## EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

I Main Topics
A Motivation
B Inverse and determinant of a matrix (review)
C Eigenvalue problems, eigenvectors, and eigenvalues
D Diagonalization of a matrix
E Quadratic forms and the principal axes theorem (maxima/minima in 2D)
F Strain ellipses, strain ellipsoids, and principal strains
II Motivation
A To find the magnitudes and directions of the principal strains from the displacement field using linear algebra rather than graphical means or calculus (e.g., Ramsay and Huber, 1984).

B To introduce a powerful, broadly useful mathematical method
II Inverse and determinant of a matrix (review)
A Inverse $[A]^{-1}$ of a matrix: $\quad A A^{-1}=A^{-1} A=I$, where A is nxn (square)
$B$ Inverse $[A]^{-1}$ of a $2 \times 2$ matrix $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$
$1 \quad A^{-1}=\frac{1}{a d-b c}\left[\begin{array}{cc}d & -b \\ -c & a\end{array}\right]$
[A] can not have an inverse if $a d-b c=0$
$2 \quad A^{-1} A=\frac{1}{a d-b c}\left[\begin{array}{cc}d & -b \\ -c & a\end{array}\right]\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]=\frac{1}{a d-b c}\left[\begin{array}{cc}d a-b c & d b-b d \\ -c a+a c & -b c+a d\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]=I \quad \vee$
C Determinant |A| of a real matrix
1 A number that provides scaling information on a square matrix
2 Determinant of a $2 \times 2$ matrix: $|A|=\left|\begin{array}{ll}a & b \\ c & d\end{array}\right|=a d-b c$
3 Determinant of a 3x3 matrix: $|A|=\left|\begin{array}{lll}a & b & c \\ d & e & f \\ g & h & i\end{array}\right|=a\left|\begin{array}{ll}e & f \\ h & i\end{array}\right|-b\left|\begin{array}{ll}d & f \\ g & i\end{array}\right|+c\left|\begin{array}{ll}d & e \\ g & h\end{array}\right|$

4 Relevance of the inverse of a matrix to a matrix equation $A X=0$ a $|A| \neq 0$ if the matrix has an inverse (see II.B.1). This is true for nxn matrices in general (e.g., Strang, 1998, p. 232-233).
b $|\mathrm{A}|=0$ if the matrix has no inverse (see II.B.1). This is true for nxn matrices in general (e.g., Strang, 1998, p. 232-233).

5 Geometric meanings of the real matrix equation $A X=0$ (See lab4!)

| nxn | $\mathrm{IA} \mid \neq 0 ; \mathrm{A}^{-1}$ exists (see II.B.1) | $\mathrm{IA} \mid=0 ; \mathrm{A}^{-1}$ does not exist (see II.B.1) |
| :--- | :--- | :--- |
| $2 \times 2$ | Describes two lines that <br> intersect at one unique point <br> - the origin. | Describes two lines that intersect in a <br> line that passes through the origin; no <br> unique solution. The lines are co-linear. |
| $3 \times 3$ | Describes three planes that <br> intersect at one unique point <br> - the origin. | Describes three planes that intersect <br> in a line (or a plane) that passes <br> through the origin; no unique solution. |

(See appendix for graphical examples)
III Eigenvalue problems, eigenvectors and eigenvalues
A Eigenvalue problems are represented by the matrix equation $A X=\lambda X$, where $A$ is a square $n \times n$ matrix, $X$ is a non-zero vector ( $\mathrm{an} n \times 1$ column array), and $\lambda$ is a number.

B Eigenvectors and eigenvalues provide simple, elegant, and clear ways to represent solutions to a host of physical and mathematical problems [e.g., geometry, strain, stress, curvature (shapes of surfaces)]

## C Eigenvectors

1 Non-zero directional vectors that provide solutions for $A X=\lambda X$
2 Vectors that maintain their orientation when multiplied by matrix $A$
$D$ Eigenvalues: numbers $(\lambda)$ that provide solutions for $A X=\lambda X$. If $X$ is a unit vector, $\lambda$ is the length of the vector produced by AX.

E Eigenvectors have corresponding eigenvalues, and vice-versa
F In Matlab, $[\mathrm{v}, \mathrm{d}]=$ eig(A), finds eigenvectors and eigenvalues

## G Examples

1 Identity matrix (I)

$$
\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
x \\
y
\end{array}\right]=1\left[\begin{array}{l}
x \\
y
\end{array}\right]
$$

All vectors in the xy-plane maintain their orientation and length when operated on by the identity matrix, so the eigenvalue for $/$ is 1 , and all vectors are eigenvectors.

2 A 2D rotation matrix for rotations in the xy plane
All non-zero real vectors rotate and maintain their length. A 2D rotation matrix thus has no real eigenvectors and hence no real eigenvalues; its eigenvectors and eigenvalues are imaginary.

3 A 3D rotation matrix. The only vectors that are not rotated are along the axis of rotation, so the one real eigenvector of a 3D rotation matrix gives the orientation of the axis of rotation.
4 A symmetric matrix: $A=\left[\begin{array}{ll}0 & 2 \\ 2 & 0\end{array}\right]$

| Matrix equation | Eigenvalue | Eigenvector |
| :--- | :--- | :--- |
| $A\left[\begin{array}{l}1 \\ 1\end{array}\right]=\left[\begin{array}{ll}0 & 2 \\ 2 & 0\end{array}\right]\left[\begin{array}{l}1 \\ 1\end{array}\right]=\left[\begin{array}{l}2 \\ 2\end{array}\right]=2\left[\begin{array}{l}1 \\ 1\end{array}\right]$. | 2 | $\left[\begin{array}{l}1 \\ 1\end{array}\right]$ |
| $A\left[\begin{array}{c}1 \\ -1\end{array}\right]=\left[\begin{array}{ll}0 & 2 \\ 2 & 0\end{array}\right]\left[\begin{array}{c}1 \\ -1\end{array}\right]=\left[\begin{array}{c}-2 \\ 2\end{array}\right]=-2\left[\begin{array}{c}1 \\ -1\end{array}\right]$. | -2 | $\left[\begin{array}{c}1 \\ -1\end{array}\right]$ |

5 Another symmetric matrix: $A=\left[\begin{array}{ll}9 & 3 \\ 3 & 1\end{array}\right]$

| Matrix equation | Eigenvalue | Eigenvector |
| :--- | :--- | :--- |
| $A\left[\begin{array}{l}-3 \sqrt{0.1} \\ -\sqrt{0.1}\end{array}\right]=\left[\begin{array}{ll}9 & 3 \\ 3 & 1\end{array}\right]\left[\begin{array}{l}-3 \sqrt{0.1} \\ -\sqrt{0.1}\end{array}\right]=\left[\begin{array}{l}-30 \sqrt{0.1} \\ -10 \sqrt{0.1}\end{array}\right]=10\left[\begin{array}{l}-3 \sqrt{0.1} \\ -\sqrt{0.1}\end{array}\right]$ | 10 | $\left[\begin{array}{l}-3 \sqrt{0.1} \\ -\sqrt{0.1}\end{array}\right]$ |
| $A\left[\begin{array}{l}\sqrt{0.1} \\ -3 \sqrt{0.1}\end{array}\right]=\left[\begin{array}{ll}9 & 3 \\ 3 & 1\end{array}\right]\left[\begin{array}{c}\sqrt{0.1} \\ -3 \sqrt{0.1}\end{array}\right]=\left[\begin{array}{l}0 \\ 0\end{array}\right]=0\left[\begin{array}{c}\sqrt{0.1} \\ -3 \sqrt{0.1}\end{array}\right]$. | 0 | $\left[\begin{array}{c}\sqrt{0.1} \\ -3 \sqrt{0.1}\end{array}\right]$ |

