INTERSECTIONS OF PLANES

I Main Topics
- A Equation of a plane
- B Pole to a plane using cross-products
- C Intersection of two planes in a line
- D Intersection of three planes in a point (solution of simultaneous linear equations)

II Equation for plane P1: $\mathbf{n} \bullet \mathbf{v} = d$
- A Here d is the distance from a known reference point to plane P1 as measured in the direction of a unit vector **n** that is normal to the plane. The distance d is positive if the unit normal points from the reference point towards the plane; the distance d is negative if the unit normal points from the plane towards the reference point.
- B Solution for $\mathbf{n} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$
  1. $\alpha = \cos\phi\cos\theta$, $\beta = \cos\phi\sin\theta$, $\gamma = \sin\phi$ if x=north, y=east, z=down
  2. $\alpha = \cos\phi\sin\theta$, $\beta = \cos\phi\cos\theta$, $\gamma = -\sin\phi$ if x=east, y=north, z=up
     where θ is the pole trend and ϕ is the pole plunge
  3. Matlab: [alpha,beta,gamma] = sph2cart(trend, plunge,1)
     Remember that the trend and plunge must be in radians.
- C Solution for **v**
  This is just the x,y,z position of a point (any point) on the plane
- D Solution for d
  1. $d = \alpha x + \beta y + \gamma z$
  2. d = n*v' (In Matlab, where **n** and **v** are 1x3 vectors)

III Pole to a plane using cross-products

Consider three non-colinear points that lie in a plane: A (xA,yA,zA), B (xB,yB,zB), and C (xC,yC,zC).
- A (**A-C**)x(**B-C**) is normal to the plane
- B (**A-C**)x(**B-C**)/|(**A-C**)x(**B-C**)| = **n**, a unit normal to the plane.

