

ROTATIONS

I Main Topics

- A Uses of rotation in geology (and engineering) I
- B Concepts behind rotation
- C Rotations using a stereonet
- D Comments on rotations using stereonets and matrices
- E Uses of rotation in geology (and engineering) II

II Uses of rotation in geology (and engineering) I

- A Return tilted bedding to horizontal to examine:
 - 1 pre-tilting orientation of sedimentary structures (e.g. ripples)
 - 2 pre-tilting orientation of beds below angular unconformities
 - 3 pre-tilting orientation of paleomagnetic orientations
- B To determine the orientations of features from drill cores
- C Need to consider whether object is rotated and coordinate axes are fixed or whether the object is fixed and coordinate axes are rotated.
This affects the sign(s) and sequence of the angle(s) of rotation.

III Concepts behind rotation

- A Any orthogonal coordinate system with axes x_1, x_2, x_3 can be rotated to coincide with another orthogonal coordinate system x_1', x_2', x_3' by using the direction cosines relating the axes of the two systems.

Dir. cosines	x_1 axis	x_2 axis	x_3 axis
x_1' axis	$a_{11}' = a_{1'1}$	$a_{12}' = a_{2'1}$	$a_{13}' = a_{3'1}$
x_2' axis	$a_{21}' = a_{1'2}$	$a_{22}' = a_{2'2}$	$a_{23}' = a_{3'2}$
x_3' axis	$a_{31}' = a_{1'3}$	$a_{32}' = a_{2'3}$	$a_{33}' = a_{3'3}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Right-hand rule: If the right thumb is along a rotation axis, then fingers of the right hand curl in positive θ direction

B Any orthogonal coordinate system with axes x_1, x_2, x_3 can always be rotated to coincide with another orthogonal coordinate system by the following **three** steps:

1 Rotate the xyz system about the x_1 axis by angle ω_1 ,

so $x_1, x_2, x_3 \rightarrow x_1', x_2', x_3'$ (Note that $x_1 = x_1'$)

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & -\sin\omega_1 & \cos\omega_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2 Rotate the x_1', x_2', x_3' system about the x_2' axis by angle ω_2 ,

so $x_1', x_2', x_3' \rightarrow x_1'', x_2'', x_3''$ (Note that $x_2' = x_2''$)

$$\begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix} = \begin{bmatrix} a_{1''1} & a_{1''2} & a_{1''3} \\ a_{2''1} & a_{2''2} & a_{2''3} \\ a_{3''1} & a_{3''2} & a_{3''3} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 1 & 0 \\ \sin\omega_2 & 0 & \cos\omega_2 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

3 Rotate the x_1'', x_2'', x_3'' system about the x_3'' axis by angle ω_3 ,

so $x_1'', x_2'', x_3'' \rightarrow x_1''', x_2''', x_3'''$ (Note that $x_3'' = x_3'''$)

$$\begin{bmatrix} x_1''' \\ x_2''' \\ x_3''' \end{bmatrix} = \begin{bmatrix} a_{1'''1} & a_{1'''2} & a_{1'''3} \\ a_{2'''1} & a_{2'''2} & a_{2'''3} \\ a_{3'''1} & a_{3'''2} & a_{3'''3} \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix} = \begin{bmatrix} \cos\omega_3 & \sin\omega_3 & 0 \\ -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix}$$

Note that the rotation matrices can be obtained by permutation:

$x_1 \rightarrow x_2$, $x_2 \rightarrow x_3$, and $x_3 \rightarrow x_1$.