H Alternative form of an eigenvalue equation
$1[\mathrm{~A}][\mathrm{X}]=\lambda[\mathrm{X}]$
Since $\lambda[\mathrm{X}]=\lambda[\mathrm{IX}]$, subtracting $\lambda[\mathrm{IX}]$ from both sides of (H.1) yields:
$2[A-1 \lambda][X]=0$
(same form as $[B][X]=0$; see ll.C.4)
I Solution conditions and connections with determinants
1 Unique trivial solution of $[X]=0$ if and only if $|A-|\lambda| \neq 0$ (see II.C.4).
2 Eigenvector solutions $([X] \neq 0)$ if and only if $|A-|\lambda|=0$ (see II.C.4).
$J$ Characteristic equation: $|A-|\lambda|=0$
1 The roots of the polynomial equation produced by expanding the characteristic equation are the eigenvalues (from l.2).
2 General 2-D solution for eigenvalues Let $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right], \operatorname{tr}(A)=a+d,\left|\begin{array}{ll}a & b \\ c & d\end{array}\right|=a d-b c$. Then
(1) $|A-I \lambda|=\left|\begin{array}{cc}a-\lambda & b \\ c & d-\lambda\end{array}\right|=0$

$$
\begin{align*}
& \left|\begin{array}{cc}
a-\lambda & b \\
c & d-\lambda
\end{array}\right|=(a-\lambda)(d-\lambda)-(b)(c)=\lambda^{2}-(a+d) \lambda+a d-b c=0  \tag{2}\\
& |A-I \lambda|=\lambda^{2}-(a+d) \lambda+a d-b c=\lambda^{2}-\operatorname{tr}(A) \lambda+|A|=0 \tag{3}
\end{align*}
$$

where $\operatorname{tr}(\mathrm{A})$ is the trace (sum of main diagonal terms) of $A$. The roots of (3), found with the quadratic equation, are the eigenvalues.

$$
\begin{equation*}
\lambda_{1}, \lambda_{2}=\frac{(a+d) \pm \sqrt{(a+d)^{2}-4(a d-b c)}}{2}=\frac{\operatorname{tr}(A) \pm \sqrt{(\operatorname{tr}(A))^{2}-4|A|}}{2} \tag{4}
\end{equation*}
$$

a In general, an $\mathrm{n} \times \mathrm{n}$ matrix has n eigenvalues, some of which might be identical.
b Eigenvalues can be zero even though eigenvectors can not.
c For both eigenvalues to be real and distinct, $\left((\operatorname{tr}(A))^{2}>4|A|\right.$
d For both eigenvalues to be positive, $\left((\operatorname{tr}(A))^{2}>4|A| \&|A|>0\right.$.
The latter is because $\operatorname{tr}(\mathrm{A})$ must exceed the radical term in (4).
e $\quad \lambda_{1}+\lambda_{2}=\left(\frac{\operatorname{tr}(A)+\sqrt{(\operatorname{tr}(A))^{2}-4|A|}}{2}\right)+\left(\frac{\operatorname{tr}(A)-\sqrt{(\operatorname{tr}(A))^{2}-4|A|}}{2}\right)=\operatorname{tr}(A)$
f $\quad \lambda_{1} \lambda_{2}=\left(\frac{\operatorname{tr}(A)+\sqrt{(\operatorname{tr}(A))^{2}-4 \mid A}}{2}\right)\left(\frac{\operatorname{tr}(A)-\sqrt{(\operatorname{tr}(A))^{2}-4|A|}}{2}\right)=|A|$
g Eigenvalues of a real symmetric $2 \times 2$ matrix $A=\left[\begin{array}{ll}a & b \\ b & d\end{array}\right]$ (i.e., $\mathrm{b}=\mathrm{c}$ )
Substituting $b=c$ in equation J.2.(4) yields
(1) $\quad \lambda_{1}, \lambda_{2}=\frac{(a+d) \pm \sqrt{(a+d)^{2}-4\left(a d-b^{2}\right)}}{2}$
(2) $\lambda_{1}, \lambda_{2}=\frac{(a+d) \pm \sqrt{\left(a^{2}+2 a d+d^{2}\right)-4 a d+4 b^{2}}}{2}$
(3) $\lambda_{1}, \lambda_{2}=\frac{(a+d) \pm \sqrt{\left(a^{2}-2 a d+d^{2}\right)+4 b^{2}}}{2}$
(4) $\quad \lambda_{1}, \lambda_{2}=\frac{(a+d) \pm \sqrt{(a-d)^{2}+4 b^{2}}}{2}$ (See Il.G. 4 for examples)

The term under the radical sign can not be negative, so the eigenvalues for a real symmetric $2 \times 2$ matrix must be real.
3 Eigenvectors for a $2 \times 2$ matrix (See III.G. 4 for examples)
Start with the eigenvalue equation

$$
\begin{equation*}
[A-\mid \lambda][X]=0 \tag{1}
\end{equation*}
$$

For a $2 \times 2$ matrix this is
(2) $\quad\left[\begin{array}{cc}a-\lambda & b \\ c & d-\lambda\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]=0$

Solving this yields the eigenvectors via their slopes
(5) $\frac{y}{x}=\frac{(\lambda-a)}{b}$ or $\frac{y}{x}=\frac{c}{(\lambda-d)}$

4 Distinct eigenvectors of a symmetric $2 \times 2$ matrix are perpendicular The eigenvectors $X_{1}$ and $X_{2}$ are perpendicular if $X_{1} \bullet X_{2}=0$.

Proof

$$
\begin{equation*}
A X_{1}=\lambda_{1} X_{1} \tag{1a}
\end{equation*}
$$

(1b) $A X_{2}=\lambda_{2} X_{2}$
Equation (1a) yields a vector parallel to $X_{1}$, and equation (1b) yields a vector parallel to $X_{2}$. Dotting the vector of (1a) by $X_{2}$ and the vector of (1b) by $X_{1}$ will test whether $X_{1}$ and $X_{2}$ are orthogonal.
(2a) $\quad X_{2} \bullet A X_{1}=X_{2} \bullet \lambda_{1} X_{1}=\lambda_{1}\left(X_{2} \bullet X_{1}\right)(2 b) \quad X_{1} \bullet A X_{2}=X_{1} \bullet \lambda_{2} X_{2}=\lambda_{2}\left(X_{1} \bullet X_{2}\right)$
If A is symmetric, then $X_{2} \bullet A X_{1}=X_{1} \bullet A X_{2}$, as the following lines show:

$$
\begin{align*}
& {\left[\begin{array}{l}
x_{1} \\
y_{1}
\end{array}\right] \cdot\left[\begin{array}{ll}
a & b \\
b & d
\end{array}\right]\left[\begin{array}{l}
x_{2} \\
y_{2}
\end{array}\right]=\left[\begin{array}{l}
x_{1} \\
y_{1}
\end{array}\right] \cdot\left[\begin{array}{l}
a x_{2}+b y_{2} \\
b x_{2}+d y_{2}
\end{array}\right]=a x_{1} x_{2}+b x_{1} y_{2}+b y_{1} x_{2}+d y_{1} y_{2}}  \tag{3a}\\
& {\left[\begin{array}{l}
x_{2} \\
y_{2}
\end{array}\right] \cdot\left[\begin{array}{ll}
a & b \\
b & d
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
y_{1}
\end{array}\right]=\left[\begin{array}{l}
x_{2} \\
y_{2}
\end{array}\right] \cdot\left[\begin{array}{l}
a x_{1}+b y_{1} \\
b x_{1}+d y_{1}
\end{array}\right]=a x_{1} x_{2}+b y_{1} x_{2}+b x_{1} y_{2}+d y_{1} y_{2}}
\end{align*}
$$

Since (3a) equals (3b), the left sides of (2a) and (2b) are equal, and so the right sides of (2a) and (2b) must be equal too. Hence

$$
\begin{equation*}
\lambda_{1}\left(X_{2} \bullet X_{1}\right)=\lambda_{2}\left(X_{1} \bullet X_{2}\right) \tag{4}
\end{equation*}
$$

Now subtract the right side of (4) from the left side:

$$
\begin{equation*}
\left(\lambda_{1}-\lambda_{2}\right)\left(X_{2} \cdot X_{1}\right)=0 \tag{5}
\end{equation*}
$$

The eigenvalues generally are different, so $\lambda_{1}-\lambda_{2} \neq 0$, therefore $\left(X_{2} \cdot X_{1}\right)=0$ and hence the eigenvectors $X_{1}$ and $X_{2}$ are orthogonal.

