation of Stresses

B Contribution of σ_{xx} to $\sigma_{x'x'}$

Start with the definition of a stress vector:

1   $\sigma_{x'x'}^{(1)} = F_{x'}^{(1)} / A_{x'}$

First find $F_x^{(1)}$ associated with σ_{xx}

2   $F_x^{(1)} = \sigma_{xx} A_x$

Find $F_{x'}^{(1)}$, the component of $F_x^{(1)}$ in the x'-direction

3   $F_{x'}^{(1)} = F_x^{(1)} \cos \theta_{x'x}$

Now find A_x in terms of $A_{x'}$

$A_x = A_{x'} \cos \theta_{x'x}$

4   $A_{x'} = A_x / \cos \theta_{x'x}$

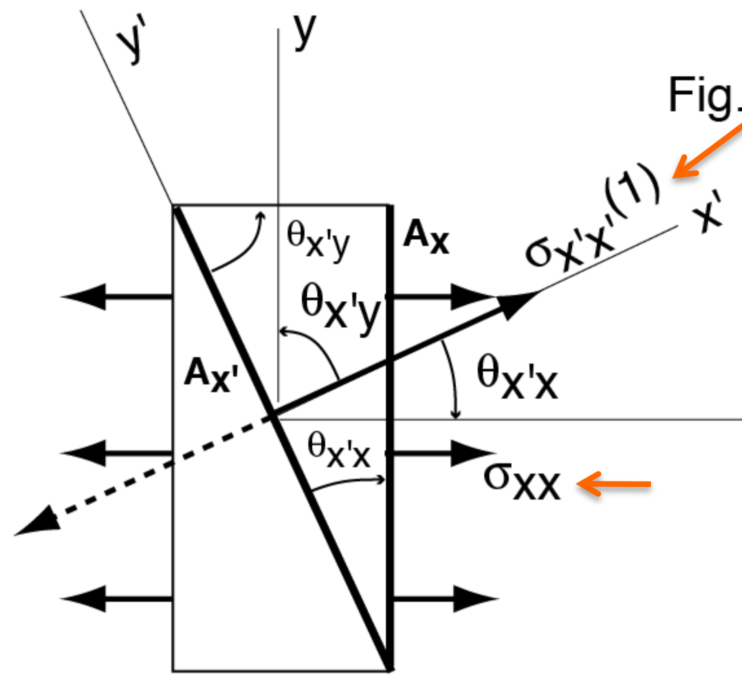
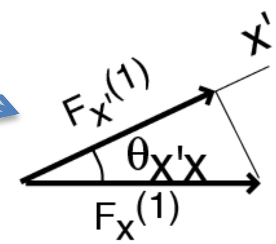


Fig. 18.1



18. Tensor Transformation of Stresses

B Contribution of σ_{xx} to $\sigma_{x'x'}$ (cont.)

3 $F_{x'}^{(1)} = F_x^{(1)} \cos \theta_{x'x}$

4 $A_{x'} = A_x / \cos \theta_{x'x}$

Now substitute:

5a $\sigma_{x'x'}^{(1)} = F_{x'}^{(1)} / A_{x'}$

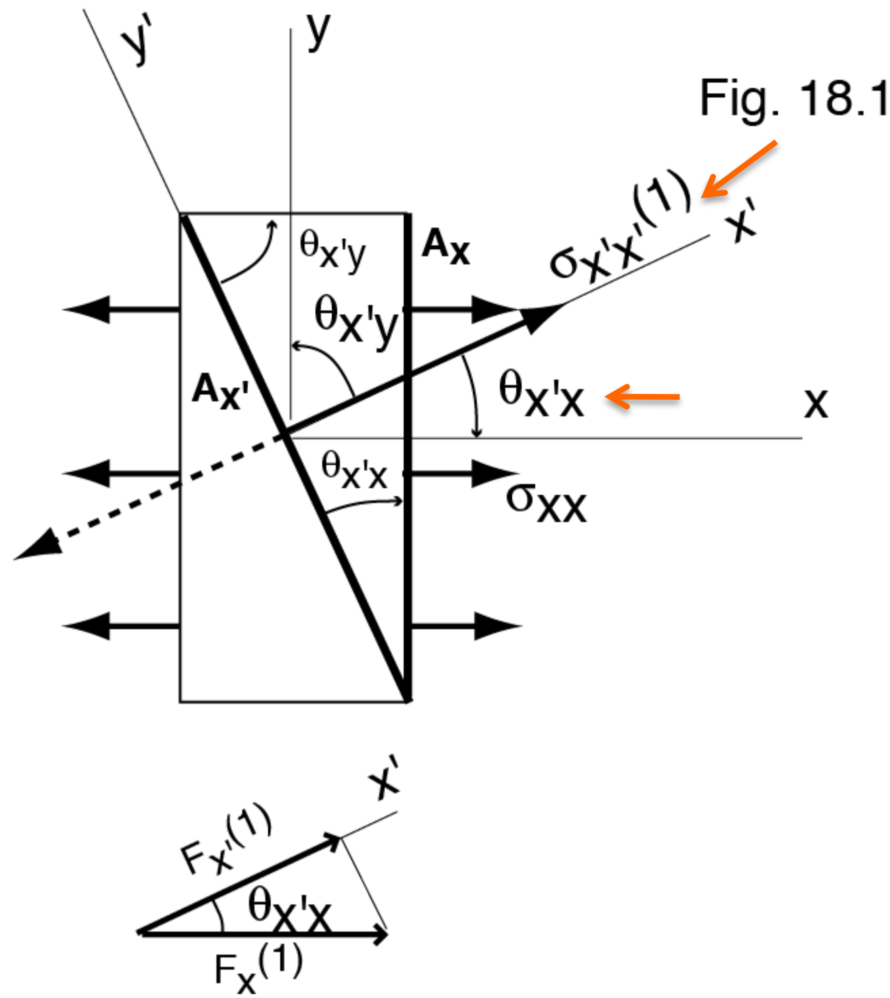
5b $\sigma_{x'x'}^{(1)} = F_x^{(1)} \cos \theta_{x'x} / (A_x / \cos \theta_{x'x})$