The direction cosines relating the x''' and x systems are:

Dir. cosines	x_1 axis	x_2 axis	x_3 axis
x_1''' axis	$a_{1'''1} = a_{11}'''$	$a_{1'''2} = a_{21}'''$	$a_{1'''3} = a_{31}'''$
x_2''' axis	$a_{2'''1} = a_{12}'''$	$a_{2'''2} = a_{22}'''$	$a_{2'''3} = a_{32}'''$
x_3''' axis	$a_{3'''1} = a_{13}'''$	$a_{3'''2} = a_{23}'''$	$a_{3'''3} = a_{33}'''$

C Any orthogonal coordinate system with axes x_1, x_2, x_3 can always be rotated to coincide with another orthogonal coordinate system by one rotation about a specially chosen axis (Euler's theorem). This single rotation can be found by successively applying the three rotations listed above:

$$\begin{bmatrix} x_1''' \\ x_2''' \\ x_3''' \end{bmatrix} = \begin{bmatrix} a_1'''1 & a_1'''2 & a_1'''3 \\ a_2'''1 & a_2'''2 & a_2'''3 \\ a_3'''1 & a_3'''2 & a_3'''3 \end{bmatrix} \left(\begin{bmatrix} a_1''1 & a_1''2 & a_1''3 \\ a_2''1 & a_2''2 & a_2''3 \\ a_3''1 & a_3''2 & a_3''3 \end{bmatrix} \left(\begin{bmatrix} a_1'1 & a_1'2 & a_1'3 \\ a_2'1 & a_2'2 & a_2'3 \\ a_3'1 & a_3'2 & a_3'3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \right)$$

$$\begin{bmatrix} x_1''' \\ x_2''' \\ x_3''' \end{bmatrix} = \begin{bmatrix} a_1'''1 & a_1'''2 & a_1'''3 \\ a_2'''1 & a_2'''2 & a_2'''3 \\ a_3'''1 & a_3'''2 & a_3'''3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

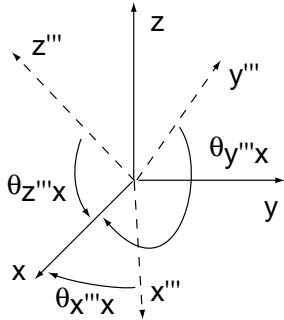
Dir. cosines	x1 axis	x2 axis	x3 axis
x1''' axis	$+\cos\omega_3 \cos\omega_2$	$+\sin\omega_3 \cos\omega_2$	$-\sin\omega_2$
x2''' axis	$-\sin\omega_3 \cos\omega_1$ $+\cos\omega_3 \sin\omega_2 \sin\omega_1$	$+\cos\omega_3 \cos\omega_1$ $+\sin\omega_3 \sin\omega_2 \sin\omega_1$	$+\cos\omega_2 \sin\omega_1$
x3''' axis	$+\sin\omega_3 \sin\omega_1$ $+\cos\omega_3 \sin\omega_2 \cos\omega_1$	$-\cos\omega_3 \sin\omega_1$ $+\sin\omega_3 \sin\omega_2 \cos\omega_1$	$+\cos\omega_2 \cos\omega_1$

Let the axis of rotation be called axis **N** and the direction cosines relating the N axis to either the x_1, x_2, x_3 system or the x_1''', x_2''', x_3''' system be $a_{xN}, a_{yN},$ and $a_{zN},$ respectively. If the angle of rotation about **N** is $\omega,$ then the nine direction cosines in the rotation matrix above can be expressed as:

Dir. cosines	x1 axis	x2 axis	x3 axis
x1''' axis	$a_{xN} a_{xN} (1-\cos\omega)$ $+\cos\omega$	$a_{xN} a_{yN} (1-\cos\omega)$ $+\eta_3 \sin\omega$	$a_{xN} a_{zN} (1-\cos\omega)$ $-a_{yN} \sin\omega$
x2''' axis	$a_{yN} a_{xN} (1-\cos\omega)$ $-a_{zN} \sin\omega$	$a_{yN} a_{yN} (1-\cos\omega)$ $+\cos\omega$	$a_{yN} a_{zN} (1-\cos\omega)$ $+a_{xN} \sin\omega$
x3''' axis	$a_{zN} a_{xN} (1-\cos\omega)$ $+a_{yN} \sin\omega$	$a_{zN} a_{yN} (1-\cos\omega)$ $-a_{xN} \sin\omega$	$a_{zN} a_{zN} (1-\cos\omega)$ $+\cos\omega$