See III.G. 4 for examples

IV Diagonalization of real symmetric nxn matrices
A A real symmetric matrix [A] can be diagonalized (converted to a matrix with zeros for all elements off the main diagonal) by premultiplying by the inverse of the matrix of its eigenvectors and postmultiplying by the matrix of its eigenvectors. The resulting diagonal matrix [ $\Lambda$ ] contains eigenvalues along the main diagonal.

## B Proof

First "bookshelf" stack the eigenvectors to form a square matrix S
$1 S=\left[X_{1}: X_{2}\right] \quad$ (the notation here can be extended to nxn )
Now pre-multiply S by A and recall that $\mathrm{AX}=\lambda \mathrm{X}$
$2 A S=A\left[X_{1}: X_{2}\right]=\left[\lambda_{1} X_{1}: \lambda_{2} X_{2}\right]=\left[X_{1}: X_{2}\right]\left[\begin{array}{cc}\lambda_{1} & 0 \\ 0 & \lambda_{2}\end{array}\right]=S \Lambda$
Now pre-multiply both sides of (2) by $\mathrm{S}^{-1}$ to yield diagonal matrix $\Lambda$ :
$3 S^{-1} A S=S^{-1} S \Lambda=I \Lambda=\Lambda$
This proves statement A
Post-multiplying both sides of (2) by $\mathrm{S}^{-1}$ yields
$4 A=S \Lambda S^{-1}$
This shows how to obtain $A$ from $\Lambda$ and $S$.
C Example (See III.G.4: $A=\left[\begin{array}{ll}0 & 2 \\ 2 & 0\end{array}\right], \lambda_{1}=2, \lambda_{2}=-2, X_{1}=\left[\begin{array}{l}\sqrt{2} / 2 \\ \sqrt{2} / 2\end{array}\right], \quad X_{2}=\left[\begin{array}{c}-\sqrt{2} / 2 \\ \sqrt{2} / 2\end{array}\right]$ ),
$1 S=\left[X_{1}: X_{2}\right]=\left[\begin{array}{cc}\sqrt{2} / 2 & -\sqrt{2} / 2 \\ \sqrt{2} / 2 & \sqrt{2} / 2\end{array}\right]$
$2 A S=\left[\begin{array}{ll}0 & 2 \\ 2 & 0\end{array}\right]\left[\begin{array}{cc}\sqrt{2} / 2 & -\sqrt{2} / 2 \\ \sqrt{2} / 2 & \sqrt{2} / 2\end{array}\right]=\left[\begin{array}{cc}\sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2}\end{array}\right]=S \Lambda=\left[\begin{array}{cc}\sqrt{2} / 2 & -\sqrt{2} / 2 \\ \sqrt{2} / 2 & \sqrt{2} / 2\end{array}\right]\left[\begin{array}{cc}2 & 0 \\ 0 & -2\end{array}\right]$
$3 S^{-1}[A S]=\left[\begin{array}{cc}\sqrt{2} / 2 & \sqrt{2} / 2 \\ -\sqrt{2} / 2 & \sqrt{2} / 2\end{array}\right]\left[\begin{array}{cc}\sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2}\end{array}\right]=\left[\begin{array}{cc}2 & 0 \\ 0 & -2\end{array}\right]=\Lambda \quad$ OK. This is indeed $\Lambda$.
$4 A=[S \Lambda] S^{-1}=\left[\begin{array}{cc}\sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2}\end{array}\right]\left[\begin{array}{cc}\sqrt{2} / 2 & \sqrt{2} / 2 \\ -\sqrt{2} / 2 & \sqrt{2} / 2\end{array}\right]=\left[\begin{array}{ll}0 & 2 \\ 2 & 0\end{array}\right]=\mathrm{A} \quad$ This checks.
By inspection, S and $\mathrm{S}^{-1}$ resemble rotation matrices, so diagonalization apparently can occur if the reference frame is rotated.

V Principal Axes Theorem (Spectral Theorem) in Two Dimensions
A A "quadratic (second-order) form" with no linear terms (e.g., $a x^{2}+$ $2 b x y+d y^{2}$ ) in one reference frame (e.g., the $x y$ frame) can be written in a simpler form (e.g., $\lambda_{1} x^{2}+\lambda_{2} y^{22}$ ) in a different reference frame such that it has a clear geometric meaning and takes advantage of symmetry. The $x^{\prime}$ and $y^{\prime}$ axes are the principal axes (eigenvectors), and $\lambda_{1}$ and $\lambda_{2}$ are the principal values (or eigenvalues).

B If $\lambda_{1}>0$ and $\lambda_{2}>0$, then the new quadratic form is that of an ellipse. The semi-axes of an ellipse are the maximum and minimum lengths of line segments connecting the center of an ellipse to its perimeter, providing a way to find maxima and minima (and their directions) without resorting to calculus.

C General Example: Equation of an ellipse centered at the origin
$1 a x^{2}+2 b x y+d y^{2}=1$
$2\left[\begin{array}{ll}x & y\end{array}\right]\left[\begin{array}{ll}a & b \\ b & d\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]=X^{T} A X=1$
Now replace $A$ with its diagonalized equivalent using eq. IV.C. 4
$3 X^{T}[A] X=X^{T}\left[S \Lambda S^{-1}\right] X=X^{T}\left[S \Lambda S^{T}\right] X=\left[X^{T} S\right] \Lambda\left[S^{T} X\right]=\left[S^{T} X\right]^{T} \Lambda\left[S^{T} X\right]$
$4 X^{T}[A] X=\left[S^{T} X\right]^{T} \Lambda\left[S^{T} X\right]=\left[X^{\prime}\right]^{T} \Lambda\left[X^{\prime}\right]=1$
So the equation of the ellipse can be written as
$5 \quad\left[\begin{array}{ll}x^{\prime} & y^{\prime}\end{array}\right]\left[\begin{array}{ll}\lambda_{1} & 0 \\ 0 & \lambda_{2}\end{array}\right]\left[\begin{array}{l}x^{\prime} \\ y^{\prime}\end{array}\right]=\lambda_{1} x^{\prime 2}+\lambda_{2} y^{\prime 2}=\left(\frac{x^{\prime}}{1 / \sqrt{\lambda_{1}}}\right)^{2}+\left(\frac{y^{\prime}}{1 / \sqrt{\lambda_{2}}}\right)^{2}=1$

```
% Matlab script gg303_lab_09_example_V
A = [2 1;1 1];
a=A(1,1); b = A(1,2); c = A(2,1); d = A(2,2);
theta = 0:2*pi/360:2*pi;
x = cos(theta); y = sin(theta); % coordinates of pts. on unit
circle
X = [x;y];
Xp = A*X; % Xp represents X' in the notes
xp = Xp(1,:); yp = Xp(2,:);
% Find matrix of eigenvectors (vA) and eigenvalues (dA)
[vA,dA] = eig(A)
% Find indices for the maximum and minimum eigenvalues
[i,j] = find(dA == max(dA(1,1),dA(2,2))); % maximum
[k,l] = find(dA == min(dA(1,1),dA(2,2))); % minimum
% Plot ellipse and its major and minor semi-axes
plot(xp,yp); % ellipse
hold on
plot(dA(j,j)*[0,vA(1,j)],dA(j,j)*[0,vA(2,j)],'--'); % major semi-
axis (x')
plot(dA(l,l)*[0,vA(1,l)],dA(l,l)*[0,vA(2,l)],'r'); % minor semi-
axis (y')
hold off
axis equal; xlabel('x'); ylabel('y')
title ('Principal semi-axes for quadric form given by A')
% Find orientation, in degrees, of major semi-axis
theta_xxp = mod(atan2(vA(2,j),vA(1,j))*180/pi,180)
print -dpng lab_09_example_v.png
>> gg303_lab_09_example_V
vA =
0.5257 -0.8507
    -0.8507 -0.5257
dA =
0.3820 0
    O.6180
theta_xxp =
31.7175
```



