6. SCALARS, VECTORS, AND TENSORS (FOR ORTHOGONAL COORDINATE SYSTEMS)

I Main Topics

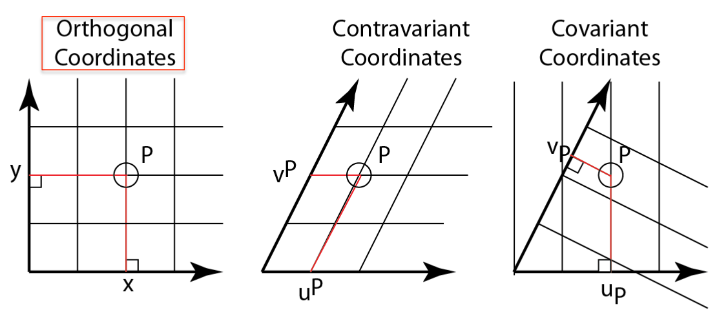
A What are scalars, vectors, and tensors?

B Order of scalars, vectors, and tensors

C Linear transformation of scalars and vectors (and tensors)

D Matrix multiplication

6. SCALARS, VECTORS, AND TENSORS



Coordinate axes meet at right angles; are parallel and perpendicular to reference axes

Coordinate axes meet at oblique angles, are parallel to reference axes

Coordinate axes meet at oblique angles, are perpendicular to reference axes

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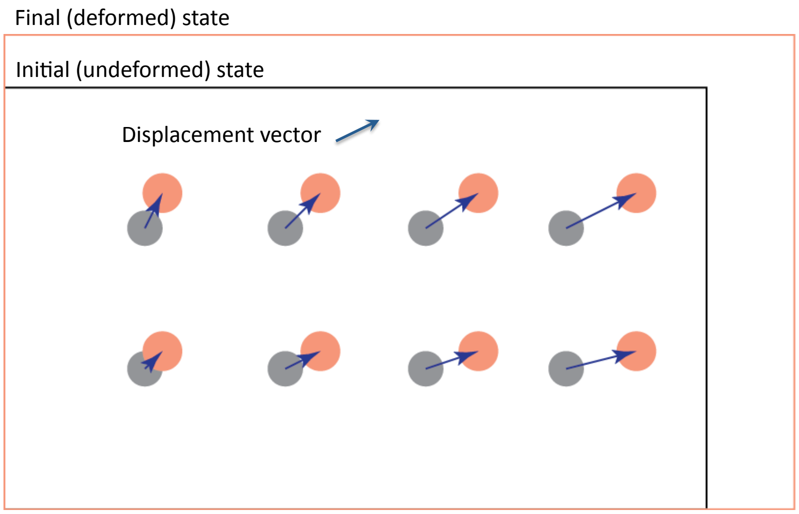

6. SCALARS, VECTORS, AND TENSORS

II What are scalars, vectors, and tensors?

- A Quantities with associated directions
- B Tensors
  - 1 Broaden our perspectives; geologists unacquainted with them are handicapped
  - 2 For multi-dimensional thinking and communication
  - 3 They can be extremely useful
  - 4 http://www.grc.nasa.gov/WWW/k-12/Numbers/Math/documents/Tensors_TM2002211716.pdf

6. SCALARS, VECTORS, AND TENSORS

III Order of scalars, vectors, and tensors

A Scalars (magnitudes)

1 Numbers with no associated direction (zero-order tensors)

2 No subscripts in notation

3 Examples: Time, mass, length volume

4 Matrix representation: 1x1 matrix [x]

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III Order of scalars, vectors, and tensors (cont.)

B Vectors (magnitude and a direction)

1 Quantities with one associated direction (first-order tensors)

2 One subscript in notation (e.g., u_x)

3 Examples: Displacement, velocity, acceleration

6. SCALARS, VECTORS, AND TENSORS

III <u>Order</u> of scalars, vectors, and tensors (cont.)