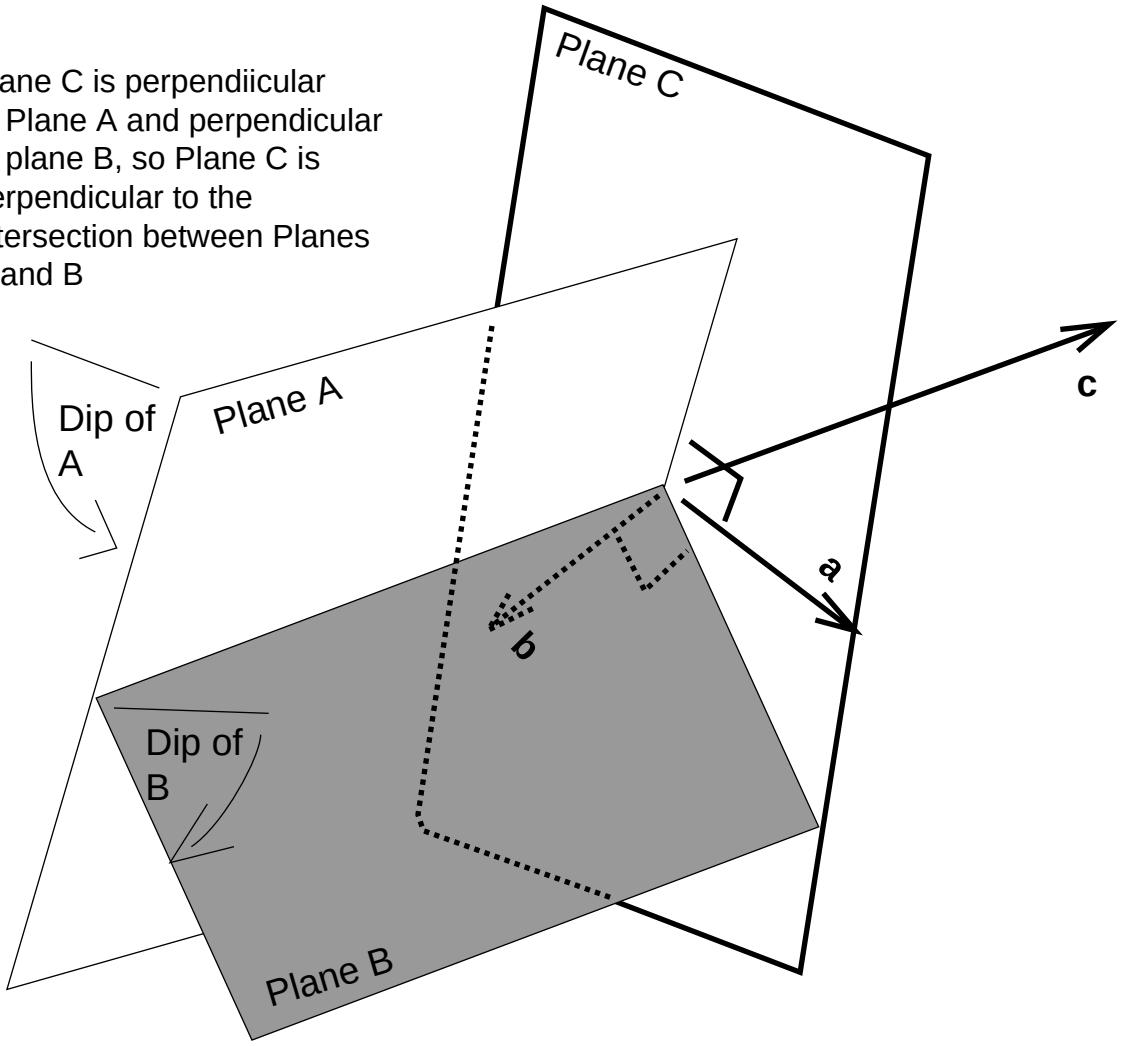
IV Intersection of two planes in a line

A Two planes P1 and P2 intersect in a line. The equations of those two planes define the line.

B The direction of intersection is along the vector that is the cross product of a vector normal to plane P1 and a vector normal to plane P2.

Direction of Intersection of Two Planes

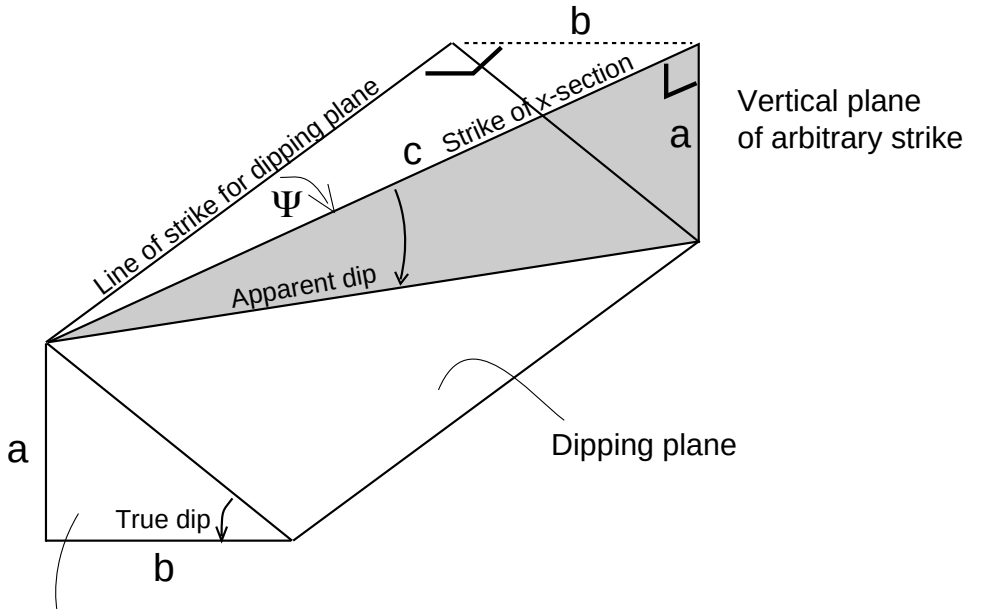
Plane C is perpendiicular to Plane A and perpendicular to plane B, so Plane C is perpendicular to the intersection between Planes A and B



a is unit normal to Plane A and lies in Plane C
b is unit normal to Plane B and lies in Plane C
c is unit normal to Plane C
c= axb/|axb|

C Apparent dip
1 How the plane of interest appears to dip as seen through the "window" of a cross section plane.
2 Plunge of the line of intersection between a vertical cross section plane and the plane of interest.
3 A vector along the apparent dip is given by the cross product of a vector normal to the plane of interest and a vector normal to the cross section plane.