D Specific Example: Equation of an ellipse centered at the origin
$12 x^{2}+2 x y+1 y^{2}=1$
$2\left[\begin{array}{ll}x & y\end{array}\right]\left[\begin{array}{ll}2 & 1 \\ 1 & 1\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]=2 x^{2}+2 x y+1 y^{2}=1 \quad$ Let $A=\left[\begin{array}{ll}2 & 1 \\ 1 & 1\end{array}\right]$ (A is symmetric)
The eigenvalues and corresponding eigenvectors of A are [use either Matlab or equation J.2.g.(4)]
$3 \lambda_{1}=\frac{3+\sqrt{5}}{2} \approx 2.6180$
$\lambda_{2}=\frac{3-\sqrt{5}}{2} \approx 0.3820$
$4 X_{1}=\left[\begin{array}{l}\frac{1+\sqrt{5}}{\sqrt{10+2 \sqrt{5}}} \\ \frac{2}{\sqrt{10+2 \sqrt{5}}}\end{array}\right] \approx\left[\begin{array}{l}0.8507 \\ 0.5257\end{array}\right]$,

$$
X_{2}=\left[\begin{array}{c}
\frac{2}{\sqrt{10+2 \sqrt{5}}} \\
-\frac{1+\sqrt{5}}{\sqrt{10+2 \sqrt{5}}}
\end{array}\right] \approx\left[\begin{array}{c}
0.5257 \\
-0.8507
\end{array}\right]
$$

So the equation of the ellipse can be written as
$5 \quad\left[\begin{array}{ll}x^{\prime} & y^{\prime}\end{array}\right]\left[\begin{array}{cc}2.6180 & 0 \\ 0 & 0.3820\end{array}\right]\left[\begin{array}{l}x^{\prime} \\ y^{\prime}\end{array}\right]=2.6180 x^{\prime 2}+0.3820 y^{\prime 2}=1$
where $\theta_{x x^{\prime}}$, the angle from the $x$-axis to the $x^{\prime}$-axis, $\approx \tan ^{-1}\left(\frac{0.5257}{0.8507}\right) \approx 31.7^{\circ}$
$E$ Meaning of solution of $[F][d X]=[\lambda][d X]$ if $F$ is a $2 x 2$ symmetric matrix

1 Solutions for $\lambda$ give maximum and minimum distance from the origin
2 The eigenvector solutions for dX give the directions of the principal axes of the ellipse and directions of lines that maintain their orientation as a unit circle deforms to an ellipse.

VI Strain ellipses, strain ellipsoids, and principal strains
A The strain ellipse and strain ellipsoid
1 Strains may be inhomogeneous over a large region but approximately homogeneous locally

2 For homogeneous strain, a unit circle in the undeformed state deforms to an ellipse called the strain ellipse.

3 For homogeneous strain, a unit sphere in the undeformed state deforms to an ellipsoid called the strain ellipsoid.

4 In general, the strain ellipse characterizes 2-D strain at a position in space and a point it time; it can vary with $x, y, z$, and $t$ (time).
5 Distortion in the neighborhood of a point due to non-deformational processes (e.g., chemical precipitation) is not a strain.

6 Characterization of the strain ellipse
a An ellipse has a major semi-axis (a) and a minor semi-axis (b). An ellipse can be characterized by the (relative) length, orientation, and rotation of these axes
b The eigenvectors of a symmetric F-matrix give the principal axes of the strain ellipse, but the eigenvectors of a nonsymmetric Fmatrix, representing irrotational strain, do not (see diagram on next page). To find the principal strains from a nonsymmetric Fmatrix, the rotation of eigenvectors needs to be accounted for; one seeks the eigenvalues and eigenvectors for a symmetric matrix that corresponds to F.

B The reciprocal strain ellipse
1 For homogeneous strain, a unit circle in the undeformed state "retro-deforms" to an ellipse (the reciprocal strain ellipse).

2 For homogeneous strain, a unit sphere in the undeformed state "retro-deforms" to an ellipsoid (the reciprocal strain ellipsoid).

Irrotational Strain


A forward deformation deforms a unit circle to a strain ellipse. For irrotational deformation, the axes which transform to become the principal strain axes (dashed) are not rotated.

Rotational Strain


A forward deformation again deforms a unit circle to a strain ellipse.


The inverse deformation to retro-deform the strain ellipse back to a unit circle causes lines along the principal strain axes in the strain ellipse (dashed) to be rotated. The angular difference between the orientation of the principal strain axes (dashed) and the retro-deformed axes (solid) in the initial state is the rotaton.


The eigenvectors for a deformation with a symmetric F-matrix will be in the orientation of the axes of the strain ellipse, but this will not be the case for deformation described by a non-symmetric F-matrix (see the example on the bottom of the previous page). In general the axes of a unit circle that transform to the axes of the strain ellipse will be stretched and rotated (see the example on the bottom of the previous page). Another way to say this is that the axes of reciprocal strain ellipse (see the diagram above) generally will have an orientation different from the axes of the strain ellipse (see the diagram above). The rotation of the axes of the strain ellipse refers to the rotation needed to bring the axes of the reciprocal strain ellipse into coincidence with the axes of the strain ellipse.

C Strain ellipse for general homogeneous rotational strain
1 Decomposition of $\mathrm{F}=\mathrm{VR}^{*}$ by method of Ramsay and Huber (for 2D)* Consider the effect of an irrotational strain that follows a pure rigid rotation of the object (not a rigid rotation of the reference frame)

$$
\begin{align*}
& F=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]=\left[\begin{array}{ll}
A & B \\
B & D
\end{array}\right]\left[\begin{array}{cc}
\cos \omega & -\sin \omega \\
\sin \omega & \cos \omega
\end{array}\right]=V R^{*}  \tag{1}\\
& {\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]=\left[\begin{array}{ll}
A \cos \omega+B \sin \omega & -A \sin \omega+B \cos \omega \\
B \cos \omega+D \sin \omega & -B \sin \omega+D \cos \omega
\end{array}\right]}
\end{align*}
$$

By inspection of $(2), c-b=(A+D) \sin \omega$ and $a+d=(A+D) \cos \omega$. So

$$
\begin{equation*}
\frac{c-b}{a+d}=\tan \omega \quad \text { If } \mathrm{c}=\mathrm{b} \text { (i.e., } \mathrm{F} \text { is symmetric), then } \omega=0 . \tag{3}
\end{equation*}
$$

Post-multiplying both sides of (1) by $\mathrm{R}^{*-1}=\mathrm{R}^{* T}$ yields V .

$$
\left[\begin{array}{ll}
a & b  \tag{4}\\
c & d
\end{array}\right]\left[\begin{array}{cc}
\cos \omega & -\sin \omega \\
\sin \omega & \cos \omega
\end{array}\right]^{-1}=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]\left[\begin{array}{cc}
\cos \omega & \sin \omega \\
-\sin \omega & \cos \omega
\end{array}\right]=\left[\begin{array}{ll}
A & B \\
B & D
\end{array}\right]=V
$$

The eigenvalues and eigenvectors of V give the principal stretches.
2 Decomposition of $\mathrm{F}=\mathrm{R} * \mathrm{U}$ by method of Ramsay and Huber (for 2D)* Consider an irrotational strain (described by a symmetric matrix U) followed by an orthogonal rotation (described by matrix R)

$$
\begin{align*}
& F=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]=\left[\begin{array}{cc}
\cos \omega & -\sin \omega \\
\sin \omega & \cos \omega
\end{array}\right]\left[\begin{array}{cc}
A^{*} & B^{*} \\
B^{*} & D^{*}
\end{array}\right]=R U  \tag{1}\\
& {\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]=\left[\begin{array}{ll}
A^{*} \cos \omega-B^{*} \sin \omega & B^{*} \cos \omega-D^{*} \sin \omega \\
A^{*} \sin \omega+B^{*} \cos \omega & B^{*} \sin \omega+D^{*} \cos \omega
\end{array}\right]}
\end{align*}
$$

Again, $c-b=(A+D) \sin \omega$ and $a+d=(A+D) \cos \omega$. So

$$
\begin{equation*}
\frac{c-b}{a+d}=\tan \omega \tag{3}
\end{equation*}
$$

Both sides of (1) can be pre-multiplied by $\mathrm{R}^{\star-1}=\mathrm{R}^{\star \top}$ to yield U .

$$
\left[\begin{array}{cc}
\cos \omega & \sin \omega  \tag{4}\\
-\sin \omega & \cos \omega
\end{array}\right]\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]=\left[\begin{array}{ll}
A^{*} & B^{*} \\
B^{*} & D^{*}
\end{array}\right]=U
$$

The eigenvalues of $U$ give the principal stretches, but the principal stretch directions are not the eigenvectors of U !

