

## 15. FINITE STRAIN & INFINITESIMAL STRAIN (AT A POINT)

### I Main Topics

A The finite strain tensor  $E$

B Infinitesimal strain and its tensor  $\epsilon$

C Deformation paths for finite strain

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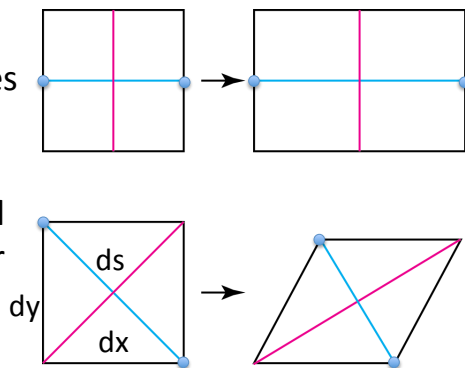
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## 15. FINITE STRAIN & INFINITESIMAL STRAIN

### II The finite strain tensor $E$

A Used to find the changes in the squares of distances  $(ds)^2$  between points in a deformed body based on differences in their initial positions



Sides of lower boxes maintain their length,  
but the diagonals change length

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II The finite strain tensor  $E$

Consider vector  $d\mathbf{X}$  and its length

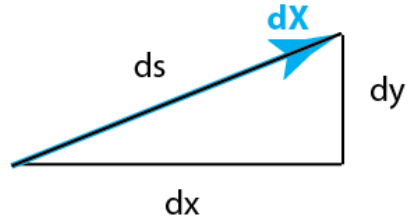
B Derivation of  $[E]$

$$1 \quad (ds)^2 = (dx)^2 + (dy)^2$$

$$2 \quad (ds)^2 = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$3 \quad \begin{bmatrix} dx \\ dy \end{bmatrix} = [dX]$$

$$4 \quad \begin{bmatrix} dx & dy \end{bmatrix} = [dX]^T$$



$$(dx)^2 + (dy)^2 = (ds)^2 = d\mathbf{X} \cdot d\mathbf{X}$$

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II The finite strain tensor  $E$

$$5 \quad (ds)^2 = [dX]^T [dX] = [dX]^T [I] [dX],$$

B Derivation of  $[E]$

$$1 \quad (ds)^2 = (dx)^2 + (dy)^2$$

$$2 \quad (ds)^2 = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$3 \quad \begin{bmatrix} dx \\ dy \end{bmatrix} = [dX]$$

$$4 \quad \begin{bmatrix} dx & dy \end{bmatrix} = [dX]^T$$

Now consider vector  $d\mathbf{X}'$

$$6 \quad [dX'] = [F][dX] \quad \text{From lecture 14:}$$

$$7 \quad (ds')^2 = \begin{bmatrix} dx' & dy' \end{bmatrix} \begin{bmatrix} dx' \\ dy' \end{bmatrix} \\ = [dX']^T [dX']$$

$$8 \quad (ds')^2 = [[F][dX]]^T [[F][dX]]$$

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### B Derivation of [E] (cont.)

Now find the difference between the two different dot products, noting that  $[[F][dX]]^T = [dX]^T [F]^T$

$$8 \quad (ds')^2 = [[F][dX]]^T [[F][dX]]$$

$$9 \quad (ds')^2 - (ds)^2 = [dX]^T [F]^T [F][dX] - [dX]^T [I]^T [dX]$$

$$10 \quad (ds')^2 - (ds)^2 = [dX]^T [[F]^T [F] - [I]][dX]$$

$$11 \quad \frac{\{(ds')^2 - (ds)^2\}}{2} = \frac{[dX]^T [[F]^T [F] - [I]][dX]}{2}$$

$$12 \quad \frac{1}{2}\{(ds')^2 - (ds)^2\} = [dX]^T [E][dX] \quad [E] \equiv \frac{1}{2}[[F]^T [F] - [I]]$$

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### C Meaning of [E]

$$1 \quad \frac{1}{2}\{(ds')^2 - (ds)^2\} = [dX]^T [E][dX]$$

Given E and dX (the difference in initial positions) of points), then one can find (half) the difference in the squares of the lengths of lines connecting the points before and after deformation

$$2 \quad [E] \equiv \frac{1}{2}[[F]^T [F] - [I]]$$

$$3 \quad [E] \equiv \frac{1}{2}[[J_u + I]^T [J_u + I] - [I]]$$

[E] gives half the change of squares of line segment lengths, but what do the terms of [E] mean?