Apparent Dip



(1) a/b = tan (true dip). Now a can be solved for:

(2) a = b tan (true dip), and a = c tan (apparent dip).

(3) b tan (true dip) = c tan (apparent dip)

By comparing the strike lines for the dipping plane and the vertical plane of arbitrary strike, the angle between these lines being Ψ, we obtain

(4) b/c = sin Ψ. Now c can be solved for:

(5) c = b / sin Ψ. Now insert (5) into the right side of (3):

(6) b tan (true dip) = (b / sin Ψ) tan (apparent dip). Solving for the apparent dip

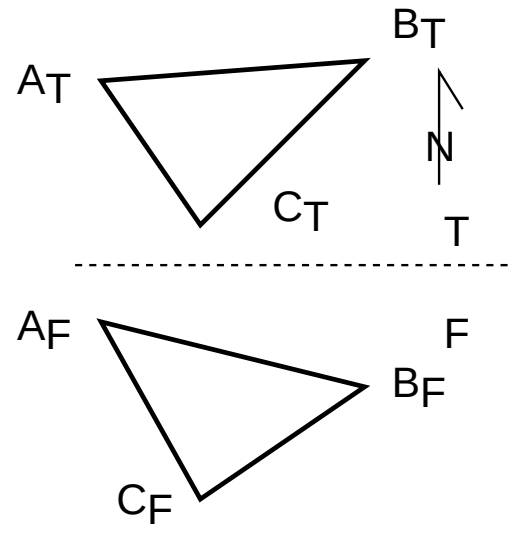
(7) apparent dip = $\tan^{-1}$ { [tan(true dip)] [sin Ψ] }.

Note: sin Ψ † 1, so apparent dip † true dip

If y = 90¡, the apparent dip = true dip

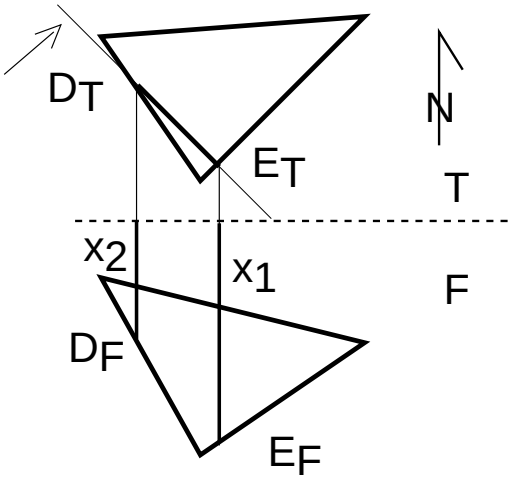
APPARENT DIPS & ORTHOGRAPHIC PROJECTIONS

Find the apparent dip of plane ABC in a vertical cross section that strikes N45W.



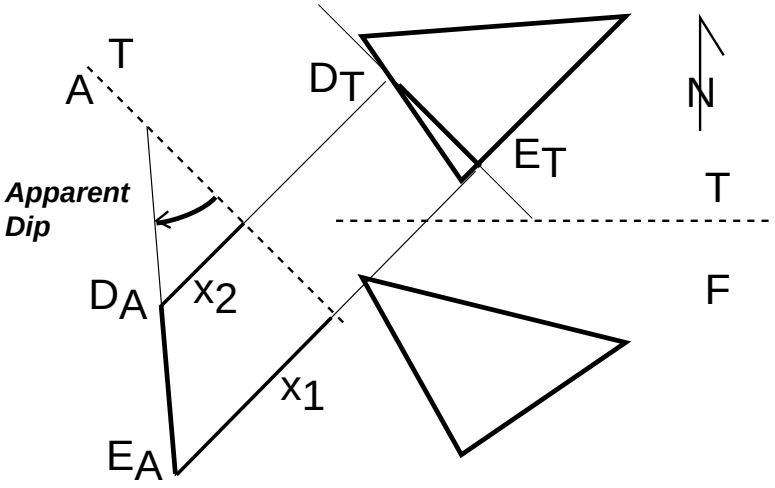
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(1) In the top view intersect the cross section plane with plane ABC. Call the line of intersection DE. Point D is on line AC. Point E is on line BC. Next we find the plunge of line DE.