Methods of Rotation

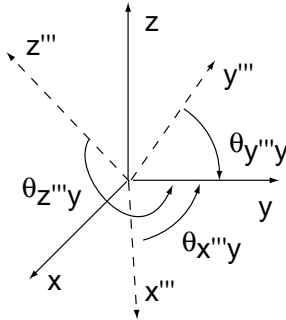
1 All the angles between the x,y,z axes and x''',y''',z''' axes are known



$$a_{x'''x} = \cos \theta_{x'''x}$$

$$a_{y'''x} = \cos \theta_{y'''x}$$

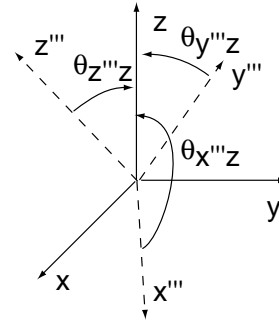
$$a_{z'''x} = \cos \theta_{z'''x}$$



$$a_{x'''y} = \cos \theta_{x'''y}$$

$$a_{y'''y} = \cos \theta_{y'''y}$$

$$a_{z'''y} = \cos \theta_{z'''y}$$

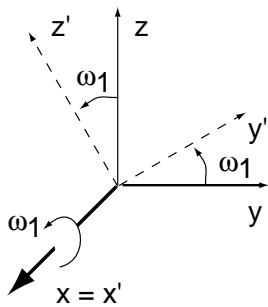


$$a_{x'''z} = \cos \theta_{x'''z}$$

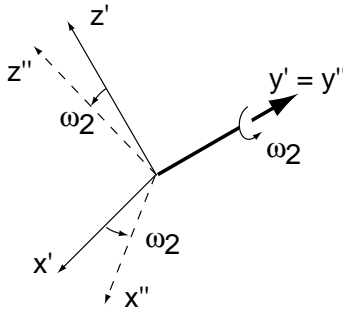
$$a_{y'''z} = \cos \theta_{y'''z}$$

$$a_{z'''z} = \cos \theta_{z'''z}$$

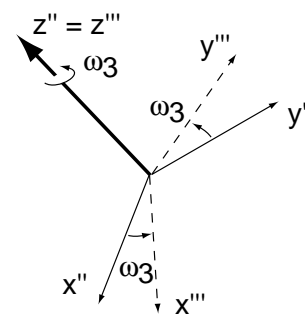
2 Method of three rotations



A: Rotate about x-axis

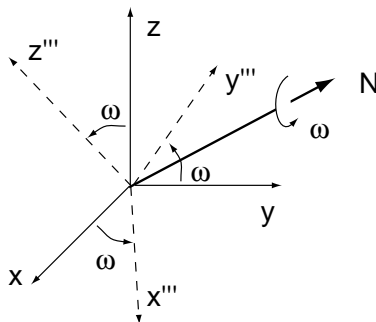


B: Rotate about y'-axis



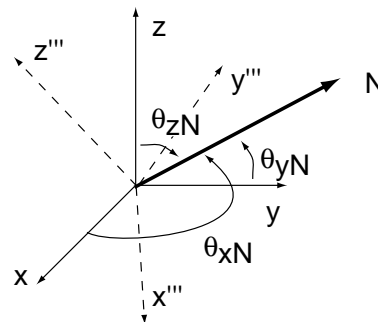
C: Rotate about z''-axis

3 Method of one rotation



This shows the angle of rotation about the rotation axis

$$a_{xN} = \cos \theta_{xN}$$



This shows the orientation of the rotation axis relative to the three coordinate axes

$$a_{yN} = \cos \theta_{yN}$$

$$a_{zN} = \cos \theta_{zN}$$

IV Rotations using a stereonet

A Best uses

- 1 Rotation axis is vertical (but this case is trivial)
- 2 Rotation axis is horizontal (e.g., to restore tilted beds)