- B Vectors (magnitude and a direction) (cont.)
  - 4 Matrix representation: 1xn row matrix, or nx1 column matrix, with n components
    - a Two-dimensional vector (n=2 components):

      $[x\ y]$ or $[x_1\ x_2]$ 1 row, 2 columns

      $\begin{bmatrix} x \\ y \end{bmatrix} or \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ 2 rows, 1 column

      x = component in x-direction, y = component in y-direction

      x_1 = component in x-direction, x_2 = component in y-direction
    - b Three-dimensional vector (n=3 components):

      $[x\ y\ z]$ or $[x_1\ x_2\ x_3]$ 1 row, 3 columns
  - 5 Don't confuse the dimensionality of a tensor with its order

6. SCALARS, VECTORS, AND TENSORS

III <u>Order</u> of scalars, vectors, and tensors (cont.)

- C Tensors (magnitude and two directions)

  (for the 2nd-order tensors we will consider)
  - 1 Quantities with two associated direction (second-order tensors)
  - 2 Two subscripts in notation (e.g., σ_{xx})
  - 3 Examples: Stress, strain, permeability

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III "Order" of scalars, vectors, and tensors (cont.)
- C Tensors (magnitude and two directions) (cont.)
  - 4 Matrix representation: nxn matrix, with n^2 components
    - a Two-dimensional tensor (4 components):

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} or \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

2 rows, 2 columns

    - b Three-dimensional tensor (3 components):

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} or \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

3 rows, 3 columns

  - 5 An n-dimensional 2^{nd}-order tensor consists of n rows of n-dimensional vectors

6. SCALARS, VECTORS, AND TENSORS

IV Linear transformations
- A "Transformations" refers to how components change when the coordinate system changes.
- B "Linear" means the transformation depends on the length of the components, not, for example, on the square of the component lengths.
- C Transformations are used to when we change reference frames in order to present physical quantities from a different (clearer) perspective.
- D Transformations of tensors not covered today

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IV Linear transformations (cont.)

E Linear transformations of scalars

1 Scalar quantities don't change in response to a transformation of coordinates; they are invariant

2 Examples (independent of reference frame orientation)

a Mass

b Volume

c Density

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IV Linear transformations (cont.)

F Linear transformations of vectors (cont.)

1 Vector components change with a transformation of coordinates

a $\mathbf{V} = \mathbf{v}_x + \mathbf{v}_y = v_x\mathbf{i} + v_y\mathbf{j}$

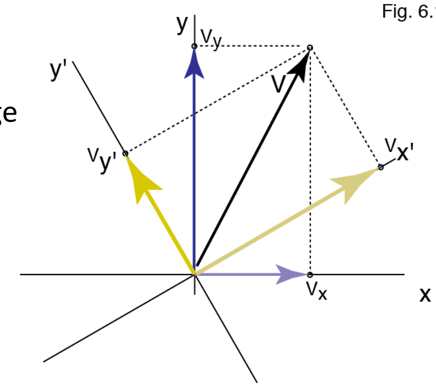
b $\mathbf{V} = \mathbf{v}_{x'} + \mathbf{v}_{y'} = v_{x'}\mathbf{i'} + v_{y'}\mathbf{j'}$

c Vector component: $\mathbf{v}_c$

d Scalar component: v_c

Bold: vector components
Unbolded: Scalar components

Fig. 6.1



i, **j**, **i'**, and **j'** are unit basis vectors along the x,y,x', and y' axes, respectively

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IV Linear transformations (cont.)

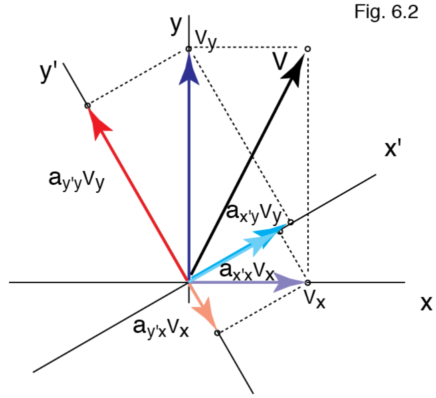
F Linear transformations of vectors (cont.)

2 Every component in the unprimed reference frame contributes linearly to each component in the primed reference frame.

$v_{x'} = a_{x'x} v_x + a_{x'y} v_y$

$v_{y'} = a_{y'x} v_x + a_{y'y} v_y$

Fig. 6.2



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IV Linear transformations (cont.)