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(2) Project the line of intersection DE onto an auxilliary view that is taken parallel to the line of cross section. The plunge of line DE is the apparent dip of plane ABC as seen in the cross section.



V Intersection of three planes in a point

A Intersection of two lines (L1 and L2) at a point

1 Intersection criteria

a The lines are <u>not</u> parallel (i.e., **L1** x **L2** ≠ 0)

b The lines lie in the same plane

2 Equations for two lines in a plane

a $a_{11} x_1 + a_{12} x_2 = b_1$ or $a_{11} x + a_{12} y = b_1$

b $a_{21} x_1 + a_{22} x_2 = b_2$ or $a_{21} x + a_{22} y = b_2$
where the subscripted "a" terms are direction cosines.

c $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ (in matrix form)

d Solution of simultaneous equations (i.e., determining where the lines intersect) by Cramer's Rule

$$x_1 = \frac{D_1}{D} \qquad x_2 = \frac{D_2}{D}$$

where

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \qquad D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \qquad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

B Intersection of three planes

1 Graphical solution: intersect two of the planes to find the line of intersection, and then intersect that line with the third plane.

2 Equations of the planes

Equation for plane P1: $\mathbf{n_1 \cdot x} = d_1$

Equation for plane P2: $\mathbf{n_2 \cdot x} = d_2$

Equation for plane P3: $\mathbf{n_3 \cdot x} = d_3$

In matrix form

$$\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$ or [A][X] = [B]

Note that the dimensions are consistent!

3 Solution by Cramer's rule

$$x_1 = \frac{D_1}{D} \qquad x_2 = \frac{D_2}{D} \qquad x_3 = \frac{D_3}{D} \qquad \text{where}$$

$$D_1 = \begin{vmatrix} d_1 & \beta_1 & \gamma_1 \\ d_2 & \beta_2 & \gamma_2 \\ d_3 & \beta_3 & \gamma_3 \end{vmatrix} \qquad D_2 = \begin{vmatrix} \alpha_1 & d_1 & \gamma_1 \\ \alpha_2 & d_2 & \gamma_2 \\ \alpha_3 & d_3 & \gamma_3 \end{vmatrix} \qquad D_3 = \begin{vmatrix} \alpha_1 & \beta_1 & d_1 \\ \alpha_2 & \beta_2 & d_2 \\ \alpha_3 & \beta_3 & d_3 \end{vmatrix}$$

$$D = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \neq 0$$

4 Solution using Matlab

If the matrix of direction cosines for the poles to the planes is A, and the matrix (vector) of distances from the origin to the planes is B, then the coordinates of the intersection point (X) is given by X = A\B.

5 Example: three planes that are a distances of 1, 2, and 3 from the origin as measured along the x, y, and z axes, respectively.

```
»alpha1 = 1; beta1 = 0; gamma1 = 0;
»alpha2 = 0; beta2 = 1; gamma2 = 0;
»alpha3 = 0; beta3 = 0; gamma3 = 1;
»A = [alpha1, beta1, gamma1; alpha2, beta2, gamma2; alpha3, beta3,
   gamma3]

A =
    1     0     0
    0     1     0
    0     0     1

»d1 = 1; d2 = 2; d3 = 3;

»B = [d1;d2;d3]

B =
    1
    2
    3

»X=A\B

X =
    1
    2
    3
```

C Intersection of a line and a plane
  1 A line (L1) and a plane (P1) intersect at a point
  2 Point of intersection can be viewed as the intersection of 3 planes
    a Plane P1
    b Plane P2; P2 intersects plane P3 to give line L1. Plane P2 can be a vertical plane containing L1 (i.e., P2 strikes parallel to the trend of L1)
    c Plane P3; P3 intersects plane P2 to give line L1. Plane P3 can be an inclined plane that contains L1 (i.e., the dip of P3 equals the plunge of L1, and the strike of P3 is 90° from the trend of L1).