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D Expansion of [E]  $[E] \equiv \frac{1}{2} \left[ [J_u + I]^T [J_u + I] - [I] \right]$

$$1 \quad J_u = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$2 \quad [E] \equiv \frac{1}{2} \left[ \left[ \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]^T \left[ \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

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D Expansion of [E]  $[E] \equiv \frac{1}{2} \left[ [J_u + I]^T [J_u + I] - [I] \right]$

$$3 \quad [E] \equiv \frac{1}{2} \left[ \left[ \begin{bmatrix} \frac{\partial u}{\partial x} + 1 & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + 1 \end{bmatrix} \right]^T \left[ \begin{bmatrix} \frac{\partial u}{\partial x} + 1 & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} + 1 \end{bmatrix} \right] - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$4 \quad [E] \equiv \frac{1}{2} \left[ \begin{array}{cc} \left( \frac{\partial u}{\partial x} + 1 \right) \left( \frac{\partial u}{\partial x} + 1 \right) + \left( \frac{\partial v}{\partial x} \right) \left( \frac{\partial v}{\partial x} \right) - 1 & \left( \frac{\partial u}{\partial x} + 1 \right) \left( \frac{\partial u}{\partial y} \right) + \left( \frac{\partial v}{\partial x} \right) \left( \frac{\partial v}{\partial y} + 1 \right) \\ \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial u}{\partial x} + 1 \right) + \left( \frac{\partial v}{\partial y} + 1 \right) \left( \frac{\partial v}{\partial x} \right) & \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial u}{\partial y} \right) + \left( \frac{\partial v}{\partial y} + 1 \right) \left( \frac{\partial v}{\partial y} + 1 \right) - 1 \end{array} \right]$$

The meanings of the terms in [E] still are not intuitive

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### III Infinitesimal strain

#### A Derivation of infinitesimal strain tensor $[\epsilon]$

$$1 \quad [E] \equiv \frac{1}{2} \begin{bmatrix} \left(\frac{\partial u}{\partial x} + 1\right)\left(\frac{\partial u}{\partial x} + 1\right) + \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) - 1 & \left(\frac{\partial u}{\partial x} + 1\right)\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial y} + 1\right) \\ \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x} + 1\right) + \left(\frac{\partial v}{\partial y} + 1\right)\left(\frac{\partial v}{\partial x}\right) & \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial y} + 1\right)\left(\frac{\partial v}{\partial y} + 1\right) - 1 \end{bmatrix}$$

If the displacement derivatives are  $\ll 1$ , their products with each other can be neglected

$$2 \quad [\epsilon] \equiv \frac{1}{2} \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial u}{\partial x}\right) & \left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right) \\ \left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right) & \left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial v}{\partial y}\right) \end{bmatrix} = \frac{1}{2} [J_u] + [J_u]^T$$

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### IV Infinitesimal strain

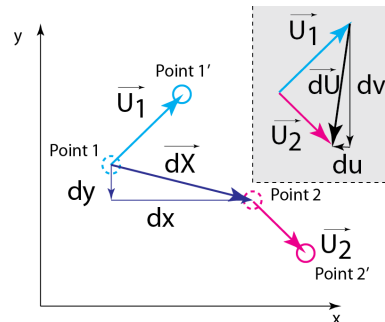
#### B Taylor series expansion

We seek  $\mathbf{U}_2$  given  $\mathbf{U}_1$  and  $d\mathbf{X}$

$$1 \quad u_2 = u_1 + du = u_1 + \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) + \dots$$

$$2 \quad v_2 = v_1 + dv = v_1 + \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right) + \dots$$

$$3 \quad \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} + \dots \Rightarrow [U_2] = [U_1] + [dU] \approx [U_1] + [J_u][dX]$$



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### B Taylor series expansion (cont.)