G Closing comments
1 Our solutions so far depend on knowing the displacement field.
2 With satellite imaging we can get an approximate value for the displacement field at the surface of the Earth for current deformations

3 This option is not available for past deformations unless certain assumptions are made that constrain the displacement field.

4 Alternative approach: formulation and solution of boundary value problems to solve for the displacement and strain fields.
5 References*
a Ramsay, J.G., and Huber, M.I., 1983, The techniques of modern structural geology, volume 1: strain analysis: Academic Press, London, 307 p. (See equations of section 5, p. 291).
b Ramsay, J.G., and Lisle, M.I., 1983, The techniques of modern structural geology, volume 3: applications of continuum mechanics in structural geology: Academic Press, London, 307 p. (See especially sessions 33 and 36).
c Malvern, L.E., 1969, Introduction to the mechanics of a continuous medium: Prentice-Hall, Englewood Cliffs, New Jersey, 713 p. (See equations 4.6.1, 4.6.3 a, 4.6.3b on p. 172-174).

## Appendix 1

Symmetry of $C=F^{\top} F$
$F=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right], F^{T}=\left[\begin{array}{ll}a & c \\ b & d\end{array}\right]$
So
$C=F^{T} F=\left[\begin{array}{ll}a & c \\ b & d\end{array}\right]\left[\begin{array}{ll}a & b \\ c & d\end{array}\right],=\left[\begin{array}{ll}a^{2}+c^{2} & a b+c d \\ a b+c d & b^{2}+d^{2}\end{array}\right]$
So $C$ is symmetric $\left(C^{\top}=C\right)$.

Similarly, $B=F^{\top}$, so
$B=F F^{T}=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]\left[\begin{array}{ll}a & c \\ b & d\end{array}\right],=\left[\begin{array}{ll}a^{2}+b^{2} & a c+b d \\ a b+c d & c^{2}+d^{2}\end{array}\right]$
So $B$ is symmetric ( $B^{\top}=B$ ).

This means $X^{\top} C X=c$ and $X^{\top} B X=c^{*}$, where $c$ and $c^{*}$ are constants, are both equations of an ellipse. Therefore, the eigenvalues of $C$ and $B$ are the lengths of the semi-axes of the corresponding ellipses, and the eigenvectors of $C$ and $B$ are the directions of the semi-axes of the corresponding ellipses.

## Appendix 2

## Example 1

If two lines intersect at a unique point (the origin), then their slopes must differ. The following example with the lines $y=2 x$ and $y=-x$ illustrates this.

| $2 x-y$ | $=0$ |
| ---: | :--- |
| $x+y$ | $=0$ |

In matrix form these become:
(b) $\left[\begin{array}{cc}2 & -1 \\ 1 & 1\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]=0\left[\begin{array}{l}x \\ y\end{array}\right]$

Since $\lambda=0,|A-|\lambda|$ equals $| A \mid$ :
(c) $\left|\begin{array}{ll}2 & -1 \\ 3 & -1\end{array}\right|=(2)(1)-(-1)(1)=3 \neq 0$


The first equation in (a) stems from $y=2 x$. Replacing $y$ by $2 x$ in the second equation in (a) yields $3 x=0$, so $x=0$. Since $y=2 x, y=0$ also. Thus $x=0, y=0$ is the solution, as the graph above confirms, and this solution is unique. Note the determinant of $A$, in (c) is indeed not zero.

Eigenvector solution for $[\mathrm{A}][\mathrm{X}]=\lambda[\mathrm{X}]$
Eigenvectors, in contrast to trivial solutions, are required to be non-zero solutions to (1) or (2). If an eigenvector solution exists in addition to $\mathrm{X}=0$, then the solution is not unique, hence $|A-|\lambda|=0$; this requirement also means that the rows in $|A-|\lambda|$ are not linearly independent.

## Example 2

Consider the lines $2 \mathrm{y}=-2 \mathrm{x}$ and $\mathrm{y}=-\mathrm{x}$. These lines plot on top of each other.

(d) | $2 x+2 y=0$ |
| :--- |
| $x+y=0$ |

In matrix form these become:
(e) $\quad\left[\begin{array}{ll}2 & 2 \\
1 & 1\end{array}\right]\left[\begin{array}{l}x \\
y\end{array}\right]=0\left[\begin{array}{l}x \\
y\end{array}\right]$
Here again $\lambda=0$, so $|\mathrm{A}-|\lambda|$ equals $| \mathrm{Al}:$
(f) $\quad\left|\begin{array}{ll}2 & 2 \\
1 & 1\end{array}\right|=(2)(1)-(2)(1)=0$
The determinant $|\mathrm{A}-|\lambda|=0$, and the
slopes of the lines are equal. Any
points along the direction $\mathrm{y} / \mathrm{x}=-1$
provide a solution. So here $\lambda=0$ is
the eigenvalue and the corresponding
eigenvector is given by the direction
$\mathrm{y} / \mathrm{x}=-1$.

## Appendix 3

Geometric meaning of a real symmetric $2 \times 2$ matrix
(1a) $\left[X^{\prime}\right]=[A][X] \Rightarrow$ (
1b) $\left[A^{-1}\right]\left[X^{\prime}\right]=\left[A^{-1}\right][A][X] \Rightarrow$ (1c) $\left[A^{-1}\right]\left[X^{\prime}\right]=[X]$

Suppose the points ( $\mathrm{x}, \mathrm{y}$ ) lie along a unit circle
(2a) $x^{2}+y^{2}=1 \Rightarrow$
(2b) $\left[\begin{array}{ll}x & y\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]=1 \Rightarrow(2 \mathrm{c})\left[X^{T}\right][X]=1$
Substituting the expression for $[\mathrm{X}]$ in (1c) into (2c) yields

$$
\begin{equation*}
\left[\left[A^{-1}\right]\left[X^{\prime}\right]\right]^{T}\left[\left[A^{-1}\right]\left[X^{\prime}\right]\right]=1 \Rightarrow(3 b)\left[\left[X^{\prime}\right]^{T}\left[A^{-1}\right]^{T}\right]\left[\left[A^{-1}\right]\left[X^{\prime}\right]\right]=1 \tag{3a}
\end{equation*}
$$

If $A=\left[\begin{array}{ll}a & b \\ b & d\end{array}\right]$, then $\mathrm{A}=\mathrm{A}^{\top}, \mathrm{A}^{-1}=\frac{1}{a d-b^{2}}\left[\begin{array}{cc}d & -b \\ -b & a\end{array}\right]$; by inspection $\mathrm{A}^{-1}=$ $\left[\mathrm{A}^{-1}\right]^{\top}$.
(4) $\left[\left[X^{\prime}\right]^{T}\left[\frac{1}{|A|^{2}}\left[A^{-1}\right]^{2}\right]\right]\left[X^{\prime}\right]=1$
(5) $\left.\frac{1}{|A|^{2}}\left[\begin{array}{ll}x^{\prime} & y^{\prime}\end{array}\right]\left[\begin{array}{cc}d^{2}+b^{2} & -d b-b a \\ -d b-b a & a^{2}+b^{2}\end{array}\right]\right]\left[\begin{array}{l}x^{\prime}\end{array}\right]\left[\begin{array}{l}y^{\prime}\end{array}\right]=1$
(6) $\frac{1}{|A|^{2}}\left[x^{\prime}\left(d^{2}+b^{2}\right)-y^{\prime}(d b+b a) \quad x^{\prime}(-d b-b a)+y^{\prime}\left(a^{2}+b^{2}\right)\right]\left[\begin{array}{l}x^{\prime} \\ y^{\prime}\end{array}\right]=1$
(7) $\frac{1}{|A|^{2}}\left[x^{\prime 2}\left(d^{2}+b^{2}\right)-2 x y^{\prime}(d b+b a)+y^{\prime}\left(a^{2}+b^{2}\right)\right]=1$

Since the terms multiplying $x$ ' and $y^{\prime}$ are both positive, this equation describes an ellipse. For a real, symmetric matrix, the eigenvectors of the matrix coincide with the major and minor axes of the ellipse.