D Intersection of two lines (L1 and L2)
  1 Let line (L1) and line L2 intersect at a point
  2 Point of intersection can be viewed as the intersection of 3 planes
    a Plane P3 that contains both lines. Normal to P3 is along the cross product of **L1** and **L2**.
    b Plane P1 that intersects plane P3 to give line L1
    c Plane P2 that intersects plane P3 to give line L2
    (Planes P1 and P2 usually can both be vertical planes

Two Math Handbook References

Gellert, W., Küstner, H., Hellwich, M,. and Kästner, H., 1977, The VNR concise encyclopedia of mathematics: Van Nostrand Reinhold, New York, 760 p.

Tuma, J.J., 1979, Engineering mathematics handbook: McGraw-Hill, New ork, 394 p.

Lab 4

Exercise 1: Apparent dip and equation of a plane **(62 points total)**

1) For plane ABC on the next page, graphically determine its strike and dip. Then find the apparent dip of plane ABC as seen in vertical cross section plane P2, which strikes N50°W. Check the apparent dip you obtain graphically against the apparent dip you obtain trigonometrically using the expression in the course notes. Enter your results in the table. Show your trigonometric calculations in the box below the table.

Strike (graphical) **(3 pts)**	True dip (graphical) **(3 pts)**	Apparent dip (Graphical) **(3 pts)**	Apparent dip (Trig.) **(3 pts)**

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2) Find the x,y,z coordinates of points A, B, and C in terms of the dimensionless units on the axes. Give the coordinates to the nearest 0.1 units. NOTE THAT POSITIVE Z POINTS DOWN! **(1 point/box)**