B **Construction technique**

- 1 Find orientation of rotation axis
- 2 Find angle of rotation and rotate **pole** to plane (or a linear feature) along a **small circle** perpendicular to the rotation axis.
 - a For a horizontal rotation axis the small circle is vertical
 - b For a vertical rotation axis the small circle is horizontal
- 3 **WARNING**: DON'T rotate a plane by rotating its dip direction vector; this doesn't work

V Comments on rotations using stereonets and matrices

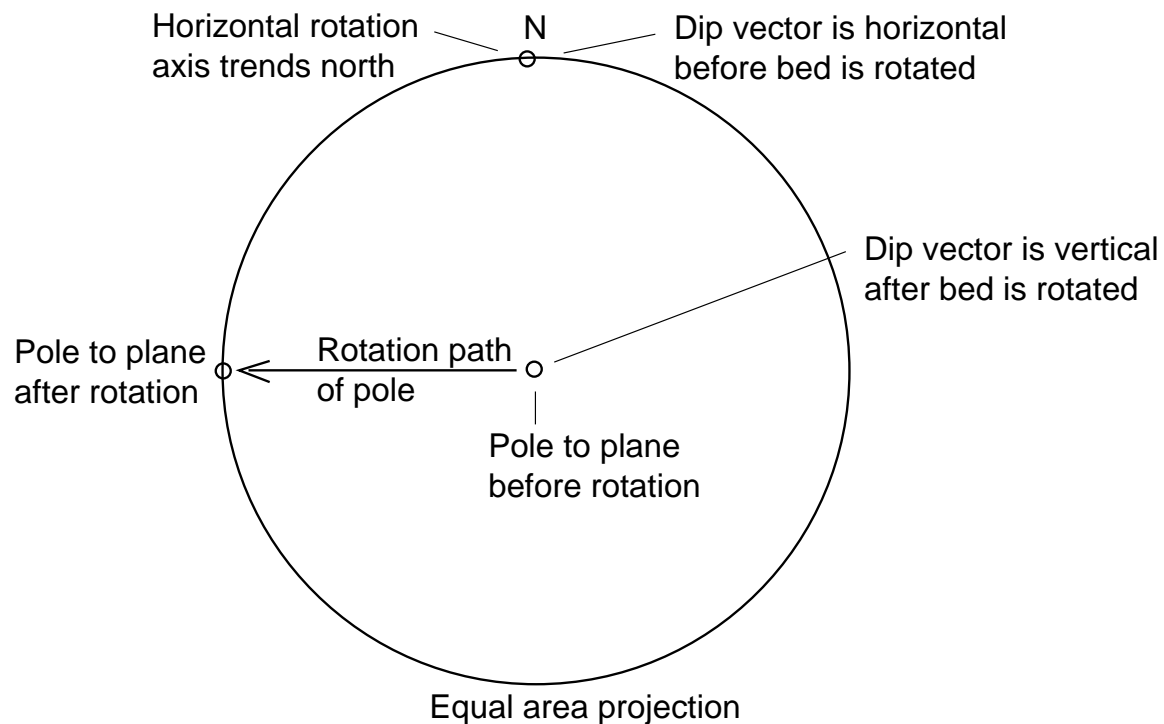
	Advantages	Disadvantages
Stereonets	Good for visualization Can bring into field	Relatively slow Need good stereonets, paper
Matrices	Speed and flexibility Good for multiple rotations	Computer really required to cut down on errors

Rotations of planes using poles;

Problem with attempted rotation of planes using dip vectors

Consider a horizontal plane. We will consider it to strike to the north and dip to the east at 0° ; these directions are consistent with a right-hand rule. The pole to the plane trends west (270°) and plunges 90° . The dip vector trends east (90°) and plunges 0° .

Suppose we wish to rotate the plane by $+90^\circ$ about a horizontal axis that trends north. We can visualize that after the rotation the plane will still strike to the north but will dip 90° . How do the pole to the plane and the dip vector rotate?