5c $\sigma_{x'x'}^{(1)} = \cos \theta_{x'x} \cos \theta_{x'x} (F_x^{(1)} / A_x)$

5d $\sigma_{x'x'}^{(1)} = a_{x'x} a_{x'x} \sigma_{xx}$

Weighting factor $w^{(1)}$

6 $w^{(1)} = a_{x'x} a_{x'x}$



18. Tensor Transformation of Stresses

IV Derivation

C Contribution of σ_{xy} to $\sigma_{x'x'}$

w = dimensionless weighting factor

1 $$\sigma_{x'x'} = w^{(1)}\sigma_{xx} + \underline{w^{(2)}}\,\underline{\sigma_{xy}} + w^{(3)}\sigma_{yx} + w^{(4)}\sigma_{yy}$$

2 $$\frac{F_{x'}}{A_{x'}} = \left(\frac{A_x}{A_{x'}}\frac{F_{x'}^{(1)}}{F_x^{(1)}}\right)\frac{F_x^{(1)}}{A_x} + \underline{\left(\frac{A_x}{A_{x'}}\frac{F_{x'}^{(2)}}{F_y^{(2)}}\right)}\,\underline{\frac{F_y^{(2)}}{A_x}} + \left(\frac{A_y}{A_{x'}}\frac{F_{x'}^{(3)}}{F_x^{(3)}}\right)\frac{F_x^{(3)}}{A_y} + \left(\frac{A_y}{A_{x'}}\frac{F_{x'}^{(4)}}{F_y^{(4)}}\right)\frac{F_y^{(4)}}{A_y}$$

3 $$\sigma_{x'x'} = a_{x'x}a_{x'x}\sigma_{xx} + \underline{a_{x'x}a_{x'y}}\,\underline{\sigma_{xy}} + a_{x'y}a_{x'x}\sigma_{yx} + a_{x'y}a_{x'y}\sigma_{yy}$$

18. Tensor Transformation of Stresses

C Contribution of σ_{xy} to $\sigma_{x'x'}$

Start with the definition of a stress vector:

1 $\sigma_{x'x'}^{(1)} = F_{x'}^{(2)})/ A_{x'}$

First find $F_y^{(2)}$ associated with σ_{xy}

2 $F_y^{(2)} = \sigma_{xy} A_x$

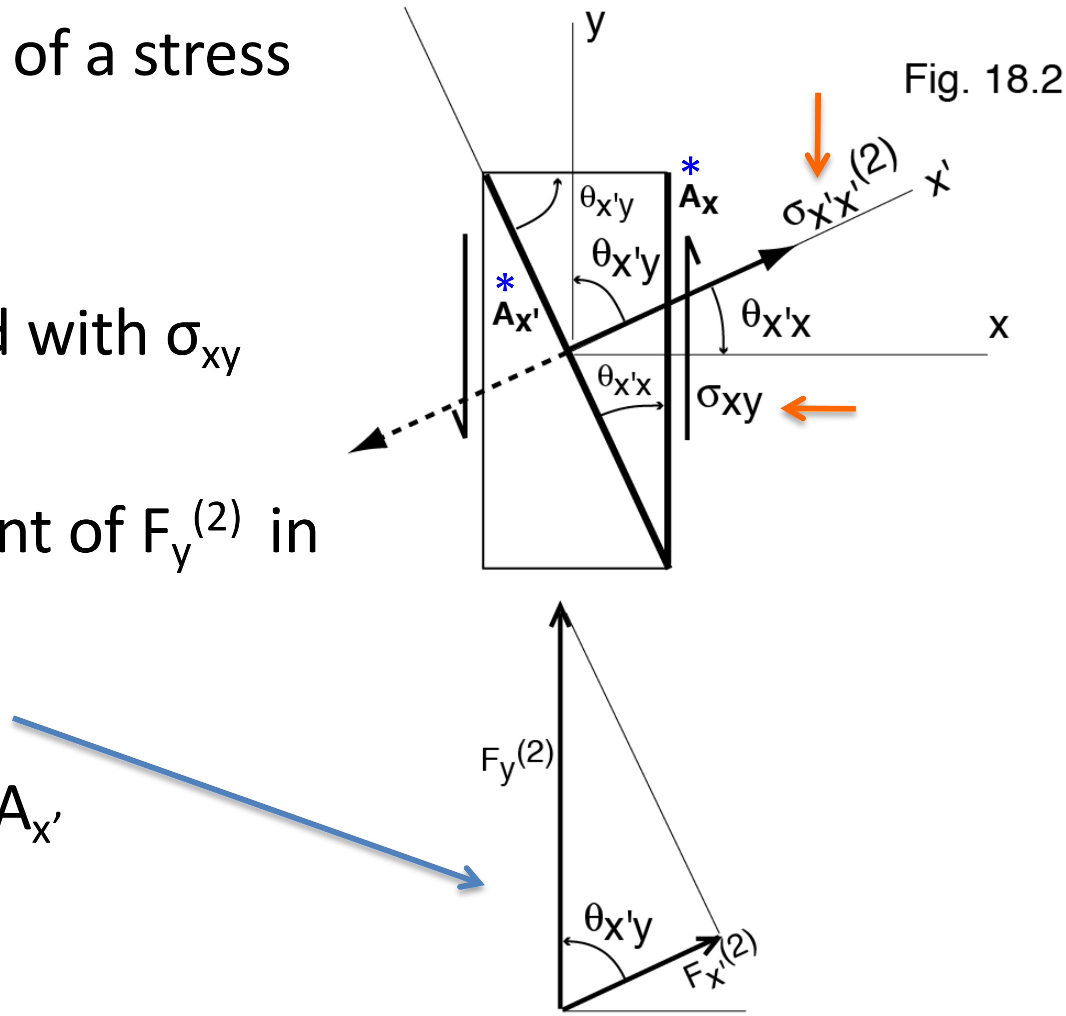
Find $F_{x'}^{(2)}$, the component of $F_y^{(2)}$ in the x'-direction