Now split  $[J_u]$  into two matrices: the infinitesimal strain matrix  $[\varepsilon]$  and the anti-symmetric rotation matrix  $[\omega]$

$$[J_u] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \quad [J_u]^T = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix} \frac{\partial v}{\partial x}$$

$$[\varepsilon] = \frac{[J_u] + [J_u]^T}{2} = \frac{1}{2} \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \end{bmatrix} \quad [\omega] = \frac{[J_u] - [J_u]^T}{2} = \frac{1}{2} \begin{bmatrix} 0 & \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} & 0 \end{bmatrix}$$

Symmetric  $\rightarrow$   $[\varepsilon]$        $[\omega]$       Anti-symmetric

$$[\varepsilon] + [\omega] = [J_u]$$

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### B Taylor series expansion (cont.)

$$[\varepsilon] = \frac{1}{2} \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix}; \varepsilon_{xy} = \varepsilon_{yx} \quad \text{Symmetric}$$

$$[\omega] = \frac{1}{2} \begin{bmatrix} 0 & \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \omega_{xy} \\ \omega_{yx} & 0 \end{bmatrix}; \omega_{xy} = -\omega_{yx} \quad \text{Anti-symmetric}$$

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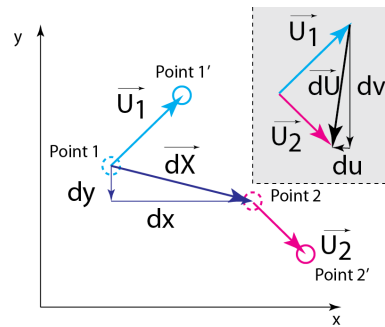
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## IV Infinitesimal strain

### C Taylor series expansion

$$[U_2] \approx [U_1] + [J_u][dX] = [U_1] + [[\epsilon] + [\omega]][dX]$$

Infinitesimal deformation can be decomposed into:  
 a translation (given by  $[U_1]$ ),  
 a strain (given by  $[\epsilon]$ ),  
 and  
 a rotation (given by  $[\omega]$ )

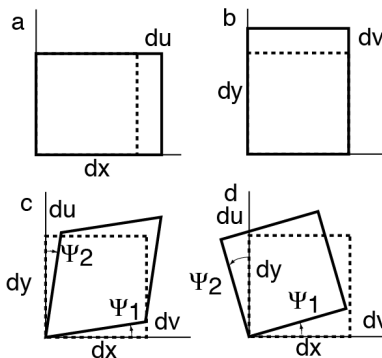


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$$\epsilon_{xx} = \frac{\partial u}{\partial x} \quad \epsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}(\Psi_1 - \Psi_2) = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

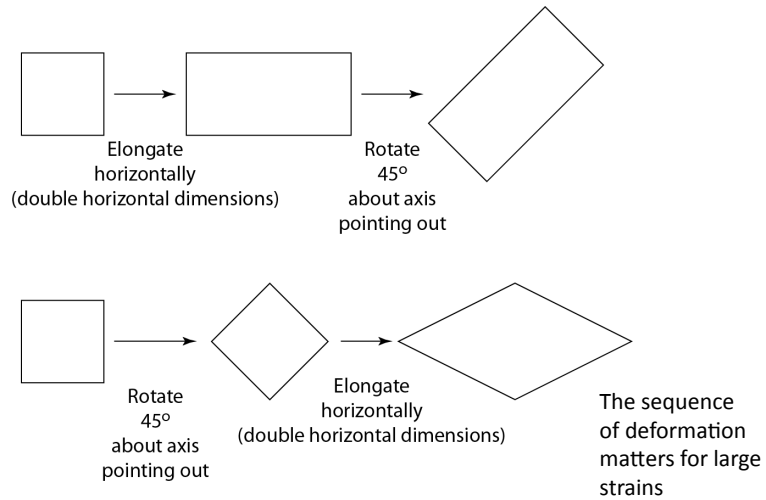
$$\omega_{xy} = -\omega_{yx} = \frac{1}{2}(\Psi_1 + \Psi_2) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$



For small angles,  $\Psi = \tan\Psi$

Positive angles are measured about the z-axis using a right hand rule. In (c) the angle  $\Psi_2$  is clockwise (negative), but  $du$  is positive. In (d)  $\Psi_2$  is counter-clockwise, and  $du < 0$ .