The pole will rotate about the rotation axis and yield a result consistent with the final orientation of the the rotated plane. **The original dip vector will not rotate about the rotation axis**, so there is no rotation path to link the pre-rotation dip vector for the plane to the post-rotation dip vector for the plane.

Bottom line: do rotations with poles, not dip vectors

VI Uses of rotation in geology (and engineering) II

A **General comments** about stereonet vs. rotation matrices

- 1 **With stereonet, an object is usually considered to be rotated and the coordinate axes are held fixed.**
- 2 **The rotation matrices, an object is usually considered to be held fixed and the coordinate axes are rotated.**

B To return tilted bedding to horizontal choose a rotation axis that coincides with the direction of strike. The angle of rotation is the **negative** of the dip of the bedding (right-hand rule!) if the bedding is to be rotated back to horizontal and **positive** if the axes are to be rotated to the plane of the tilted bedding. Only one rotation is typically used; seldom are beds then rotated about a vertical axis.

C Orientations of features from drill cores

- 1 Orientations measured relative to core are called apparent
- 2 To determine the in-situ (true) orientations of features from the apparent orientations, two rotations are needed
 - a Rotate the core (or the coordinate axes) about a vertical axis. The rotation angle involves the trend of the core.
 - b Rotate the core (or the coordinate axes) about a horizontal axis perpendicular to the trend of the core . The rotation angle is the involves the plunge of the core .

c The rotations are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \cos(-[90-\phi]) & 0 & -\sin(-[90-\phi]) \\ 0 & 1 & 0 \\ \sin(-[90-\phi]) & 0 & \cos(-[90-\phi]) \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \right)$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos(-\theta)\cos(-[90-\phi]) & \sin(-\theta) & -\cos(-\theta)\sin(-[90-\phi]) \\ -\sin(-\theta)\cos(-[90-\phi]) & \cos(-\theta) & \sin(-\theta)\sin(-[90-\phi]) \\ \sin(-[90-\phi]) & 0 & \cos(-[90-\phi]) \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos \theta \sin \phi & -\sin \theta & \cos \theta \cos \phi \\ \sin \theta \sin \phi & \cos \theta & \sin \theta \cos \phi \\ -\cos \phi & 0 & \sin \phi \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

3 In the approach described in the figure below the primed axes are rotated to coincide with the unprimed axes. An alternative to rotate the unprimed axes to coincide with the primed axes. This can be accomplished by the following two steps:

a Rotate the unprimed axes about the vertical x_3 axis by angle θ .

The x_2 axis becomes the x_2^a axis.

b Rotate the unprimed axes about the x_2^a axis by angle $(90^\circ - \phi)$.

c The rotations are given by

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cos(90^\circ - \phi) & 0 & -\sin(90^\circ - \phi) \\ 0 & 1 & 0 \\ \sin(90^\circ - \phi) & 0 & \cos(90^\circ - \phi) \end{bmatrix} \left(\begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

