## 7. VECTORS, TENSORS, AND MATRICES

I Main Topics
A Vector length and direction B Vector Products

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III Vector length and direction A Vector length:

$$
|\mathbf{A}|=\sqrt{A_{x}{ }^{2}+A_{y}{ }^{2}+A_{z}{ }^{2}}
$$

B Vector A can be defined by its length $|\mathbf{A}|$ and the direction of a parallel unit vector a (which is given by its direction cosines).
$A=|A| a$.
C Unit vector a has vector components $A_{x} i /|A|, A_{y} j /|A|$, and $A_{2} \mathbf{k} /|\mathbf{A}|$, where $\mathbf{i}, \mathrm{j}$, and $\mathbf{k}$ are unit vectors along the $x, y$, and $z$ axes, respectively.

C Example

$$
\mathbf{A}=1.5 \mathbf{i}+2 \mathbf{j}+0 \mathbf{k}
$$

$$
|\mathbf{A}|=\sqrt{1.5^{2}+2^{2}+0^{2}}=\sqrt{6.25}=2.5
$$

$$
\mathbf{a}=\frac{1.5}{2.5} \mathbf{i}+\frac{2}{2.5} \mathbf{j}+\frac{0}{2.5} \mathbf{k}
$$

$$
\mathbf{a}=\frac{3}{5} \mathbf{i}+\frac{4}{5} \mathbf{j}+0 \mathbf{k}=0.6 \mathbf{i}+0.8 \mathbf{j}+0 \mathbf{k}
$$



## 7. VECTORS, TENSORS, AND MATRICES

III Vector Products
A Dot product: $\mathbf{A} \cdot \mathbf{B}=\mathrm{M}$
1 M is a scalar
2 If $B$ is a unit vector $b$, then $\mathbf{A} \bullet \mathbf{b}(o r \mathbf{b} \cdot \mathbf{A})$ is the projection of $\mathbf{A}$ onto the direction
 defined by b

## 7. VECTORS, TENSORS, AND MATRICES

A Dot product (cont.)
3 Let unit vectors $\mathbf{a}$ and $\mathbf{b}$ parallel vectors $\mathbf{A}$ and $\mathbf{B}$, respectively. The angle between $\mathbf{a}$ and $\mathbf{b}$ (and $\mathbf{A}$ and $\mathbf{B}$ ) is $\theta_{\mathrm{ab}}$. Convenient/ $\cos \left(\theta_{a b}\right)=\cos \left(-\theta_{a b}\right)$.
a $\mathbf{a} \bullet \mathbf{b}=\cos \left(\theta_{\mathrm{ab}}\right)=\mathbf{b} \cdot \mathbf{a}$

b $\mathbf{A}=|\mathbf{A}| \mathbf{a}$, and $\mathbf{B}=|\mathbf{B}| \mathbf{b}$
From (a) and (b)
c $\mathbf{A} \bullet \mathbf{B}=|\mathbf{A}||\mathbf{B}| \cos \left(\theta_{\mathrm{ab}}\right)$

## $\mathbf{a} \cdot \mathbf{b}$

$|A|=$ length of $A$

## 7. VECTORS, TENSORS, AND MATRICES

A Dot product (cont.)
4 Dot Product Tables of Cartesian Vectors

|  | i | j | k |
| :--- | :--- | :--- | :--- |
| i• | 1 | 0 | 0 |
| j• | 0 | 1 | 0 |
| k• | 0 | 0 | 1 |


|  | $B_{x} i$ | $B_{y} j$ | $B_{z} k$ |
| :--- | :--- | :--- | :--- |
| $A_{x} i \bullet$ | $A_{x} B_{x}$ | 0 | 0 |
| $A_{y} j \bullet$ | 0 | $A_{y} B_{y}$ | 0 |
| $A_{z} k \bullet$ | 0 | 0 | $A_{z} B_{z}$ |