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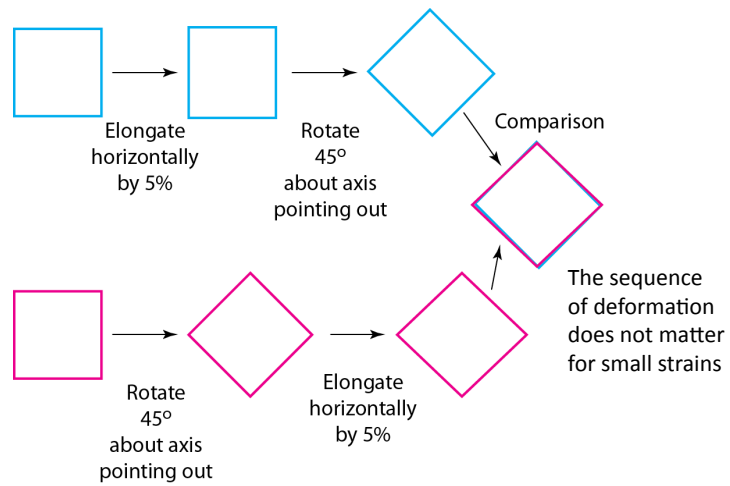


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## 15. FINITE STRAIN & INFINITESIMAL STRAIN



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## 15. FINITE STRAIN & INFINITESIMAL STRAIN

### III Deformation paths for finite strain

Consider two different deformations

#### A Deformation 1

$$[F_1] = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

#### B Deformation 2

$$[F_2] = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

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## 15. FINITE STRAIN & INFINITESIMAL STRAIN

### III Deformation paths for finite strain

Consider two different deformations

#### A Deformation 1

$$[F_1] = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

#### B Deformation 2

$$[F_2] = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

#### C F2 acts on F1

$$[F_2][F_1] = \begin{bmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \end{bmatrix}$$

#### D F1 acts on F2

$$[F_1][F_2] = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}$$

E The sequence of finite deformations matters – unless off-diagonal terms are small

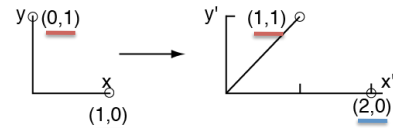
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[F1]



This shows the effect of [F1] all by itself

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

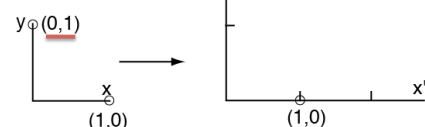
[F1]

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

[F1]

Two vectors, <1,0> and <0,1>, can be handled at the same time

[F2]



This shows the effect of [F2] all by itself

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

[F2]

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

[F2]

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Now consider [F2] acting after [F1] (i.e., [F2] acts on [F1])

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = [F_2][F_1]$$

[F2]      [F1]

Now consider [F1] acting after [F2] (i.e., [F1] acts on [F2])

$$\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = [F_1][F_2]$$

[F1]      [F2]

$$\boxed{[F_2][F_1] \neq [F_1][F_2]}$$

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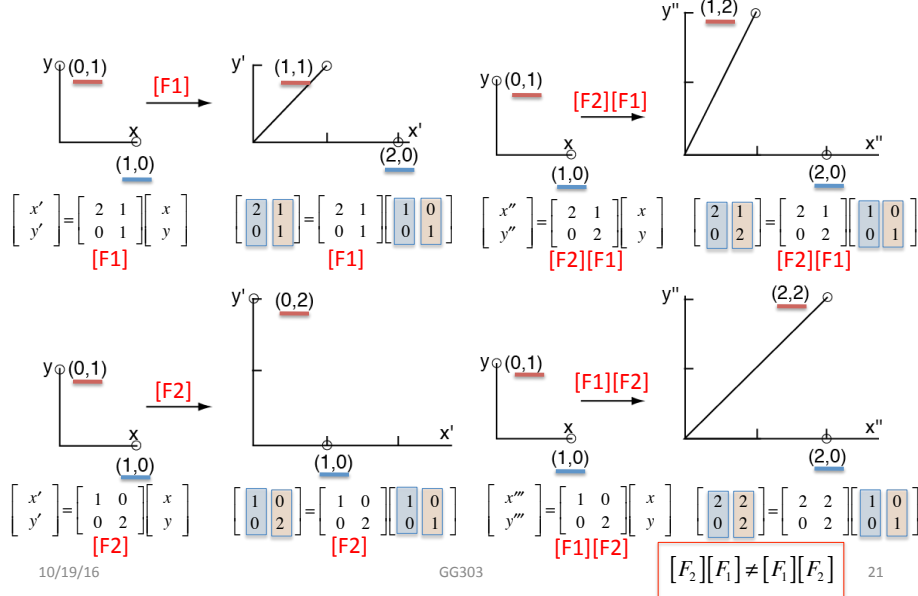
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Individual effects