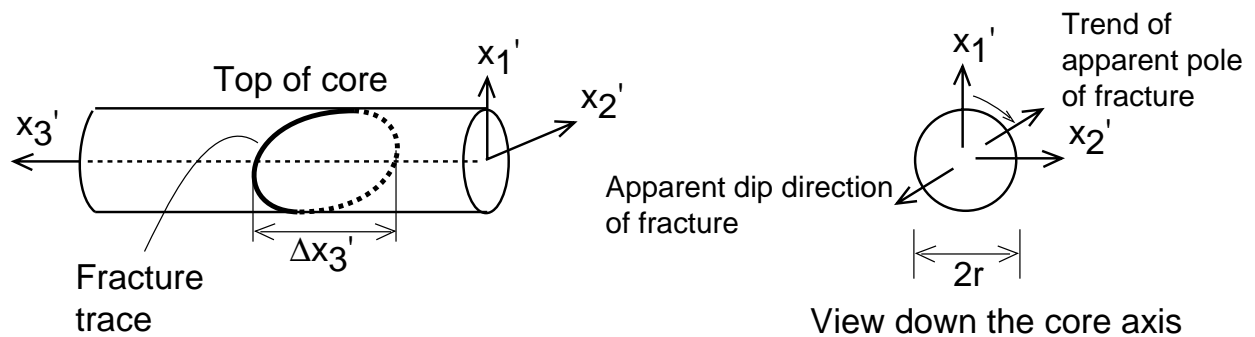
or

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \sin \phi \cos \theta & \sin \phi \sin \theta & -\cos \phi \\ -\sin \theta & \cos \theta & 0 \\ \cos \phi \cos \theta & \cos \phi \sin \theta & \sin \phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

d From (2c), if $[\mathbf{x}] = [\mathbf{a}][\mathbf{x}']$, then, from (3c), $[\mathbf{x}'] = [\mathbf{a}]^T[\mathbf{x}]$

Conversion of apparent orientation scheme of Goodman to in-situ orientations (I)

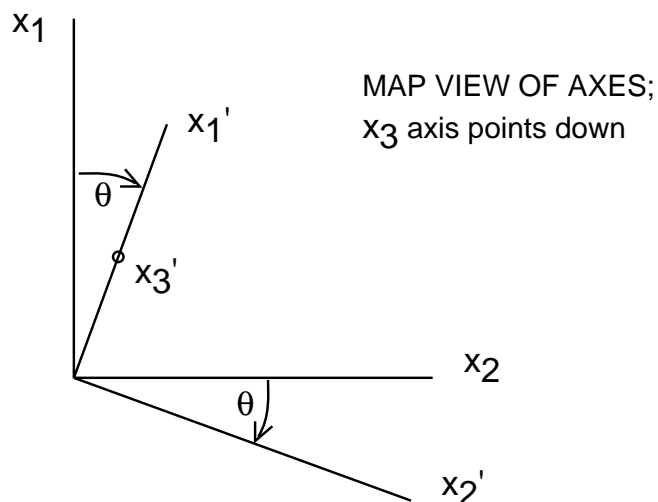
Let the coordinate system for the apparent orientations be the x_1', x_2', x_3' system, where x_3' points down the core axis and x_1' points towards a line scribed on the top surface of the core. In this reference frame the apparent dip direction of a fracture = α and the apparent dip angle of the fracture = β .



In practice, this means the "apparent orientations" are measured with the core held vertically and its top line facing north.

If the radius of the core is r , and the limits of the fracture trace in the x_3' direction is Δx_3 , then $\beta = \tan^{-1} (\Delta x_3' / 2r)$

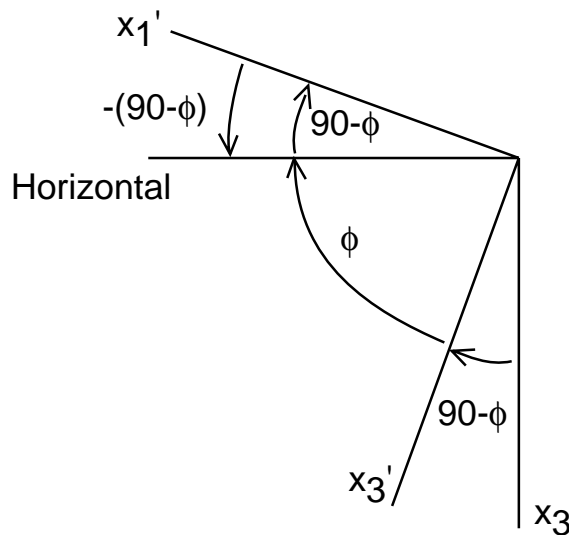
Suppose the in-situ geographic coordinate system is chosen to be $x_1 =$ north, $x_2 =$ east, and $x_3 =$ down. The vertical plane containing the borehole contains the x_1' and x_3' axes, and the x_2' axis is horizontal. The trend and plunge of the x_3' axis coincide with the trend (θ) and plunge (ϕ) of the borehole.