3 $F_{x'}^{(2)} = F_y^{(2)} \cos \theta_{x'y}$

Now find A_x in terms of $A_{x'}$

$A_x = A_{x'} \cos \theta_{x'x}$

4 $A_{x'} = A_x/\cos \theta_{x'x}$



18. Tensor Transformation of Stresses

C Contribution of σ_{xy} to $\sigma_{x'x'}$ (cont.)

3 $F_{x'}^{(2)} = F_y^{(2)} \cos\theta_{x'y}$

4 $A_{x'} = A_x / \cos\theta_{x'x}$

Now substitute:

5a $\sigma_{x'x'}^{(2)} = F_{x'}^{(2)} / A_{x'}$

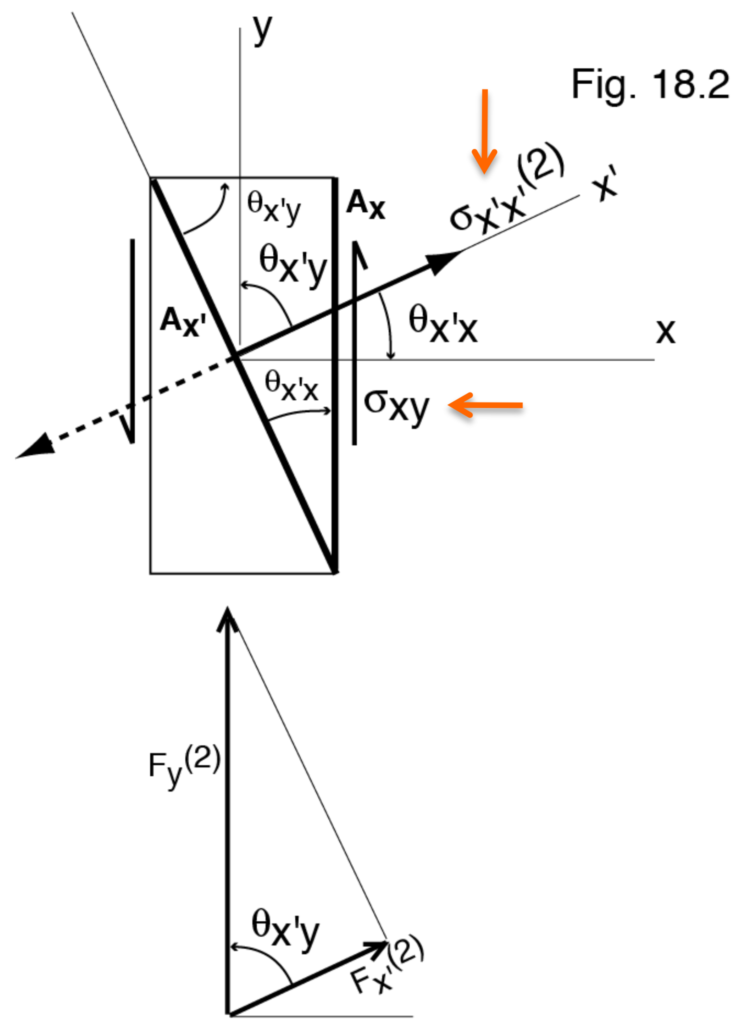
5b $\sigma_{x'x'}^{(2)} = F_y^{(2)} \cos\theta_{x'y} / (A_x / \cos\theta_{x'x})$

5c $\sigma_{x'x'}^{(2)} = \cos\theta_{x'x} \cos\theta_{x'y} (F_y^{(2)} / A_x)$

5d $\sigma_{x'x'}^{(2)} = a_{x'x}\, a_{x'y}\, \sigma_{xy}$

Weighting factor $w^{(2)}$

6 $w^{(2)} = a_{x'x}\, a_{x'y}$



18. Tensor Transformation of Stresses

IV Derivation

D Contribution of σ_{yx} to $\sigma_{x'x'}$

w = dimensionless weighting factor

1 $$\sigma_{x'x'} = w^{(1)}\sigma_{xx} + w^{(2)}\sigma_{xy} + w^{(3)}\sigma_{yx} + w^{(4)}\sigma_{yy}$$

2 $$\frac{F_{x'}}{A_{x'}} = \left(\frac{A_x}{A_{x'}}\frac{F_{x'}^{(1)}}{F_x^{(1)}}\right)\frac{F_x^{(1)}}{A_x} + \left(\frac{A_x}{A_{x'}}\frac{F_{x'}^{(2)}}{F_y^{(2)}}\right)\frac{F_y^{(2)}}{A_x} + \left(\frac{A_y}{A_{x'}}\frac{F_{x'}^{(3)}}{F_x^{(3)}}\right)\frac{F_x^{(3)}}{A_y} + \left(\frac{A_y}{A_{x'}}\frac{F_{x'}^{(4)}}{F_y^{(4)}}\right)\frac{F_y^{(4)}}{A_y}$$

3 $$\sigma_{x'x'} = a_{x'x}a_{x'x}\sigma_{xx} + a_{x'x}a_{x'y}\sigma_{xy} + a_{x'y}a_{x'x}\sigma_{yx} + a_{x'y}a_{x'y}\sigma_{yy}$$