Combined effects



### 15. FINITE STRAIN & INFINITESIMAL STRAIN

IV Infinitesimal strain and the infinitesimal strain tensor  $[\epsilon]$

A Infinitesimal strain

Deformation where the displacement derivatives in  $[J_u]$  are small relative to one so that the products of the derivatives are very small and can be ignored.

B An approximation to finite strain

$$[\epsilon] \equiv \frac{1}{2} \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial u}{\partial x}\right) & \left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right) \\ \left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right) & \left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial v}{\partial y}\right) \end{bmatrix}$$

$$[\epsilon] = \frac{1}{2} [ [J_u] + [J_u]^T ]$$

$$[J_u] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \quad [J_u]^T = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix}$$

## 15. FINITE STRAIN & INFINITESIMAL STRAIN

### IV Infinitesimal strain and the infinitesimal strain tensor $[\epsilon]$ (cont.)

#### C Why consider $[\epsilon]$ if it is an approximation?

- 1 Relevant to important geologic deformations
  - A Fracture
  - B Earthquake deformation
  - C Volcano deformation

Gelatin Volcano Experiment



[http://www.spacegrant.hawaii.edu/class\\_acts/Webimg/gelatinVolcano.gif](http://www.spacegrant.hawaii.edu/class_acts/Webimg/gelatinVolcano.gif)

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#### C Why consider $[\epsilon]$ if it is an approximation? (cont.)

- 2 Terms of the infinitesimal strain tensor  $[\epsilon]$  have clear geometric meaning
- 3 Can apply principal of superposition (addition)
- 4 Infinitesimal deformation is essentially independent of the deformation sequence
- 5 Amenable to sophisticated mathematical treatment (e.g., elasticity theory)
- 6 Quantitative predictive ability

Gelatin Volcano Experiment



[http://www.spacegrant.hawaii.edu/class\\_acts/Webimg/gelatinVolcano.gif](http://www.spacegrant.hawaii.edu/class_acts/Webimg/gelatinVolcano.gif)

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## 15. FINITE STRAIN & INFINITESIMAL STRAIN

C Why consider  $[\epsilon]$  if it is an approximation? (cont.)

7 Infinitesimal strain example

$$F_3 = \begin{bmatrix} 1.02 & 0.01 \\ 0 & 1.01 \end{bmatrix} \longrightarrow J_{u(3)} = \begin{bmatrix} 0.02 & 0.01 \\ 0 & 0.01 \end{bmatrix}$$

$$F_4 = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.02 \end{bmatrix} \longrightarrow J_{u(4)} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}$$

$$[F_4][F_3] = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.02 \end{bmatrix} \begin{bmatrix} 1.02 & 0.01 \\ 0 & 1.01 \end{bmatrix} = \begin{bmatrix} 1.0302 & 0.0100 \\ 0.0000 & 1.0302 \end{bmatrix}$$

$$[F_3][F_4] = \begin{bmatrix} 1.02 & 0.01 \\ 0 & 1.01 \end{bmatrix} \begin{bmatrix} 1.01 & 0 \\ 0 & 1.02 \end{bmatrix} = \begin{bmatrix} 1.0302 & 0.0101 \\ 0.0000 & 1.0302 \end{bmatrix}$$

$$[I] + [[J_{u(3)}] + [J_{u(4)}]] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.02 & 0.01 \\ 0 & 0.01 \end{bmatrix} + \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix} = \begin{bmatrix} 1.0300 & 0.0100 \\ 0.0000 & 1.0300 \end{bmatrix}$$

Sequence results can be obtained regardless of the order of events, but also by superposition (bottom row)

Results nearly indistinguishable