Conversion of apparent orientation scheme of Goodman to in-situ orientations (II)

Rotation 1

Rotating the primed axes about the horizontal x_2' axis by the angle $-(90^\circ - \phi)$ will bring the x_3' and x_3 axes into coincidence.

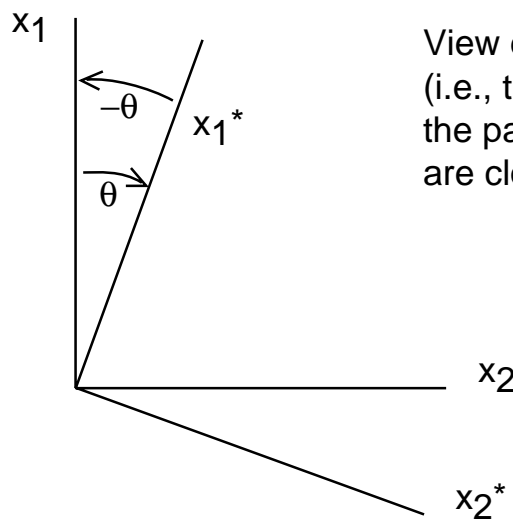


View down along the x_2' axis (i.e., the x_2' axis points into the page). Positive angles are clockwise.

The x_3' axis becomes the x_3^* axis.

Rotation 2

Rotating the starred axes about the vertical x_3^* axis by the angle $(-\theta)$ will bring all the axes into coincidence.



View down along the x_3^* axis (i.e., the x_3^* axis points into the page). Positive angles are clockwise.

Lab 7

Exercise 1: Apparent dip problem (28 points total)

1 a) An apparent dip of 62° to the northwest is measured for a bedding plane in a vertical cross section that strikes $N40^\circ W$ (call this apparent dip vector 1). An apparent dip of 34° to the southwest is measured for a bedding plane in a vertical cross section that strikes $S20^\circ W$ (call this apparent dip vector 2). What is the strike of the bedding plane and the true dip of the bedding plane? Solve this problem requires you to find the common plane that contains two intersecting lines. Solve the problem using an equal-angle spherical projection **(7 points total; 2 points for plotting each line, 1 pt for graphically identifying the common plane, and 2 pts for getting the strike and dip of the common plane)**

Strike	Dip

1 b) Solve the problem using cross products using Matlab. Include a copy of your Matlab printout. **(21 points total, 1 point per box)**

	Trend	Plunge	α	β	γ
Vector v1					
Vector v2					

	V_x	V_y	V_z	$ V $
$N = v2 \times v1$				

	α	β	γ	Pole Trend	Pole Plunge	Plane strike	Plane dip
$n = N / N$							

Exercise 2: Rotation problem 2 (15 points total)

2) An outcrop displays a regular set of current ripple marks. The axes of the ripple marks pitch 32° north in the bedding plane, and the bedding plane strikes 325° and dips 20° NE. Determine the direction of the original paleocurrents responsible for the ripple marks by restoring the beds back to horizontal (assume the current flowed perpendicular to the axes of the ripples. Before you answer that question, first determine the orientation of the rotation axis **N** and the angle of rotation θ . Neatly label your stereonet to show how the relevant features rotate (i.e., the ripple axis and the pole to bedding). **(1 pt/box here, 5 subjective points for clarity of stereonet work)**