18. Tensor Transformation of Stresses

D Contribution of σ_{yx} to $\sigma_{x'x'}$

Start with the definition of a stress vector:

1 $\sigma_{x'x'}^{(3)} = F_{x'}^{(3)} / A_{x'}$

First find $F_x^{(3)}$ associated with σ_{yx}

2 $F_x^{(3)} = \sigma_{yx} A_y$

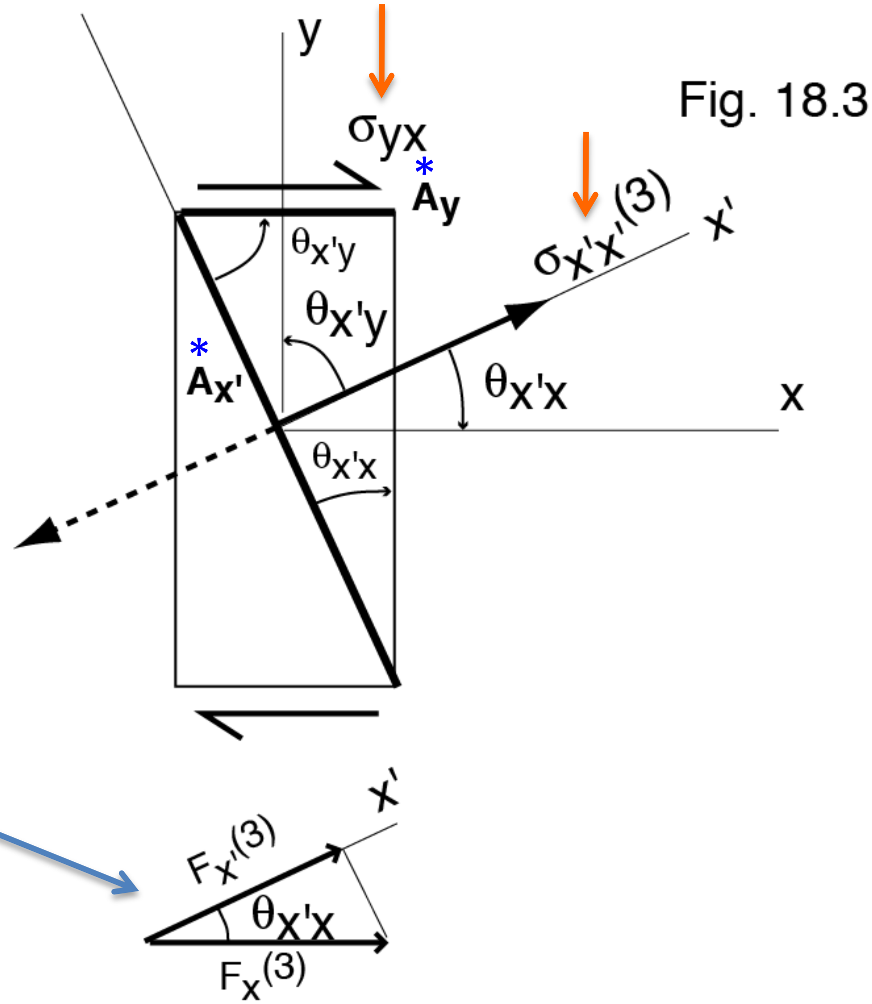
Find $F_{x'}^{(3)}$, the component of $F_x^{(3)}$ in the x'-direction

3 $F_{x'}^{(3)} = F_x^{(3)} \cos\theta_{x'x}$

Find A_y in terms of $A_{x'}$

$A_y = A_{x'} \cos\theta_{x'y}$

4 $A_{x'} = A_y / \cos\theta_{x'y}$



18. Tensor Transformation of Stresses

D Contribution of σ_{yx} to $\sigma_{x'x'}$ (cont.)

3 $F_{x'}^{(3)} = F_x^{(3)} \cos\theta_{x'x}$

4 $A_{x'} = A_y / \cos\theta_{x'y}$

Now substitute:

5a $\sigma_{x'x'}^{(3)} = F_{x'}^{(3)} / A_{x'}$

5b $\sigma_{x'x'}^{(3)} = F_x^{(3)} \cos\theta_{x'x} / (A_y / \cos\theta_{x'y})$

5c $\sigma_{x'x'}^{(3)} = \cos\theta_{x'y} \cos\theta_{x'x} (F_x^{(3)} / A_y)$

5d $\sigma_{x'x'}^{(3)} = a_{x'y}\, a_{x'x}\, \sigma_{yx}$

Weighting factor $w^{(3)}$

6 $w^{(3)} = a_{x'y}\, a_{x'x}$

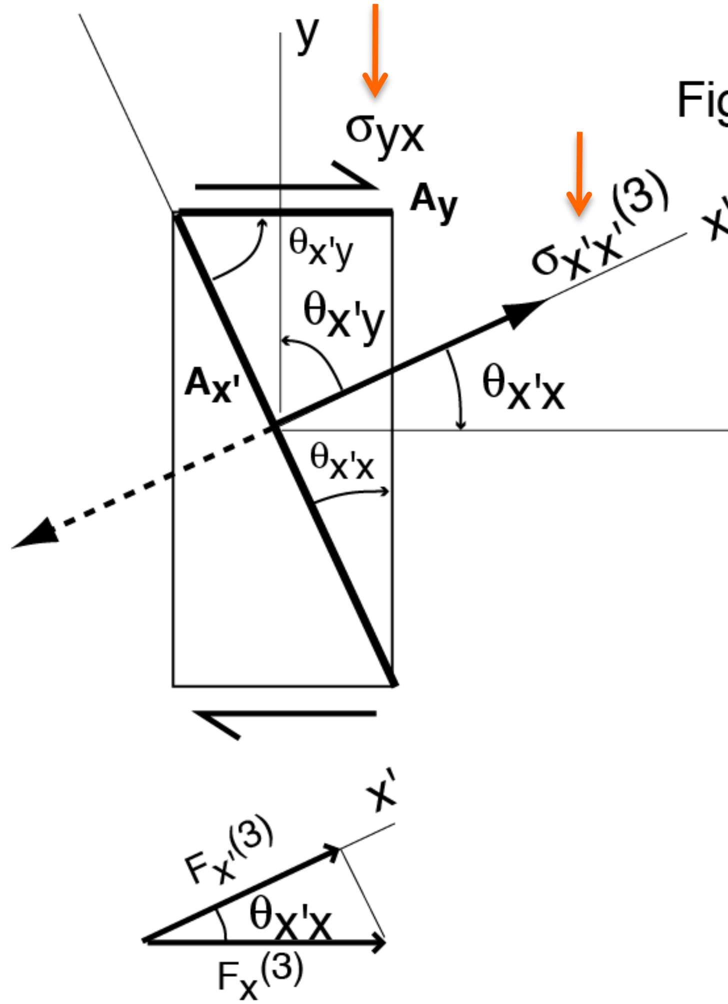


Fig. 18.3

18. Tensor Transformation of Stresses

IV Derivation

E Contribution of σ_{yy} to $\sigma_{x'x'}$

w = dimensionless weighting factor

1 $$\sigma_{x'x'} = w^{(1)}\sigma_{xx} + w^{(2)}\sigma_{xy} + w^{(3)}\sigma_{yx} + \underline{w^{(4)}}\,\underline{\sigma_{yy}}$$

2 $$\frac{F_{x'}}{A_{x'}} = \left(\frac{A_x}{A_{x'}}\frac{F_{x'}^{(1)}}{F_x^{(1)}}\right)\frac{F_x^{(1)}}{A_x} + \left(\frac{A_x}{A_{x'}}\frac{F_{x'}^{(2)}}{F_y^{(2)}}\right)\frac{F_y^{(2)}}{A_x} + \left(\frac{A_y}{A_{x'}}\frac{F_{x'}^{(3)}}{F_x^{(3)}}\right)\frac{F_x^{(3)}}{A_y} + \underline{\left(\frac{A_y}{A_{x'}}\frac{F_{x'}^{(4)}}{F_y^{(4)}}\right)}\,\underline{\frac{F_y^{(4)}}{A_y}}$$

3 $$\sigma_{x'x'} = a_{x'x}a_{x'x}\sigma_{xx} + a_{x'x}a_{x'y}\sigma_{xy} + a_{x'y}a_{x'x}\sigma_{yx} + \underline{a_{x'y}a_{x'y}}\,\underline{\sigma_{yy}}$$

18. Tensor Transformation of Stresses

E Contribution of σ_{yy} to $\sigma_{x'x'}$

Start with the definition of a stress vector:

1 $\sigma_{x'x'}^{(4)} = F_{x'}^{(4)} / A_{x'}$

First find $F_y^{(4)}$ associated with σ_{yy}

2 $F_y^{(4)} = \sigma_{yy} A_y$

Find $F_{x'}^{(4)}$, the component of $F_x^{(4)}$ in the x'-direction

3 $F_{x'}^{(4)} = F_y^{(4)} \cos \theta_{x'y}$

Find A_y in terms of $A_{x'}$

$A_y = A_{x'} \cos \theta_{x'y}$

4 $A_{x'} = A_y / \cos \theta_{x'y}$

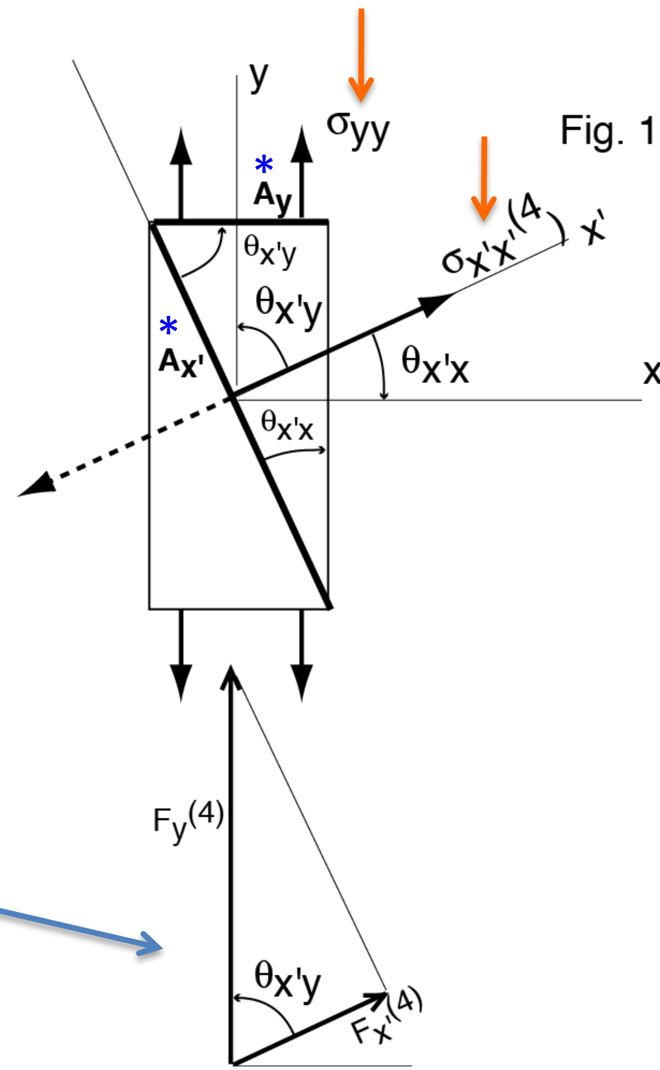


Fig. 18.4

18. Tensor Transformation of Stresses

E Contribution of σ_{yy} to $\sigma_{x'x'}$

3 $F_{x'}^{(4)} = F_y^{(4)} \cos\theta_{x'y}$

4 $A_{x'} = A_y / \cos\theta_{x'y}$

Now substitute:

5a $\sigma_{x'x'}^{(4)} = F_{x'}^{(4)} / A_{x'}$

5b $\sigma_{x'x'}^{(4)} = F_y^{(4)} \cos\theta_{x'y} / (A_y / \cos\theta_{x'y})$

5c $\sigma_{x'x'}^{(4)} = \cos\theta_{x'y} \cos\theta_{x'y} (F_y^{(4)}/A_y)$

5d $\sigma_{x'x'}^{(4)} = a_{x'y}\, a_{x'y}\, \sigma_{yy}$

Weighting factor $w^{(4)}$

6 $w^{(4)} = a_{x'y}\, a_{x'y}$

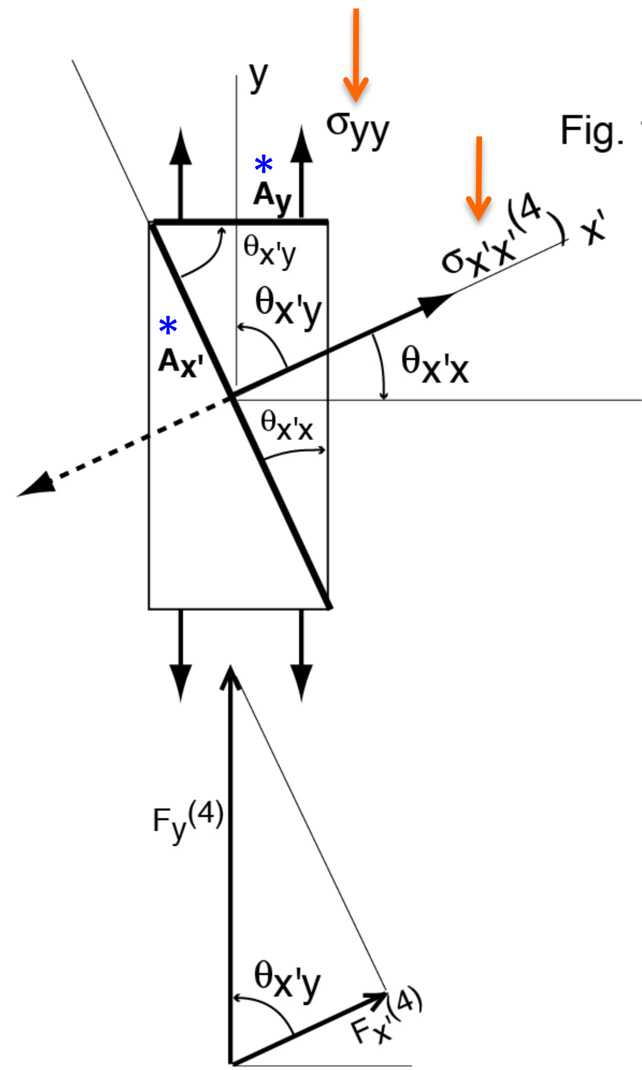


Fig. 18.4

18. Tensor Transformation of Stresses

IV Derivation

F Formulas for $\sigma_{x'x'}$, $\sigma_{x'y'}$, $\sigma_{y'x'}$, and $\sigma_{y'y'}$

1 $$\sigma_{x'x'} = a_{x'x}a_{x'x}\sigma_{xx} + a_{x'x}a_{x'y}\sigma_{xy} + a_{x'y}a_{x'x}\sigma_{yx} + a_{x'y}a_{x'y}\sigma_{yy}$$

2 $$\sigma_{x'y'} = a_{x'x}a_{y'x}\sigma_{xx} + a_{x'x}a_{y'y}\sigma_{xy} + a_{x'y}a_{y'x}\sigma_{yx} + a_{x'y}a_{y'y}\sigma_{yy}$$

3 $$\sigma_{y'x'} = a_{y'x}a_{x'x}\sigma_{xx} + a_{y'x}a_{x'y}\sigma_{xy} + a_{y'y}a_{x'x}\sigma_{yx} + a_{y'y}a_{x'y}\sigma_{yy}$$

4 $$\sigma_{y'y'} = a_{y'x}a_{y'x}\sigma_{xx} + a_{y'x}a_{y'y}\sigma_{xy} + a_{y'y}a_{y'x}\sigma_{yx} + a_{y'y}a_{y'y}\sigma_{yy}$$