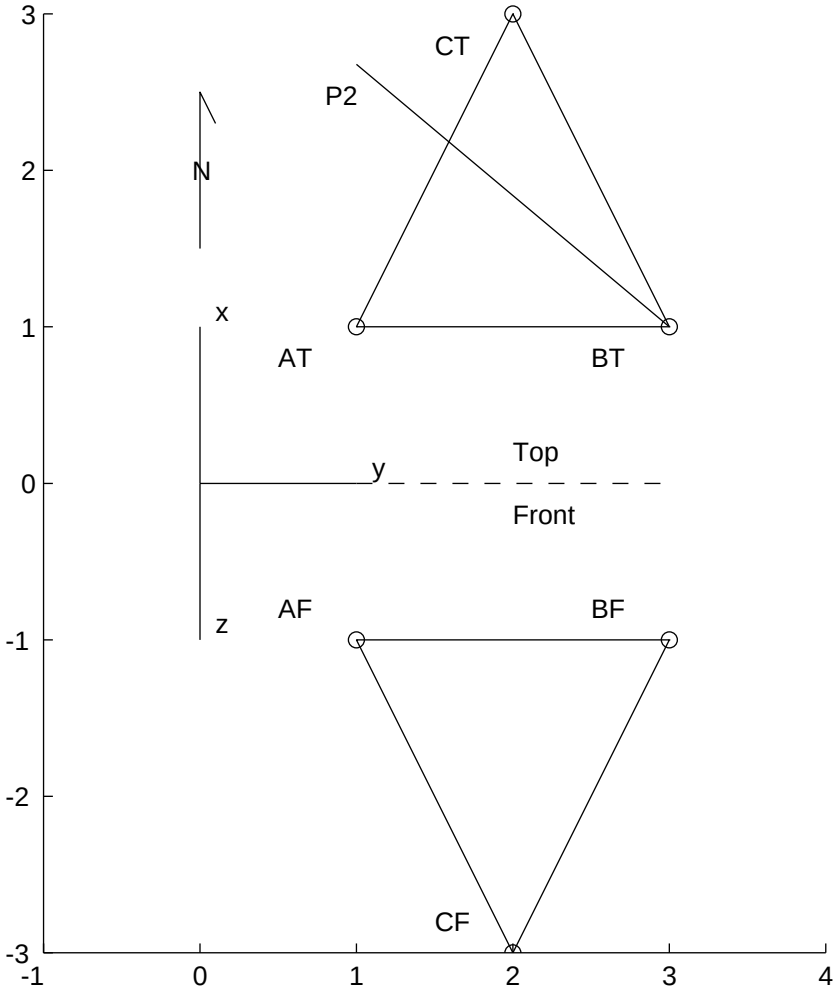
	x	y	z
Point A			
Point B			
Point C			

You will now find the orientation of the line of intersection between plane ABC and the cross section plane from the cross product of the poles to the planes. This check your graphical solution is to be done on Matlab. One-line Matlab commands can address each question. Include a printout of your work showing your answers. Annotate the printout as you see fit.

3) Enter the x,y,z coordinates of points A, B, and C (i.e., vectors **OA**, **OB**, and **OC**, respectively) into Matlab as 1-row, 3-column arrays. For example: Q = [3,4,5]. **(3 points total)**

4) Using Matlab and vector subtraction, find vectors **AC** and **AB**, where **AC** "points" from point A to point C, and **AB** "points" from point A to point B **(2 points)**

	x	y	z
Vector AC			
Vector AB			


3
2
1
0
-1
-2
-3
-1
0
1
2
3
4
CT
P2
N
x
AT
BT
Top
y
Front
AF
BF
z
CF

For questions 5, 7, and 8 fill in the table below.

5) Using Matlab, find the x, y, and z components of **N1**, where **N1** is the cross product **AC** x **AB**. **(1 point)**

	x	y	z
Vector **N1**			

6) Type "help dot" to see how Matlab suggests taking the dot product of two vectors. **(1 point)**

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7) Using the square root function in Matlab (sqrt) and the method for obtaining the dot product of a vector with itself, find the length L1 of vector **N1**. L1 = |**N1**|. **(1 point)**

L1 =|**N1**|

8) Divide **N1** by L1 to get the components α, β, γ of unit vector **n1**, which points in the direction of **N1**. **(1 point)**

α for **n1**	β for **n1**	γ for **n1**

9) Structural geologists deal with poles that point down, not up. If **n1** points down, then the answer to question 5 is fine. If **n1** points up, then it needs to be "turned around" to point down by changing the sign of all the components of **n1**. Fill the table below with values of α, β, and γ, corrected if necessary, for a downward pointing unit normal.

α for downward normal	β for downward normal	γ for downward normal
(1 pt)	**(1 pt)**	**(1 pt)**

10) Using the appropriate equations of section IIB of this lab, derive general algebraic expressions for the trend (θ) and plunge (ϕ) of **n1** in terms of α, β, and γ; write the mathematical expressions below. **(2 pts)**

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.	

11) Using your answer to question (9) and the atan2 and asin functions in Matlab (or else Matlab's cart2sph function), find the trend and plunge of **n1 in degrees**. (**2 points**)

n1 trend (degrees)	**n1** plunge (degrees)

12) Check: is this result consistent with the graphical solution for the strike and dip of the plane for question 1, yes or no?

13) Take the dot product of **AC** and **n1** and write it below. **(1point)**

14) Take the dot product of **AB** and **n1** and write it below. **(1point)**

15) Check: the cross product of **AC** and **AB** , and hence **n1**, is supposed to be perpendicular to the ABC plane. Describe below whether or not your answers to (13) and (14) support this. **(2 points)**

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.	

