14. HOMOGENEOUS FINITE STRAIN

I Main Topics
- A Position vectors
- B Displacement vectors
- C Infinitesimal differences in position
- D Infinitesimal differences in displacement
- E Chain rule for a function of multiple variables
- F Gradient tensors and matrix representation
- G Homogenous (uniform) deformation
- H Examples

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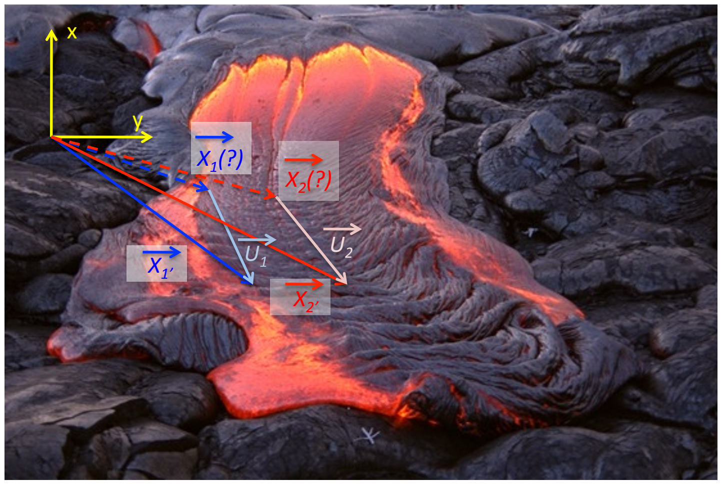

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II Position vectors

A Initial position vectors:

$$\text{Pt.1}: \vec{X}_1 = \vec{x}_1 + \vec{y}_1$$

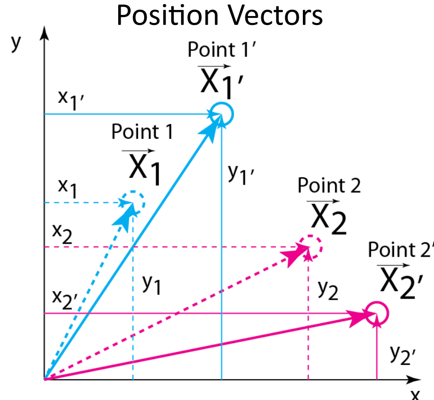
$$\text{Pt.2}: \vec{X}_2 = \vec{x}_2 + \vec{y}_2$$

B Final position vectors:

$$\text{Pt.1}': \vec{X}_{1'} = \vec{x}_{1'} + \vec{y}_{1'}$$

$$\text{Pt.2}': \vec{X}_{2'} = \vec{x}_{2'} + \vec{y}_{2'}$$

Position Vectors



Point 1 moves to Point 1′
Point 2 moves to Point 2′

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III Displacement vectors

A In terms of positions:

$$\text{Pt.1}: \vec{U}_1 = \vec{X}_{1'} - \vec{X}_1$$

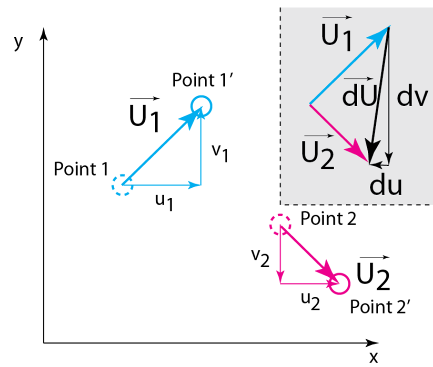
$$\text{Pt.2}: \vec{U}_2 = \vec{X}_{2'} - \vec{X}_2$$

B Displacement vector components:

$$\text{Pt.1}: \vec{U}_1 = \vec{u}_1 + \vec{v}_1$$

$$\text{Pt.2}: \vec{U}_2 = \vec{u}_2 + \vec{v}_2$$

Displacement Vectors



Point 1 moves to Point 1′
Point 2 moves to Point 2′
$\vec{U}_2$ is displaced, rotated, and "stretched" relative to $\vec{U}_1$.