5 For unit vectors $\mathbf{e}_{\mathrm{r}}$ and $\mathbf{e}_{\mathrm{s}}$ along axes of a Cartesian frame
$\mathbf{e}_{\mathrm{r}} \cdot \mathbf{e}_{\mathrm{s}}=1$ if $\mathrm{r}=\mathrm{s}$
$\mathbf{e}_{\mathrm{r}} \cdot \mathbf{e}_{\mathrm{s}}=0$ if $\mathrm{r} \neq \mathrm{s}$
$6 \mathrm{~A} \cdot \mathrm{~B}=\left(A \mathbf{i}+A_{j} \mathbf{j}+A_{k} \mathbf{k}\right) \cdot\left(B_{i} \mathbf{i}+B_{j} \mathbf{j}+B_{k} \mathbf{k}\right)$
$\mathrm{A} \cdot \mathrm{B}=A_{i} B_{i}+A_{j} B_{j}+A_{k} B_{k}$

## 7. VECTORS, TENSORS, AND MATRICES

A Dot product (cont.)
7 In Matlab, $\mathrm{c}=\mathbf{A} \bullet \mathbf{B}$ is performed as
a $\quad c=\operatorname{dot}(A, B)$
b $\quad \mathrm{c}=\mathrm{A}(:)^{*} * \mathrm{~B}(:)$
8 Uses in geology: all kinds of projections
$\gg A=[1,2,3]$
A $=$ 123
>> B=A
$B=$
123
$\gg C=\operatorname{dot}(A, B)$
c =
14
>> $\mathrm{C}=\mathrm{A}(\text { ( })^{1 * B(:)}$
$\mathrm{c}=$

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II Vector Products (cont.)
B Cross product: $\mathbf{A} \times \mathbf{B}=\mathbf{C}$
1 C is a vector
2 C is perpendicular to the $\mathbf{A B}$ plane
3 Direction of $\mathbf{C}$ by right-hand rule
$4 \mathrm{~A} \times \mathrm{B}=-\mathrm{B} \times \mathrm{A}$

Note that here, $\mathrm{A}, \mathrm{B}$, and C are points, not vectors

$\overrightarrow{A B} \times \overrightarrow{A C}$ yields downward pole $A, B, C$ form clockwise circuit

## 7. VECTORS, TENSORS, AND MATRICES

B Cross product (cont.)
5 Let unit vectors $\mathbf{a}$ and $\mathbf{b}$ parallel vectors $\mathbf{A}$ and $\mathbf{B}$, respectively. The angle between $\mathbf{a}$ and $\mathbf{b}$ (and $\mathbf{A}$ and $\mathbf{B})$ is $\theta_{a b}$.
a $\mathbf{a x b}=\sin \left(\theta_{\mathrm{ab}}\right) \mathbf{n}$, where $\mathbf{n}$ is a unit normal to the AB plane

$$
\begin{aligned}
b A \times B & =|A| a \times|B| b \\
& =|A||B|(a \times b) \\
& =|A||B| \sin \left(\theta_{a b}\right) n
\end{aligned}
$$

c Example

$$
A=2 i+0 j+0 k, B=0 i+2 j+0 k
$$

$$
|\mathbf{A}|=2,|B|=2
$$

$$
A \times B=(2)(2) \sin \left(90^{\circ}\right) k=4 k
$$



## 7. VECTORS, TENSORS, AND MATRICES

B Cross product (cont.)
$6|\mathbf{A} \times \mathbf{B}|$ is the area of the parallelogram defined by vectors $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are along adjacent side of the parallelogram. In the figure below, $\mathbf{A} \times \mathbf{B}$ points into the page, and $\mathbf{B} \times \mathbf{A}$ points out of the page.