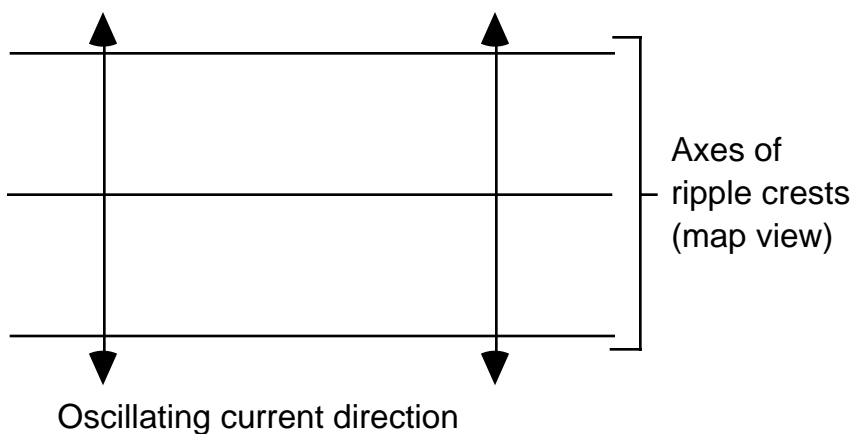
Trend of rotation axis θ	Plunge of rotation axis ϕ	Rotation angle ω

Existing ripple axis trend	Existing ripple axis plunge

Restored ripple axis trend	Restored ripple axis plunge

Original trend of current	Original plunge of current

Rake of restored ripple axis



Exercise 3: Rotation problem 1 (40 points total)

3a The beds below an angular unconformity (the “b beds” strike S30°E and dip 40° to the west. The sequence above the unconformity (the “a beds”) is tilted, with a strike of N20°E and a dip of 30°E. What was the attitude of the “b beds” before the younger beds were tilted? In other words, if the “a beds” are restored to horizontal, what is the restored orientation of the “b beds”? Before you answer that question, first determine the orientation of the rotation axis **N** and the angle of rotation θ . Neatly label your stereonet to show how the relevant features rotate (i.e., the pole(s) to the “a beds” and the pole(s) to the “b beds”).

(14 points total; 1 pt/box here, 5 subjective points for stereonet work)

Trend θ of rotation axis N	Plunge ϕ of rotation axis N	Rotation angle ω

Existing trend of pole b to “b beds”	Existing plunge of pole b to “b beds”

Restored trend of pole b' to “b beds”	Restored plunge of pole b' to “b beds”	Restored strike of “b beds”	Restored dip of “b beds”

3b) Solve the problem using a rotation matrix and Matlab. Include a copy of your Matlab printout. Know what your reference frame is! **(26 points total, 1 point per box)**

	Trend	Plunge	α	β	γ
Vector N					
Pole b					

Rotation matrix

Restored Pole α	Restored Pole β	Restored Pole γ	Restored Pole Trend	Restored Pole Plunge	Plane strike	Plane dip

Exercise 4: Borehole problem (45 points total)

4a The apparent dip of a fracture in a core is 10° and the apparent direction of dip (measured from the top line of the core; see figure on borehole rotations) is 180° . The core comes from a borehole with a trend of 60° and a plunge of 40° . Find the in-situ (in-place) orientation of the fracture. Plot the pole to the fracture and the orientation of the core at each step through this problem. Neatly label your stereonet to show how the core axis and the pole to bedding rotate. **(19 points total; 1 pt/box; 5 subjective points for stereonet)**

Trend of rotation axis1	Plunge of rotation axis1	Rotation angle ω_1	Trend of rotation axis2	Plunge of rotation axis2	Rotation angle ω_2

Apparent trend of fracture pole	Apparent plunge of fracture pole

Pole trend after first rotation	Pole plunge after first rotation	Pole trend after second rotation	Pole plunge after second rotation

In-situ strike of fracture	In-situ dip of fracture

4 b) Solve the problem using a rotation matrix and Matlab. Include a copy of your Matlab printout. Know what your reference frame is! **(26 points total, 1 point per box)**

	Trend	Plunge	α	β	γ
Vector N					
Apparent pole n					

Rotation matrix

Restored orientations

Restored Pole α	Restored Pole β	Restored Pole γ	Restored Pole Trend	Restored Pole Plunge	Plane strike	Plane dip

Equal-Angle Net (Wulff Net)

N