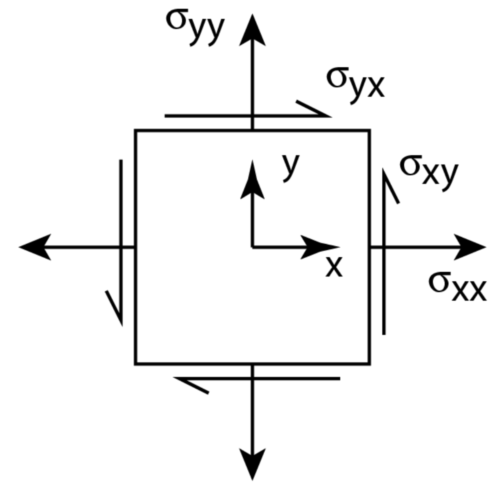
18. Tensor Transformation of Stresses

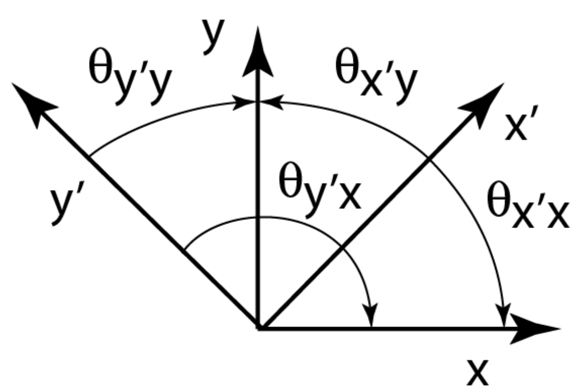
V Example

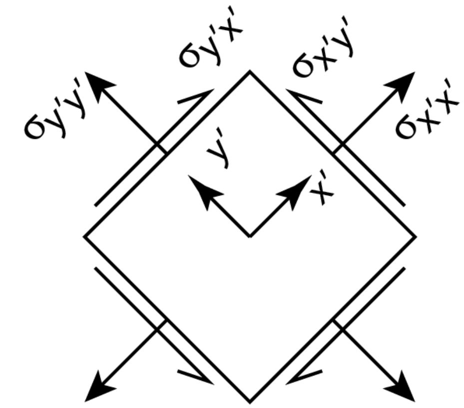
Find $$\sigma_{i'j'} = \begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \sigma_{y'x'} & \sigma_{y'y'} \end{bmatrix}$$

given $$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} = -4\,MPa & \sigma_{xy} = -4\,MPa \\ \sigma_{yx} = -4\,MPa & \sigma_{yy} = -4\,MPa \end{bmatrix}$$

$$\theta_{x'x} = -45^\circ, \theta_{x'y} = 45^\circ, \theta_{y'x} = -135^\circ, \theta_{y'y} = -45^\circ$$







18. Tensor Transformation of Stresses

V Example (cont.)

$$\sigma_{x'x'} = a_{x'x}a_{x'x}\sigma_{xx} + a_{x'x}a_{x'y}\sigma_{xy} + a_{x'y}a_{x'x}\sigma_{yx} + a_{x'y}a_{x'y}\sigma_{yy}$$

$$\sigma_{x'x'} = (-2MPa) + (-2MPa) + (-2MPa) + (-2MPa) = -8MPa$$

$$\sigma_{x'y'} = a_{x'x}a_{y'x}\sigma_{xx} + a_{x'x}a_{y'y}\sigma_{xy} + a_{x'y}a_{y'x}\sigma_{yx} + a_{x'y}a_{y'y}\sigma_{yy}$$

$$\sigma_{x'y'} = (2MPa) + (-2MPa) + (2MPa) + (-2MPa) = 0MPa$$

$$\sigma_{y'x'} = a_{y'x}a_{x'x}\sigma_{xx} + a_{y'x}a_{x'y}\sigma_{xy} + a_{y'y}a_{x'x}\sigma_{yx} + a_{y'y}a_{x'y}\sigma_{yy}$$

$$\sigma_{y'x'} = (2MPa) + (2MPa) + (-2MPa) + (-2MPa) = 0MPa$$

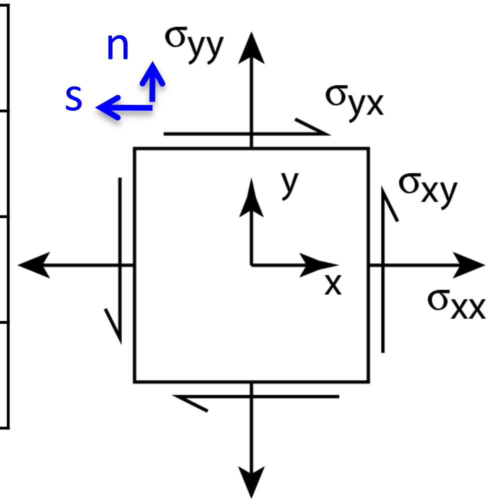
$$\sigma_{y'y'} = a_{y'x}a_{y'x}\sigma_{xx} + a_{y'x}a_{y'y}\sigma_{xy} + a_{y'y}a_{y'x}\sigma_{yx} + a_{y'y}a_{y'y}\sigma_{yy}$$