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III Infinitesimal differences in position

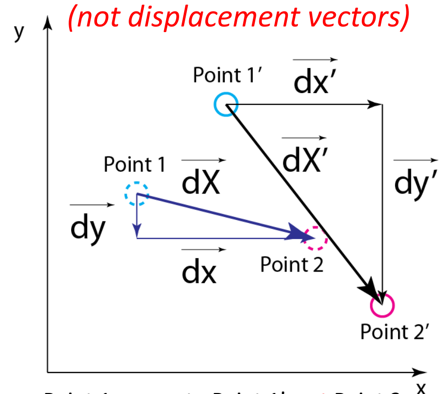
A Difference in initial positions (Pts. 1 and 2)

$$d\vec{X} = \vec{X}_2 - \vec{X}_1$$

B Difference in final positions (Pts. 1′ and 2′)

$$d\vec{X}' = \vec{X}_{2'} - \vec{X}_{1'}$$

Difference in Positions
(not displacement vectors)



Point 1 moves to Point 1′, not Point 2
Point 2 moves to Point 2′
$d\vec{X}'$ is displaced, rotated, and stretched relative to $d\vec{X}$.

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IV Infinitesimal differences in displacement

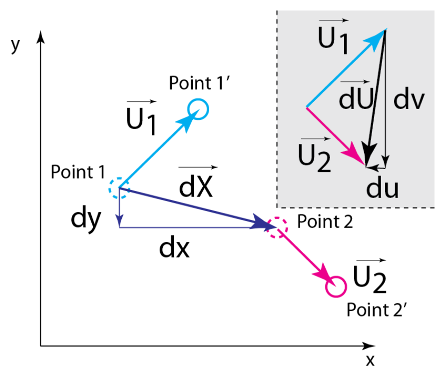
A Difference in displacement

$$d\vec{U} = \vec{U}_2 - \vec{U}_1$$

B Components of difference in displacement

$$d\vec{U} = d\vec{u} + d\vec{v}$$

Differences in Displacement



Point 1 moves to Point 1′, not Point 2

Point 2 moves to Point 2′

$\vec{dX'}$ is displaced, rotated, and stretched relative to $\vec{dX}$.

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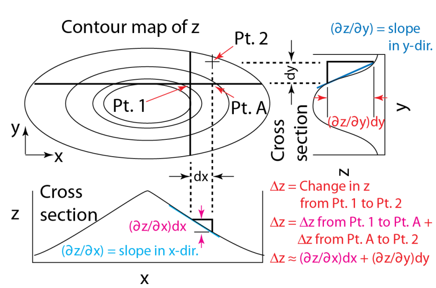
V Chain rule

A Functions of two variables

$$z = z(x,y);\ dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

The total change in a function of variables x and y equals its rate of change with respect to x, multiplied by the change in x, plus its rate of change with respect to y, multiplied by the change in y.



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V Chain rule

A Functions of two variables (cont.)

x′ = x′(x,y), y′ = y′(x,y)

u = u(x,y), v = v(x,y)

$$dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy$$

$$dy' = \frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

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V Chain rule

B 2D matrix representation

$$dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy$$

$$dy' = \frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Matrix form

$$\begin{bmatrix} dx' \\ dy' \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$[dX'] = [F][dX]$$

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$[dU] = [J_u][dX]$$

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V Chain rule

C 3D matrix representation

$$\begin{bmatrix} dx' \\ dy' \\ dz' \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

$$[dX'] = [F][dX]$$

Difference in final position in terms of difference in initial position

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

$$[dU] = [J_u][dX]$$

Difference in displacement in terms of difference in initial position

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VI Gradient tensors and matrix representation

A Deformation gradient tensor [F]: relates differences in initial position to differences in final position