## 7. VECTORS, TENSORS, AND MATRICES

A Cross product (cont.)
7 Cross Product Tables of Cartesian Vectors

|  | i | j | k |
| :--- | :--- | :--- | :--- |
| $\mathbf{i x}$ | 0 | k | - |
| jx | $-k$ | 0 | i |
| $k x$ | j | $-\mathbf{i}$ | 0 |


|  | $B_{x} i$ | $B_{y} j$ | $B_{z} k$ |
| :--- | :--- | :--- | :--- |
| $A_{x} i x$ | 0 | $A_{x} B_{y} k$ | $-A_{x} B_{j} j$ |
| $A_{y} j x$ | $-A_{y} B_{x} k$ | 0 | $A_{y} B_{z} i$ |
| $A_{z} k x$ | $A_{z} B_{x} j$ | $-A_{z} B_{y} i$ | 0 |

$$
\begin{aligned}
8 \quad \mathbf{A x} \boldsymbol{B} & =\left(A_{x} \mathbf{i}+A_{y} \boldsymbol{j}+A_{z} \mathbf{k}\right) \times\left(B_{x} \mathbf{i}+B_{y} \boldsymbol{j}+B_{z} \mathbf{k}\right) \\
\mathbf{A x} \boldsymbol{B} & =\left(A_{y} B_{z}-A_{z} B_{y}\right) \mathbf{i}+\left(A_{x} B_{z}-A_{z} B_{x}\right) \mathbf{j}+\left(A_{x} B_{y}-A_{y} B_{x}\right) \mathbf{k}
\end{aligned}
$$

$$
9 \quad \mathrm{AxB}=\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
A_{x} & A_{y} & A_{z} \\
B_{x} & B_{y} & B_{z}
\end{array}\right|
$$

Note: The red letters in the tables are j , not i

## 7. VECTORS, TENSORS, AND MATRICES

B Cross product (cont.)
10 In Matlab, $\mathbf{C}=\mathbf{A} \times \mathbf{B}$ is performed as
$\mathrm{C}=\operatorname{cross}(\mathrm{A}, \mathrm{B})$
11 Uses in geology
a Finding poles to a plane from 3 coplanar pts
$\gg A=\left[\begin{array}{lll}2 & 0 & 0\end{array}\right]$

| $\begin{aligned} & \\ & \\ & 2 \\ & \\ &\end{aligned}$ |  |
| :---: | :---: |
|  |  |
| $\gg B=\left[\begin{array}{lll}0 & 2\end{array}\right]$ |  |
| $\mathrm{B}=$ |  |
| 0 | 2 |

$\gg C=\operatorname{cross}(A, B)$
C $=$
$0 \quad 0 \quad 4$
b Finding axes of cylindrical folds from poles to bedding

## 7. VECTORS, TENSORS, AND MATRICES

III Vector Products
C Scalar triple product: $(A, B, C)=A \bullet(B \times C)=V$
1 The vector triple product is a scalar (i.e., a number) that corresponds to a
 volume the volume of a parallelepiped with edges along A, B, and C.
$2|\mathrm{~V}| \geq 0$


## 7. VECTORS, TENSORS, AND MATRICES

C Scalar triple product (cont.)
3 The determinant of a $3 \times 3$ matrix reflects
 the volume of a parallelepiped

$V=\mathrm{A} \cdot(\mathrm{BxC})=\mathrm{A} \cdot\left|\begin{array}{ccc}\mathrm{i} & \mathrm{j} & \mathrm{k} \\ B_{x} & B_{y} & B_{z} \\ C_{x} & C_{y} & C_{z}\end{array}\right|=\left|\begin{array}{lll}A_{x} & A_{y} & A_{z} \\ B_{x} & B_{y} & B_{z} \\ C_{x} & C_{y} & C_{z}\end{array}\right|$

## 7. VECTORS, TENSORS, AND MATRICES

C Scalar triple product (cont.)
4 In Matlab:
$V=A \bullet(B \times C)$ is performed as $\mathrm{V}=$ $\operatorname{dot}(\mathrm{A}, \operatorname{cross}(\mathrm{B}, \mathrm{C}))$


5 Uses in geology
a Equation of plane
b Estimating volume
 of ore bodies

## 7. VECTORS, TENSORS, AND MATRICES

D Invariants
1 Quantities that do not depend on the orientation of a reference frame

2 Examples
a Dot product (a length)
b Scalar triple product (a volume)