$$\sigma_{y'y'} = (-2MPa) + (2MPa) + (2MPa) + (-2MPa) = 0MPa$$

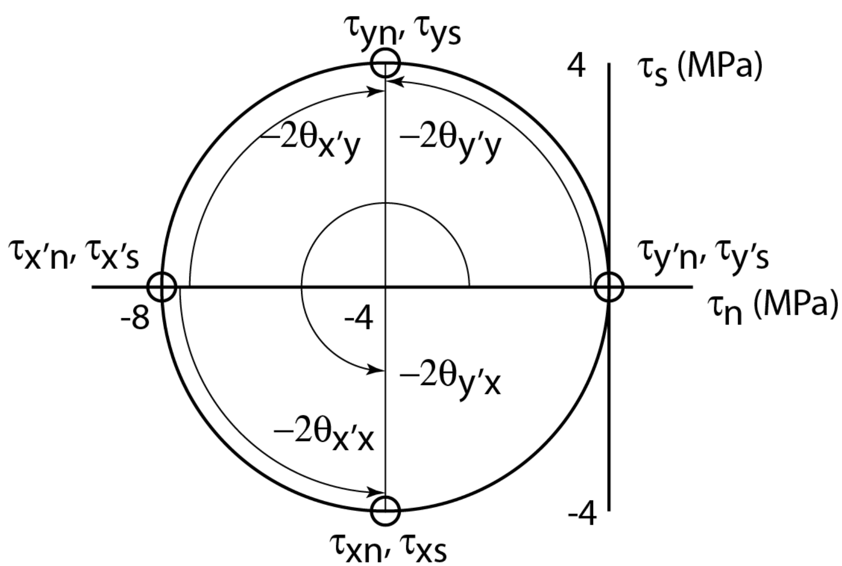
18. Tensor Transformation of Stresses

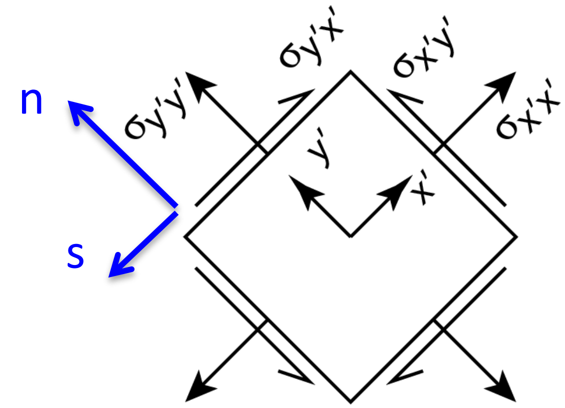
V Example (values in MPa)

σ_{xx} = - 4	τ_{xn} = - 4	$\sigma_{x'x'}$ = - 8	$\tau_{x'n}$ = - 8
σ_{xy} = - 4	τ_{xs} = - 4	$\sigma_{x'y'}$ = 0	$\tau_{x's}$ = 0
σ_{yx} = - 4	τ_{ys} = + 4	$\sigma_{y'x'}$ = -0	$\tau_{y's}$ = +0
σ_{yy} = - 4	τ_{yn} = -4	$\sigma_{y'y'}$ = 0	$\tau_{y'n}$ = 0



See slide 19 for identification of the angles





18. Tensor Transformation of Stresses

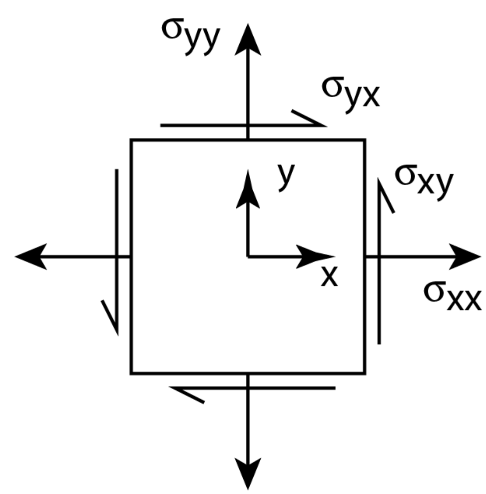
V Example

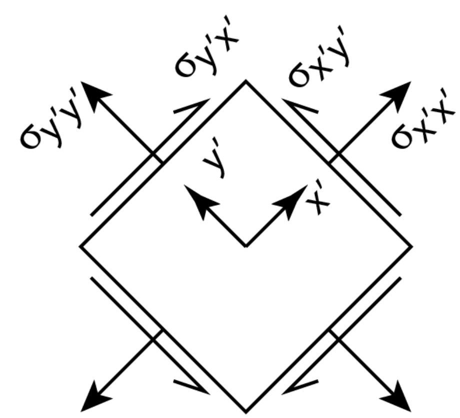
Matrix form

$$\begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \sigma_{y'x'} & \sigma_{y'y'} \end{bmatrix} = \begin{bmatrix} a_{x'x} & a_{x'y} \\ a_{y'x} & a_{y'y} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} a_{x'x} & a_{x'y} \\ a_{y'x} & a_{y'y} \end{bmatrix}^T$$

$$\left[\sigma_{i'j'}\right] = [a]\left[\sigma_{ij}\right][a]^T$$

This expression is valid in 2D and 3D!



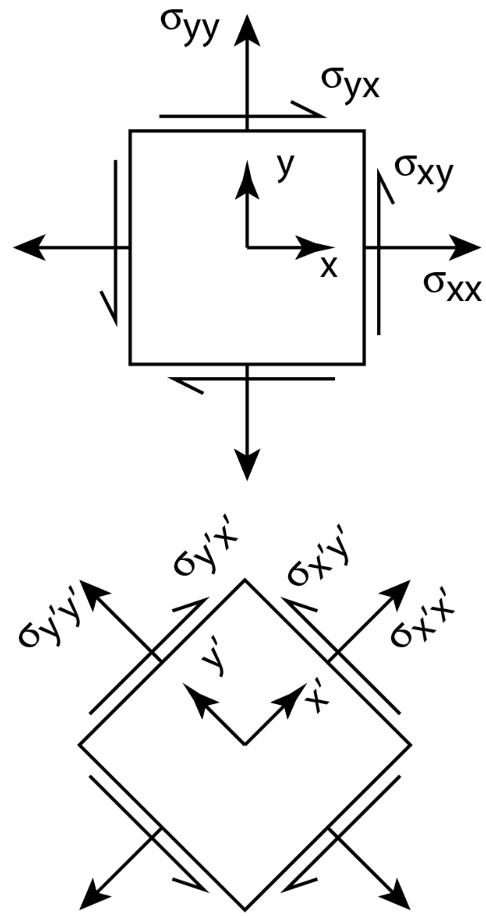


18. Tensor Transformation of Stresses

V Example

Matrix form/Matlab

$$\left[\sigma_{i'j'}\right] = [a]\left[\sigma_{ij}\right][a]^T$$




```
>> sij = [-4 -4;-4 -4]
sij =
   -4   -4
   -4   -4
>> a=[sqrt(2)/2 sqrt(2)/2; -sqrt(2)/2 sqrt(2)/2]
a =
    0.7071    0.7071
   -0.7071    0.7071
>> sipjp = a*sij*a’
sipjp =
   -8.0000         0
         0         0
```