## Appendix 4

1 Polar decomposition $\mathrm{F}=\mathrm{VR}$ (for 2D and 3D)
Consider a rotation of the axes (described by R ) followed by an irrotational strain (described by a symmetric matrix $V$ ), The rotation of the axes aligns them with the principal strain axes.

$$
\begin{equation*}
F=V R \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
F^{\top}=(V R)^{\top}=R^{\top} V^{\top} \tag{2}
\end{equation*}
$$

Since $V$ is symmetric (i.e., $V=V^{\top}$ ), (2) becomes

$$
\begin{equation*}
F^{\top}=R^{\top} V \tag{3}
\end{equation*}
$$

Since $R$ is a rotation matrix (see lecture 11 ), $R^{\top}=R^{-1}$, so

$$
\begin{equation*}
\mathrm{F}^{\top}=\mathrm{R}^{-1} \mathrm{~V} \tag{4}
\end{equation*}
$$

Combining (1) and (4) yields the definite positive matrix $\mathrm{FF}^{\top}$

$$
\begin{equation*}
\mathrm{FF}^{\top}=\mathrm{VRR}^{-1} \mathrm{~V}=\mathrm{VIV}=\mathrm{VV}=\mathrm{V}^{2} \tag{5}
\end{equation*}
$$

Taking the matrix square root of $\mathrm{FF}^{\top}$ (sqrtm in Matlab) yields V :

$$
\begin{equation*}
V=\left(F^{\top}\right)^{1 / 2} \tag{6}
\end{equation*}
$$

To obtain R , first pre-multiply both sides of (1) by $\mathrm{V}^{-1}$

$$
\begin{equation*}
V^{-1} F=V^{-1} V R=R \tag{7}
\end{equation*}
$$

Combining (6) and (7) yields R in terms of F

$$
\begin{equation*}
R=\left[\left(F F^{\top}\right)^{1 / 2}\right)^{-1} F \tag{8}
\end{equation*}
$$

The eigenvalues and eigenvectors of $V$ give the principal stretches.
2 Polar decomposition $\mathrm{F}=\mathrm{RU}$ (for 2D and 3D)*
Consider an irrotational strain (U) followed by a rotation (R)

$$
\begin{equation*}
F=R U \tag{1}
\end{equation*}
$$

(This is the same R as above)
By a procedure similar to the one above

$$
\begin{align*}
& U=\left(F^{\top} F\right)^{1 / 2}  \tag{2}\\
& R=F\left[\left(F^{\top} F\right)^{1 / 2}\right)^{-1} \tag{3}
\end{align*}
$$

The eigenvalues of $U$ give the principal stretches, but the_principal stretch directions are not the eigenvectors of $U!$

C Strain ellipse for general homogeneous rotational strain The stretch ( $S$ ) of a short line segment (line element) is
$1 S=\frac{d s^{\prime}}{d s}=\frac{d s^{\prime}-d s}{d s}+\frac{d s}{d s}=\varepsilon+1$, where $s^{\prime}=$ final length, $s=$ initial length
$2 \quad S^{2}=\left(\frac{d s^{\prime}}{d s}\right)^{2}=(\varepsilon+1)^{2}=\lambda \quad$ This is the quadratic strain.
The square of the length of an element (e.g., the radius of a unit circle) in the undeformed state is
$3 \quad(d s)^{2}=(d x)^{2}+(d y)^{2}=\left[\begin{array}{ll}d x & d y\end{array}\right]\left[\begin{array}{l}d x \\ d y\end{array}\right]=[d X]^{T}[d X]$.
Similarly, the square of the length of the deformed element (radius) is
$4 \quad\left(d s^{\prime}\right)^{2}=\left(d x^{\prime}\right)^{2}+\left(d y^{\prime}\right)^{2}=\left[\begin{array}{ll}d x^{\prime} & d y^{\prime}\end{array}\right]\left[\begin{array}{l}{\left[d x^{\prime}\right.} \\ d y^{\prime}\end{array}\right]=\left[d X^{\prime}\right]^{T}\left[d X^{\prime}\right]$
By the coordinate transformation equations, $d X^{\prime}=[F][d X]$. So
$5 \quad\left(d s^{\prime}\right)^{2}=[[F][d X]]^{T}[[F][d X]]=[d X]^{T}[F]^{T}[F][d X]$ (equation of an ellipse)
$6 \quad \frac{\left(d s^{\prime}\right)^{2}}{(d s)^{2}}=\frac{[d X]^{T}[F]^{T}[F][d X]}{[d X]^{T}[d X]}=\frac{[d X]^{T}[C][d X]}{[d X]^{T}[d X]}=\lambda$, where $[C]=[F]^{T}[F]$.
Since the numerator and denominator of (6) are scalars (numbers),
$7 \quad[d X]^{T}[C][d X]=\lambda[d X]^{T}[d X]=[d X]^{T} \lambda[d X]$
The terms on both sides of (8) are premultiplied by $[d X]^{\top}$, hence
$8 \quad[C][d X]=\lambda[d X]$
This is an eigenvalue equation. This means that the eigenvalues of $[C]$ are the principal quadratic stains and hence the square of the principal stretches. The eigenvectors of $[C]$ are the directions of the principal quadratic strains, which coincide with the directions of the principal stretches and the directions of the principal strains. Since $[C]$ is symmetric (see proof in appendix), the principal stretches must be perpendicular.

D Reciprocal strain ellipse for general homogeneous rotational strain The approach follows that for the strain ellipse.
The square of an element (e.g., the radius of a unit circle) in the deformed state is
$1 \quad\left(d s^{\prime}\right)^{2}=\left(d x^{\prime}\right)^{2}+\left(d y^{\prime}\right)^{2}=\left[\begin{array}{ll}d x^{\prime} & d y^{\prime}\end{array}\right]\left[\begin{array}{c}d x^{\prime} \\ d y^{\prime}\end{array}\right]=\left[d X^{\prime}\right]^{T}\left[d X^{\prime}\right]$.
Similarly, the square of the length of the retro-deformed element (radius) is
$2 \quad(d s)^{2}=(d x)^{2}+(d y)^{2}=\left[\begin{array}{ll}d x & d y\end{array}\right]\left[\begin{array}{c}d x \\ d y\end{array}\right]=[d X]^{T}[d X]$
By the coordinate transformation equations, $\left[F^{-1}\right]\left[d X^{\prime}\right]=[d X]$. So
3

$$
\begin{array}{ll}
3 & (d s)^{2}=\left[d X^{\prime}\right]^{T}\left[d X^{\prime}\right]=\left[\left[F^{-1}\right]\left[d X^{\prime}\right]\right]^{T}\left[\left[F^{-1}\right]\left[d X^{\prime}\right]\right]=\left[d X^{\prime}\right]^{T}\left[F^{-1}\right]^{T}\left[F^{-1}\right]\left[d X^{\prime}\right] \\
4 & \frac{(d s)^{2}}{\left(d s^{\prime}\right)^{2}}=\frac{\left[d X^{\prime}\right]^{T}\left[F^{-1}\right]^{T}\left[F^{-1}\right]\left[d X^{\prime}\right]}{\left[d X^{\prime}\right]^{T}\left[d X^{\prime}\right]}=\frac{\left[d X^{\prime}\right]^{T}\left[B^{-1}\right]\left[d X^{\prime}\right]}{\left[d X^{\prime}\right]^{T}\left[d X^{\prime}\right]}=\lambda^{*} \text {, where }\left[B^{-1}\right]=\left[F^{-1}\right]^{T}\left[F^{-1}\right] .
\end{array}
$$

Since the numerator and denominator of (4) are scalars (numbers),
$5 \quad\left[d X^{\prime}\right]^{T}\left[B^{-1}\right]\left[d X^{\prime}\right]=\lambda^{*}\left[d X^{\prime}\right]^{T}\left[d X^{\prime}\right]=\left[d X^{\prime}\right]^{T} \lambda^{*}\left[d X^{\prime}\right]$
The terms on both sides of (5) are premultiplied by $[d X]^{\top}$, hence
$6 \quad\left[B^{-1}\right]\left[d X^{\prime}\right]=\lambda^{*}\left[d X^{\prime}\right] \quad$ This too is an eigenvalue equation.
The eigenvalues of $\left[B^{-1}\right]$ are thus the principal reciprocal quadratic stains and hence the square of the reciprocal principal stretches. The igenvectors of [ $\left.B^{-3}\right]$ are the direction of the principal reciprocal quadratic strains. They coincide with the directions of the principal reciprocal stretches and principal strains. Since $\left[B^{-1}\right]$ is symmetric (see proof in appendix), the principal reciprocal stretches must be perpendicular.