B Displacement gradient tensor [J_u]: relates differences in initial position to differences in displacement

Matrix form (2D)

$$\begin{bmatrix} dx' \\ dy' \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$[dX'] = [F][dX]$$

Differences in final position; Differences in initial position

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$[dU] = [J_u][dX]$$

Differences in displacement; Differences in initial position

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VII Homogeneous (uniform) deformation

A 2D Equations

1 Chain rule

$$dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy$$
$$dy' = \frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

2 Scale

a At a *point*, derivatives have unique (constant) values; equations are linear in dx and dy in the neighborhood of the point (e.g., dx' and dy' depend on dx and dy raised to the first power.

b If the derivatives do not vary with x or y, (i.e., are constant as a function of position), then the equations are linear in dx and dy no matter how large dx and dy are. *This is the condition of homogenous strain*.

c Homogeneous strain applies at a point

d Homogeneous strain *is applied* to "small" regions

e Deformation in large regions is invariably inhomogeneous (derivatives vary spatially)

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VII Homogeneous (uniform) deformation (cont.)

B Common reformulation

1 For constant derivatives a, b, c, d, one replaces dx, dy, dx' and dy' by x, y, x', and y' (derivatives are the same for small dx or large x)

2 This distinction is seldom stated!

3 Linearity is clarified

4 Chain rule origin is obscured

$$dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy \Rightarrow$$
$$x' = ax + by$$

$$dy' = \frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy \Rightarrow$$
$$y' = cx + dy$$

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VII Homogeneous (uniform) deformation (cont.)

C Lagrangian

a $x' = ax + by$

b $y' = cx + dy$

(x', y'): Final positions; (x, y): Initial positions

D Eulerian (see derivation)

a $x = Ax' + By'$

b $y = Cx' + Dy'$

(x, y): Initial positions; (x', y'): Final positions

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Derivation of Eulerian equations

Start with Lagrangian equations

a $x' = ax + by \rightarrow y = (x'-ax)/b$

b $y' = cx + dy \rightarrow y = (y'-cx)/d$

Equate right sides above

d $(x' - ax)/b = (y' - cx)/d$

e $d(x'-ax) = b(y'- cx)$

f $cbx - adx = by'-dx'$

g $x(cb-ad) = by'-dx'$

h $x = [-d/(cb-ad)]\, x' + [b/(cb-ad)]\, y'$

i $x = Ax' + By'$ ← Eulerian equations

j $x' = ax + by \rightarrow x = (x'-by)/a$

k $y' = cx + dy \rightarrow x = (y'-dy)/c$

Equate right sides above

l $(y' - dy)/c = (x' - by)/a$

m $a(y' - dy) = c(x' - by)$

n $cby - ady = cx'-ay'$

o $y(cb-ad) = cx'-ay'$

p $y = [c/(cb-ad)]x' + [-a/(cb-ad)]y'$

Eulerian equations → q $y = Cx' + Dy'$

Bottom line: if final positions can be expressed as a linear combination of initial positions, then the initial positions can be expressed as a linear combination of initial positions

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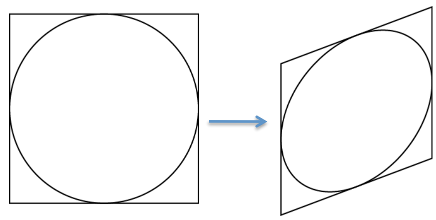
VII Homogenous (uniform) deformation (cont.)

- B Straight parallel lines remain straight and parallel (see appendix)
- C Parallelograms deform into parallelograms in 2-D;
- D Parallelepipeds deform into parallelepipeds in 3-D



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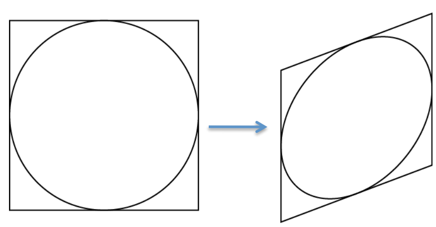
VII Homogenous (uniform) deformation (cont.)