16) Find d1 = **n1 • OA**, where **OA** goes from the origin to point A. **(1 point)**

17) Find d2 = **n1 • OB**, where **OB** goes from the origin to point B. **(1 point)**

18) Find d3 = **n1 • OC**, where **OC** goes from the origin to point C. **(1 point)**

19) What is the meaning of your answers to questions 15-17? **(1 point)**

20) Check: is this consistent with the points as plotted in the figure? **(1 point)**

21) Write the equation for plane ABC in normal form (e.g., $<\alpha,\beta,\gamma> \bullet v = d$), filling in the terms that you know. **(1 point)**

22) From the strike and dip of cross section plane P2 find the trend and plunge (spherical coordinates) of its pole. **(6 points total; 1 point per box)**

	Strike(°)	Dip(°)	Pole trend(°)	Pole plunge(°)
Plane 2				
	Strike(radians)	Dip(radians)	Pole trend(radians)	Pole plunge(radians)
Plane 2				

23) Using Matlab, find the direction cosines for the pole **n2** to plane 2. **(3 points)**

	α	β	γ
n2 pole			

For questions 24-26 , fill out the table below.

24) Using Matlab, find **V2**= **n1** x **n2** (see question 9 for **n1**). **(1 point)**

25) Find the length L2 of vector **V2**. {i.e., L1 = f(**N1**)}. **(1 point)**

26) Divide **V2** by L2 to get the components α, β, γ of unit vector **v2**. **(1 point)**

	x	y	z	L2 =|**V2**|	α for **v2**	β for **v2**	γ for **v2**
Vector **V2**							

27) Using your answer to question (10) and the atan2 and asin functions in Matlab, find the trend and plunge of **v2** in degrees. (**2 points**)

	Trend (degrees)	Plunge (degrees)
Vector **v2**		

28) Explain what the answer to question (27) is in terms of the original problem as described in question 1: what is **v2**? Write your answers below. **(2 points)**

Think about what you have done. You have prepared a graphical solution, and you checked your solution by an independent algebraic method. The graphical solution probably did not take nearly as long to do as the algebraic solution for this one problem. The graphical solution also lets you "see" what is going on visually. You can save your algebraic solution on a computer, however, and do another problem rapidly just by changing the input values of question 5. If you have many problems to do, the computer will be faster.

Exercise 2: Point-of-intersection problem **(78 pts total; 12 + 66)**
Much of the exercise below is done using Matlab. Include a printout of your Matlab work.
A scientific team plans to test whether earthquakes along the Snake Eyes Fault can be triggered by injecting water down boreholes into the fault. The team needs to know where the borehole will intersect the fault. Here is the geometric data for one borehole option:

	Location of drill pad			Borehole trend	Borehole plunge
A	0m east,	0m north,	el. = 1000m	234°	85°

The fault plane has been located at three spots:

C	1208m east	1050m north	el. = 2000m
D	2100m east	1050m north	el. = 3017m
E	2100m east	700m south	el. = 2000m

Graphical Approach

1) Plot the locations of the above 4 points on a map **(4 points)**, remembering to include a title, scale and north arrow **(3 points)**. Use a scale of 1 cm = 200 m.

2) Find the strike and dip of the fault plane (Plane 1) graphically, then find the trend and plunge of the pole to the plane, and write the values in the table below.

	Strike **(3pts/box)**	Dip **(3pts/box)**	Pole trend **(1 pt/box)**	Pole plunge **(1 pt/box)**
Plane 1 (fault)				

3) Construct a cross section along a vertical plane that goes through the drill pad of the borehole, with the strike of the cross section plane matching the trend of the borehole. In this cross section plane show the borehole **(1 point)**, the Snake Eyes fault, which will appear with an apparent dip **(4 points)**, and show the intersection of the borehole with the fault plane. **(2 points)**. The cross section should also have a title **(1 point)**, a vertical scale **(1 point)**, and something to designate its orientation **(1 point)**. This problem is similar to that of Exercise 1. (**10 points total**)

4) Measure the distance along the borehole from the drill pad to the fault. Give the distance in terms of “meters in the field”, not “cm on the piece of paper.” **(2 points)**.