E Rotation of the principal axes of strain
The rotation of the principal axes of strain is described by the orientation of the axes of the strain ellipse relative to the axes of the reciprocal strain ellipse and (see figure).

Lab 9
Eigenvectors and Eigenvalues for Strain

Exercisel Vector fields and lines strained in simple shear (23 pts)
A Print a copy of my Matlab function strain 1 and read it. (1 point)
B Use Matlab function strain1 from my web page to produce plots of squares deformed under simple shear for the following deformation gradients.
F1 = [1 0.5;0 1]
(1 point)
F2 = [1 3;01]

C Assuming that the $y$-axis is north find the following quantities. Show your work or (preferably) your Matlab script (1 point/box)

|  | Deformation 1 | Deformation 2 |
| :--- | :--- | :--- |
| Extension of AB |  |  |
| Extension of BC |  |  |
| Extension of AC |  |  |
| Extension of BD |  |  |
| Shear strain of ABC <br> (watch the sign!) |  |  |
| Shear strain of BAD <br> (watch the sign!) |  |  |
| Trend of the axis of <br> greatest extension |  |  |
| Trend of the axis of <br> least extension |  |  |
| Dilation of the square |  |  |
| Rotation angle $\omega$ |  |  |

Exercise 2 Homogenous rotational strain ( $\sim 30$ pts)
A Print a copy of my Matlab script lab_09_exc_2008.m and read it. (1 point)
B Make a copy of the Matlab script. Call it what you want.
C Modify the copy of the Matlab script so that it
(a) Shows only the original and undeformed states for the lowest box (the one with its lower left corner at the origin.
(b) Returns the magnitudes, orientations and rotation for the principal stretches.

Exercise 3 (adopted from Rowland and Duebenforder, 1994)

## (32 pts total)

A Find the dimensions of the major and minor axes of each of the six undeformed ellipses (i.e., the circles) and the deformed ellipses and enter the values in the table below. From them calculate the stretches $1+\mathrm{e} 1$ and $1+e 2$.

|  | Undeformed <br> major axis <br> $(1 \mathrm{pt})$ | Undeformed <br> minor axis <br> $(1 \mathrm{pt})$ | Deformed <br> major axis <br> $(1 \mathrm{pt} / \mathrm{box})$ | Deformed <br> minor axis <br> $(1 \mathrm{pt} / \mathrm{box})$ | $1+\mathrm{e} 1$ <br> $(1 \mathrm{pt}$ <br> (box) | $1+\mathrm{e} 2$ <br> $(1 \mathrm{pt}$ <br> box$)$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| A |  |  |  |  |  |  |
| B |  |  |  |  |  |  |
| C |  |  |  |  |  |  |
| D |  |  |  |  |  |  |
| E |  |  |  |  |  |  |
| F |  |  |  |  |  |  |
| G |  |  |  |  |  |  |

B Plot the points corresponding to each deformation on the graph below.


(1 pt for each correctly plotted point; 6 points total)

## Exercise 3 Wellman's Problem ( 35 pts total)

Using the deformed brachiopods shown below (fossilius horribilus), and assuming homogeneous deformation, use Wellman's method to determine the strain ellipse for the rock containing the fossils. Draw the strain ellipse on this page. Label the points on your strain ellipse that corresponding to the fossils. For the base points for your ellipse use two points on a horizontal line, make the distance between the points $79 \mathrm{~mm}(1 \mathrm{pt})$. From the strain ellipse ( $7 \times 4 \mathrm{pts}=\mathbf{2 8} \mathrm{pts}$ ) determine the direction of the principal strains ( $2 \mathbf{p t s}$ ) and determine the ratio of the principal stretches ( $2 \mathbf{~ p t s}$ ) .
Finally, is there anything special about fossil E? If so, what? (2 pts)


Exercise 5 Flinn diagram problem (Modified from problem 14.8 of Rowland and Duebenforder, 1994).

Consider the attached diagram of an oolitic limestone showing cuts made perpendicular to two principal strain axes.

A Measure the dimensions of six ooids in each view ( $4 \times 6=24$ pts)
B Find a mean semi-major and a mean semi-minor axis for each view ( $4 \times 1$ $=4 \mathrm{pts}$ ).
$C$ Use the answers from (B) to find ratios for $1+\mathrm{e}_{2}: 1+\mathrm{e}_{3}(1 \mathrm{pt})$ and $1+\mathrm{e}_{1}: 1+\mathrm{e}_{3}(1 \mathrm{pt})$.
D Use the answers from (C) to find the $1+e_{1}: 1+e_{2}: 1+e_{3}$ ratio for the strain ellipsoid, assuming the ooids were deformed homogenously (3 pts).

E Plot the point on the Flinn diagram that represents this strain ellipsoid (1 pt).


Exercise 6 Heterogeneous strain around a pressurized hole (50* pts) (Note that the last question, question K , is optional)
Assume that the radial (i.e., outward) displacement ( $u_{r}$ ) from the center of a pressurized circular cylindrical hole is given by the following equation:
$u_{r}=u_{0} \frac{a}{r}$
where (1) $u_{0}$ is the radial (outward) displacement away from the center of the hole that occurs at the perimeter of the hole.
(2) $a$ is the original radius of the hole.
(3) $r$ is the original distance of a point from the center of the hole (i.e., the distance before deformation).
$u_{r}=r^{\prime}-r \quad$ where $r^{\prime}$ is the radial distance after deformation

We want to see how the square deforms if the hole is inflated such that its radius doubles, given the displacement relationship above.

A In the center of a page, draw a circle with a radius of $2 \mathrm{~cm}(1 \mathrm{pt})$ and a square 2 cm on a side ( 1 pt ) that is tangent to circle, and label the points on the square as shown below ( 1 pt ):


B Fill in the table below assuming that $u 0=a$.

| Point | $\mathrm{r}(\mathrm{cm})$ <br> $(1 \mathrm{pt} / \mathrm{box})$ | $\mathrm{u}_{\mathrm{r}}(\mathrm{cm})$ <br> $(1 \mathrm{pt} / \mathrm{box})$ | $\mathrm{r}^{\prime}(\mathrm{cm})$ <br> $(1 \mathrm{pt} / \mathrm{box})$ |
| :--- | :--- | :--- | :--- |
| A |  |  |  |
| B |  |  |  |
| C |  |  |  |
| D |  |  |  |
| E |  |  |  |
| F |  |  |  |
| G |  |  |  |
| H |  |  |  |

C Draw the hole showing it after it has been inflated.
(3 pts)
D Draw the deformed "square" A'B'C'E'H'G'F'D'A'.
(8 pts)
E Are parallel lines $A B C$ and $F G H$ parallel to $A^{\prime} B^{\prime} C^{\prime}$ and $F^{\prime} G^{\prime} H^{\prime} ? ~(1 ~ p t)$
F Are parallel lines ADF and CEH parallel to A'D'F' and C'E'H'? (2 pts)
G What is the absolute value of the shear strain for lines DA and $A B$ ? To answer this you need to know the angle DAB and D'A'B'.