- E Circles deform into ellipses in 2-D (see appendix); Spheres deform into ellipsoids in 3-D
- F The shape, orientation, and rotation of the "strain" (stretch) ellipse or ellipsoid describe homogeneous deformation.
- G The rotation is the angle between the axes of the strain ellipse and their counterparts before any deformation occurred (to be elaborated upon later).



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VII Homogenous (uniform) deformation (cont.)

H Lagrangian equations for position

1 $x' = ax + by$

$y' = cx + dy$

2 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

3 $[X'] = [F][X]$

F = Deformation gradient matrix

I Lagrangian equations for displacement

1 $u = x' - x = (a-1)x + by$

$v = y' - y = cx + (d-1)y$

2 $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

3 $U = [J_u][X]$

4 $[J_u] = [F] - [I]$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

J_u= Jacobian matrix for displacements

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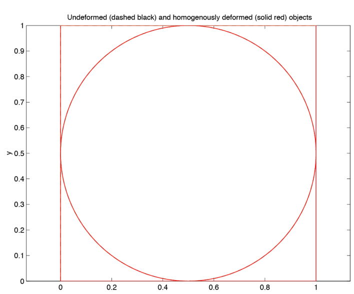
VIII Examples

A No strain



$x' = 1x + 0y$ $y' = 0x + 1y$ Position transformations (Lagrangian)	$u = 0x + 0y$ $v = 0x + 0y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Position transformations (matrix form)	$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Displacement gradient tensor J_u

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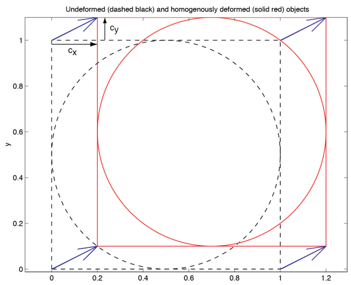
VIII Examples

B Rigid body translation



$x' = 1x + 0y + c_x$ $y' = 0x + 1y + c_y$ Position transformations (Lagrangian)	$u = 0x + 0y + c_x$ $v = 0x + 0y + c_y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix}$ Position transformations (matrix form)	$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Displacement gradient tensor J_u

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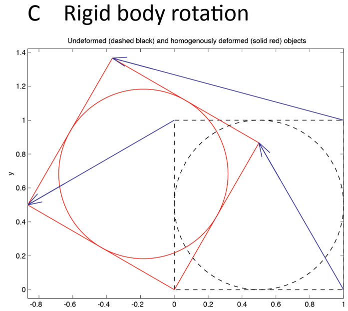
Example C: Rigid body rotation

VIII Examples

C Rigid body rotation



$x' = (\cos 60°)x - (\sin 60°)y$ $y' = (\sin 60°)x + (\cos 60°)y$ Position transformations (Lagrangian)	$u = (\cos 60° - 1)x - (\sin 60°)y$ $v = (\sin 60°)x + (\cos 60° - 1)y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Position transformations (matrix form)	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 60° - 1 & -\sin 60° \\ \sin 60° & \cos 60° - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} \cos 60° - 1 & -\sin 60° \\ \sin 60° & \cos 60° - 1 \end{bmatrix}$ Displacement gradient tensor J_u

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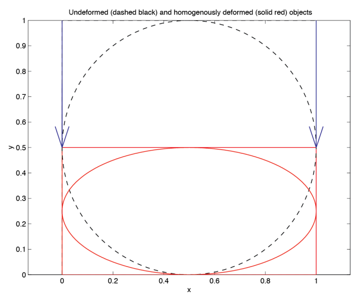
VIII Examples