Distance along the borehole from the drill pad to the fault (meters)	

Algebraic Approach

5) Read the remaining questions for this exercise. After reading them, outline in your own words in ten sentences or less your plan of attack for determining where the borehole intersects the fault. Your words should provide some insight into why you conduct the keys steps. You can update or revise your outline of your plan of attack as you go. An answer that receives full credit can be written in as few as one well-chosen sentence. Turn the answer to this question in on a separate page. **(10 points)**.

6) **Show on your map a reference frame where x=north, y=east, z=down, with the coordinate origin at an elevation of 0. A positive elevation has a negative z value! Use this reference frame for the remaining analyses. (2 points)**

For the rest of the questions here include copies of your Matlab printout. Also, angular values you enter in the tables should be in degrees, not radians (but use radians for Matlab calculations).

7) Find the direction cosines for the pole to the fault using your answers to question 2.

	α **(1 pt/box)**	β **(1 pt/box)**	γ **(1 pt/box)**
Plane 1 (fault)			

8) Time for an intermediate check on your work. Using Matlab, find the x,y,z components of vectors **CD** and **CE**, where **CD** "points" from point C to point D, and **CE** "points" from point C to point E **(1 point for each vector; 2 points total)**

	x	y	z
Vector **CD**			
Vector **CE**			

9) Using Matlab, find the cross product **N1** = **CD** x **CE**. **(1 point)**

	Nx	Ny	Nz
Vector **N1**			

10) Using Matlab, find L1, the length of vector **N1**. **(1 point)**

L1 =|**N1**|

11) Using Matlab, find the unit vector **n1** by dividing **N1** by L1. **(1 point)**

α for **n1**	β for **n1**	γ for **n1**

12) How do your answers to questions 7 and 11 compare? Explain any substantive differences. Which result do you think is more accurate? Why? **(4 points)**

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13) Find the strike and dip of the **vertical** plane (Plane 2) that contains the borehole, then find the trend and plunge of the pole to the plane, and write the values in the table below.

14) Find the strike and dip of an **inclined** plane (Plane 3) that contains the borehole and that dips at an angle equal to the borehole plunge, then find the trend and plunge of the pole to the plane, and write the values in the table below.

	Strike **(3pts/box)**	Dip **(3pts/box)**	Pole trend **(1 pt/box)**	Pole plunge **(1 pt/box)**
Plane 2 (vertical)				
Plane 3 (inclined)				

Write the answers for questions 15-17 in the table below question 17.

15) Using Matlab, find the direction cosines of the poles to each plane. Write the answers in the table below. For plane 1, use the direction cosines you think are the most accurate in the table of question 14. Make sure your answers are consistent with your choice of reference frame.

16) In the table below, write the x,y,z coordinates of a point in each plane. Make sure your answers are consistent with your choice of reference frame.

17) Using Matlab and the information in the table immediately above, find the distance (d) from the origin to each plane along the direction of the appropriate unit normals. Write the answers in the right-hand column of the table for question 10. **(3 points)**

	α **(1 pt/box)**	β **(1 pt/box)**	γ **(1 pt/box)**	x,y,z **(1 pt/box)**	d **(1 pt/box)**
Plane 1 (fault)				x: y: z:	
Plane 2 (vertical)				x y z	
Plane 3 (inclined)				x y z	

18) in light of the x=north, y=east, z=down reference frame, does **n** point in the right direction for Plane 1, and do the direction cosines for Planes 2 and 3 seem reasonable? State your reasoning below. **(3 points total)**

The remaining questions deal with the location where the borehole intersects the fault.

19) Fill out the matrices below for the equation AX = B, for which you know values. Circle the boxes that you need to solve for to find where the borehole intersects the fault. Then use Matlab to solve for the coordinates of where the borehole would intersect the fault, and fill in those coordinates in the appropriate matrix below (**1 point per box; 1 point for circling the correct matrix; 16 points total)**.

A

X

=

B

20) Using Matlab, determine the length of the borehole based on the coordinates of its drillpad and the coordinates of where it would intersect the fault. Compare the answer here with the answer from question 3 of Exercise 2. **(2 points)**.