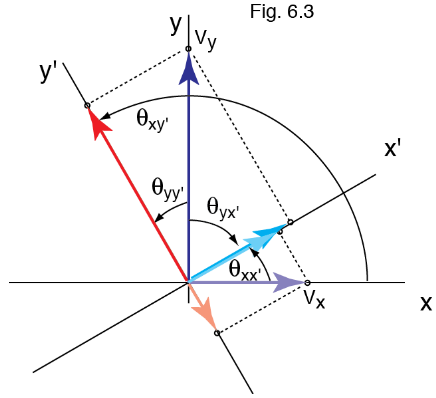
F Linear transformations of vectors (cont.)

3 The direction cosines are weighting factors that specify how much each component in one reference frame contributes to a component in the other reference frame.

$v_{x'} = a_{x'x} v_x + a_{x'y} v_y$

$v_{y'} = a_{y'x} v_x + a_{y'y} v_y$

Fig. 6.3



$a_{xx'} = \cos(\theta_{xx'}) = \cos(\theta_{x'x}) = a_{x'x}$

$a_{xy'} = \cos(\theta_{xy'}) = \cos(\theta_{y'x}) = a_{y'x}$

$a_{yx'} = \cos(\theta_{yx'}) = \cos(\theta_{x'y}) = a_{x'y}$

$a_{yy'} = \cos(\theta_{yy'}) = \cos(\theta_{y'y}) = a_{yy}$

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IV Linear transformations (cont.)

F Linear transformations of vectors (cont.)

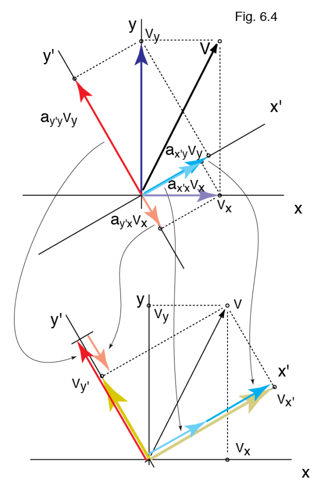
4 Transformation rule for vectors

a $v_{i'} = a_{i'j} \, v_j$

b Expanded form

$v_{x'} = a_{x'x} \, v_x + a_{x'y} v_y$

$v_{y'} = a_{y'x} \, v_x + a_{y'y} v_y$



6. SCALARS, VECTORS, AND TENSORS

IV Linear transformation of scalars, vectors, and tensors (cont.)

C Vectors (cont.)

4 Transformation rule for vectors

a $v_{i'} = a_{i'j} \, v_j$

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$v_{x'} = a_{x'x} \, v_x + a_{x'y} v_y$

$v_{y'} = a_{y'x} \, v_x + a_{y'y} v_y$

5 Matrix form

[V'] = [A][V]

(Note upper case)

- List what you know
- List what you want to know
- Add the projection terms

V'←V $$\begin{bmatrix} v_{x'} \\ v_{y'} \end{bmatrix} = \begin{bmatrix} a_{x'x} & a_{x'y} \\ a_{y'x} & a_{y'y} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

V←V' $$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} a_{xx'} & a_{xy'} \\ a_{yx'} & a_{yy'} \end{bmatrix} \begin{bmatrix} v_{x'} \\ v_{y'} \end{bmatrix}$$

6. SCALARS, VECTORS, AND TENSORS

V Matrix Multiplication - Examples

A General Rule: An nxm matrix times an mxp matrix gives a nxp matrix

B Examples

1 A 1x2 matrix times a 2x1 matrix gives a 1x1 matrix

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (1)(3)+(2)(4) \end{bmatrix} = \begin{bmatrix} 11 \end{bmatrix}$$

1 row
2 columns

2 rows
1 column

1 row
1 column

2 A 2x1 matrix times a 1x2 matrix gives a 2x2 matrix

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} (3)(1) & (3)(2) \\ (4)(1) & (4)(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$$

2 rows
1 column

1 row
2 columns

2 rows
2 columns

3 A 2x2 matrix times a 2x2 matrix gives a 2x2 matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1)(1)+(2)(0) & (1)(0)+(2)(1) \\ (3)(1)+(4)(0) & (3)(0)+(4)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$