D Uniaxial shortening



$x' = 1x + 0y$ $y' = 0x + 0.5y$ Position transformations (Lagrangian)	$u = 0x + 0y$ $v = 0x - 0.5y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Position transformations (matrix form)	$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix}$ Displacement gradient tensor J_u

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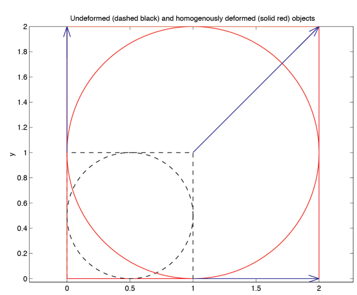
Example E: Dilation

VIII Examples

E Dilation



$x' = 2x + 0y$ $y' = 0x + 2y$ Position transformations (Lagrangian)	$u = 1x + 0y$ $v = 0x + 1y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Position transformations (matrix form)	$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Displacement gradient tensor J_u

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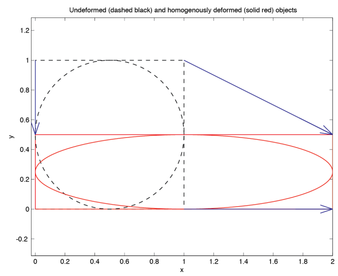
VIII Examples

F Pure shear strain (biaxial strain, no dilation



$x' = 2x + 0y$ $y' = 0x + 0.5y$ Position transformations (Lagrangian)	$u = 1x + 0y$ $v = 0x - 0.5y$ Displacement equations (Lagrangian)
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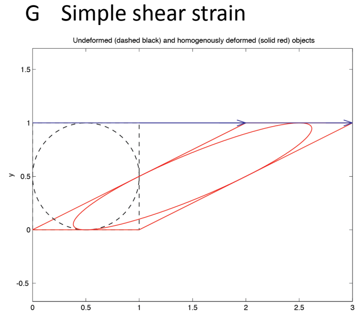
G Simple shear strain



$x' = 1x + 2y$ $y' = 0x + 1y$ Position transformations (Lagrangian)	$u = 0x + 2y$ $v = 0x + 0y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Position transformations (matrix form)	$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ Displacement gradient tensor J_u

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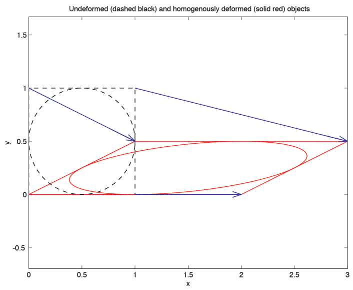
H General deformation (plane strain)



$x' = 2x + 1y$ $y' = 0x + 0.5y$ Position transformations (Lagrangian)	$u = 1x + 1y$ $v = 0x - 0.5y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Position transformations (matrix form)	$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 2 & 1 \\ 0 & -0.5 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 1 & 1 \\ 0 & -0.5 \end{bmatrix}$ Displacement gradient tensor J_u

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- Appendix: Proof that under homogenous 2D strain that an ellipse transforms into an ellipse
- The equation of an ellipse centered at the origin is

$$ax^2 + 2exy + dy^2 = 1$$

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- Appendix: Proof that under homogenous 2D strain that an ellipse transforms into an ellipse
- The equation of an ellipse centered at the origin is

$$ax^2 + 2exy + dy^2 = 1$$

- This equation can be written in matrix form as

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1$$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = 1$$

$$ax^2 + bxy + cxy + dy^2 = 1$$

$$ax^2 + (b+c)xy + dy^2 = 1$$

Letting (b+c) = 2e yields this

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$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\left[\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}\right]^T\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = 1$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}^T\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = 1$$

$$\begin{bmatrix} x' & y' \end{bmatrix}\begin{bmatrix} A & C \\ B & D \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = 1$$

All constants

$$\begin{bmatrix} x' & y' \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = 1$$

Constants

These have the same form, so both are equations of an ellipse.
So an ellipse transforms to an ellipse under homogeneous finite strain.