H What is the absolute value of the shear strain for lines BC and CE? To answer this you need to know the angle BCE and B'C'E'.
(2 pts)
I Draw a circle with a 1 cm radius inside the square ACHF and draw its deformed counterpart as best you can in figure $A^{\prime} B^{\prime} C^{\prime} E^{\prime} H^{\prime} G^{\prime} F^{\prime} D$. (4 pts)
$J \quad$ Is the deformed counterpart an ellipse?
K (Bonus* - not required!) Determine $F$ and $J_{u}$, for this deformation.
Show your work.
function [val1,val2,vec $1 \mathrm{x}, \mathrm{vec} 1 \mathrm{y}, \mathrm{vec} 2 \mathrm{x}, \mathrm{vec} 2 \mathrm{y}$ ] $=$ eig2d(a,b,c,d)
\% function [val1,val2,vec1x,vec1y,vec2x,vec2y] = eig2d(a,b,c,d)
\% Calculates the eigenvalues and eigenvectors of $2 \times 2$ matrices
\% using the method of Newton (Newton, T.A., 1990, A simple method
$\%$ for finding eigenvalues and eigenvectors for $2 \times 2$ matrices:
\% The American Mathematical Monthly, v. 97, p. 57-60.
\% The input and output are arranged as vectors rather than
\% matrices to allow the code to be used without the need for \% looping over a grid.
\% I AM HAVING TROUBLE WITH [a b; c d] = [1 $0 ; 22]$ !!!!
\% Input parameters (arguments)
$\% \mathrm{a}=$ vector of coefficients ( $\mathrm{n} \times 1$ )
$\% b=$ vector of coefficients $(n \times 1)$
$\% \mathrm{c}=$ vector of coefficients ( $\mathrm{n} \times 1$ )
$\% d=$ vector of coefficients ( $n \times 1$ )
\% Output parameters (arguments)
$\%$ val1 $=$ vector of greatest eigenvalues ( $\mathrm{n} \times 1$ )
$\%$ val2 $=$ vector of least eigenvalues ( $\mathrm{n} \times 1$ )
$\%$ vec $1 x=$ vector of $x$-components for eigenvalues for val ( $n \times 1$ )
$\%$ vec1y $=$ vector of $y$-components for eigenvalues for val1 ( $n \times 1$ )
$\% \operatorname{vec} 2 x=$ vector of $x$-components for eigenvalues for val2 ( $n \times 1$ )
$\%$ vec $2 \mathrm{y}=$ vector of y -components for eigenvalues for val2 ( $\mathrm{n} \times 1$ )
\% Examples
\% [val1,val2,vec $1 x, v e c 1 y, v e c 2 x, v e c 2 y]=$ eig2d(1,2,3,4) \% This checks!
\% [val1,val2,vec $1 \mathrm{x}, \mathrm{vec} 1 \mathrm{y}, \mathrm{vec} 2 \mathrm{x}, \mathrm{vec} 2 \mathrm{y}]=\operatorname{eig} 2 \mathrm{~d}(1,-2,-1,2)$ \% This checks!
\% [val1,val2,vec1x,vec1y,vec2x,vec2y] = eig2d(1,1,-2,3) \% Checks. Odd.
\% [val1,val2,vec $1 \mathrm{x}, \mathrm{vec} 1 \mathrm{y}, \mathrm{vec} 2 \mathrm{x}, \mathrm{vec} 2 \mathrm{y}$ ] = eig2d(3,4,4,-3) \% This checks!
\% [val1,val2,vec $1 \mathrm{x}, \mathrm{vec} 1 \mathrm{y}, \mathrm{vec} 2 \mathrm{x}, \mathrm{vec} 2 \mathrm{y}]=\operatorname{eig} 2 \mathrm{~d}(1,0,2,2)$ \% This checks!
\% [val1,val2,vec $1 \mathrm{x}, \mathrm{vec} 1 \mathrm{y}, \mathrm{vec} 2 \mathrm{x}, \mathrm{vec} 2 \mathrm{y}]=$ eig2d(2,0,0,2) $\%$ This checks!
\% [val1,val2,vec $1 \mathrm{x}, \mathrm{vec} 1 \mathrm{y}, \mathrm{vec} 2 \mathrm{x}, \mathrm{vec} 2 \mathrm{y}]=\operatorname{eig} 2 \mathrm{~d}(2,0,6,1) \quad$ \% This checks!
\% Dimension output vectors
\%val1 = zeros(length(a),1);
val1 = zeros(size(a));
val2 = val1;
vec $1 x=$ val1;
vec $1 \mathrm{y}=$ val1;
vec2x = val1;
vec2y = val1;
m 1 = val1;
m 2 = val2;
\% Three cases
\% Case a of Newton
$\mathrm{i}=$ find $(\mathrm{b} \sim=0)$;
$C=(a(i)+d(i)) / 2 ;$
$\mathrm{R}=\operatorname{sqrt}\left((\mathrm{d}(\mathrm{i})-\mathrm{a}(\mathrm{i})) . \wedge 2+4 * \mathrm{~b}(\mathrm{i}) .{ }^{*} \mathrm{c}(\mathrm{i})\right) / 2$;
$\operatorname{val} 1(\mathrm{i})=\mathrm{C}+\mathrm{R}$;
val2(i) $=\mathrm{C}-\mathrm{R}$;
m1 (i) = ( val1 (i) - a(i) )./b(i);
$m 2(i)=(\operatorname{val} 2(i)-a(i)) . / b(i) ;$
\% Find the corresponding (normalized) eigenvectors
$\operatorname{vec} 1 x(i)=1 . / \operatorname{sqrt}(1+m 1(i) . \wedge 2)$;
vec1y(i) $=\mathrm{m} 1$ (i).*vec1x(i);
vec2x(i) $=1 . /$ sqrt( $1+\mathrm{m} 2(\mathrm{i}) . \wedge 2)$;
$\operatorname{vec} 2 y(i)=m 2(i) .{ }^{*} \operatorname{vec} 2 x(i)$;
\% Case b of Newton (two identical eigenvalues)
$j=$ find ( $a==d \& b==0$ );
val1(j) = d(j);
$\operatorname{val} 2(\mathrm{j})=\operatorname{val} 1(\mathrm{j})$;
\%vec1x(j) = zeros(j,1); \% Possible vector dimension problem here vec $1 \times(\mathrm{j})=$ zeros(size(val1(j))); \% Possible vector dimension problem here \%vec1y(j) = ones(j,1); \% Possible vector dimension problem here $\operatorname{vec} 1 \mathrm{y}(\mathrm{j})=$ ones(size(val1(j))); \% Possible vector dimension problem here $\operatorname{vec} 2 x(j)=\operatorname{vec} 1 x(j)$; \% Possible vector dimension problem here $\operatorname{vec} 2 \mathrm{y}(\mathrm{j})=\mathrm{vec} 1 \mathrm{y}(\mathrm{j})$; \% Possible vector dimension problem here
\% Subcase of isotropic tensor. Eigenvectors set to be orthogonal $j 1=$ find ( $a==d \& b==0 \& c==0$ );
$\operatorname{vec} 2 x(j 1)=\operatorname{vec} 1 y(j 1)$; \% Possible vector dimension problem here $\operatorname{vec} 2 \mathrm{y}(\mathrm{j} 1)=\operatorname{vec} 1 \mathrm{x}(\mathrm{j} 1)$; \% Possible vector dimension problem here
\% Mix of Case a and Case b of Newton (two different eigenvalues)
$k 1=$ find ( $a \sim=d \& b==0 \& a>d$ );
$k 2=$ find ( $a \sim=d \& b==0 \& d>a)$;
$\operatorname{val1}(\mathrm{k} 1)=\mathrm{a}(\mathrm{k} 1)$; $\%$ Maximum eigenvalue $=\mathrm{a}$
val2(k1) = d(k1); \% Minimum eigenvalue = d
$\mathrm{m}=\mathrm{c}(\mathrm{k} 1) . /(\mathrm{a}(\mathrm{k} 1)-\mathrm{d}(\mathrm{k} 1))$;
$\operatorname{vec} 1 x(k 1)=1 . / \operatorname{sqrt}(1+m . \wedge 2)$;
vec1y(k1) = m.*vec1x(k1);
$\% \mathrm{vec} 2 \mathrm{x}(\mathrm{k} 1)=0$;

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%vec2y(k1) = 1;
vec2x(k1) = zeros(size(a(k1)));
vec2y(k1) = ones(size(a(k1)));
val1(k2) = d(k2); % Maximum eigenvalue = d
val2(k2) = a(k2); % Minimum eigenvalue =a
m = c(k2)./(a(k2)-d(k2));
%vec1x(k2) = 0;
%vec1y(k2) = 1;
vec1x(k2) = zeros(size(d(k2)));
vec1y(k2) = ones(size(d(k2)));
vec2x(k2) = 1./sqrt(1+m.^2);
vec2y(k2) = m.*vec2x(k2);
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