10. PRINCIPAL STRAINS

I Main Topics

- A Inverses and transposes of rotation matrices
- B Principal strains for irrotational deformation
- C Principal strains for rotational deformation
- D Key points for geologic applications
- E Recap

10. PRINCIPAL STRAINS

II Inverses and transposes of rotation matrices

- A <u>The inverses of rotation matrices</u> [R] <u>are their transposes</u>
- B $[R]^T = [R]^{-1}$
- C Proof is based on the fact that vectors maintain their length (and the square of their length) when the reference frame rotates

D Proof

X'•X' = **X•X**

X' = **RX**

RX•RX = X•X

Vectors in primed and unprimed frames

$[RX]^T[RX] = [X]^T[X]$

$[X]^T[R]^T\,[R][X] = [X]^{T[}[I][X]$

$[R]^T\,[R] = [I]$

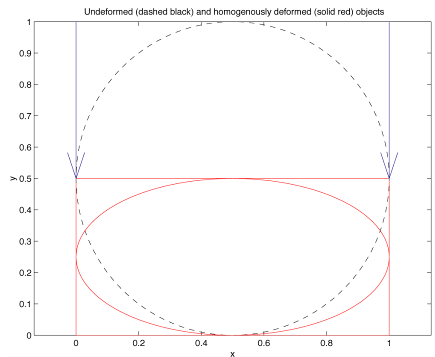
$[R]^{-1}[R] = [I]$

$[R]^T = [R]^{-1}$

10. PRINCIPAL STRAINS

III Principal strains for irrotational deformation
- A Unit circles deform into strain ellipses/ellipsoids
- B F-matrix is symmetric
- C Principal stretch (L_f/L_0) magnitudes are eigenvalues of F-matrix
- D Principal strain ($\Delta L/L_0$) magnitudes are eigenvalues of J_u-matrix
- E Principal directions are eigenvectors of F-matrix
- F Principal axes do not rotate during deformation

Example D from lecture 14



The axes referred to here as “principal strain axes” should probably be referred to as “principal elongation axes”

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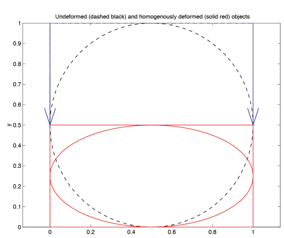
14. HOMOGENEOUS FINITE STRAIN: DISPLACEMENT & DEFORMATION GRADIENTS

V Examples
- D Uniaxial shortening



$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Position transformations (matrix form)	$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix}$ Displacement gradient tensor J_u

```
>> [vf,df]=eig(F)
vf =
    0    1
    1    0
df =
   0.5000         0
        0    1.0000
```

```
>> [vju,dju]=eig(Ju)
vju =
    0    1
    1    0
dju =
  -0.5000         0
        0         0
```

Principal stretch (L_f/L_0) magnitudes are eigenvalues of F-matrix
Principal strain ($\Delta L/L_0$) magnitudes are eigenvalues of J_u-matrix
$\varepsilon = S - 1$

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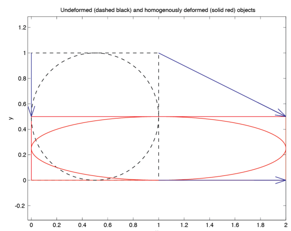
14. HOMOGENEOUS FINITE STRAIN: DISPLACEMENT & DEFORMATION GRADIENTS

V Examples

F Pure shear strain (biaxial strain, no dilation



$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Position transformations (matrix form)	$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}$ Displacement gradient tensor J_u

```
>> [vf,df]=eig(F)
vf =
    0    1
    1    0
df =
   0.5000         0
        0    2.0000
```

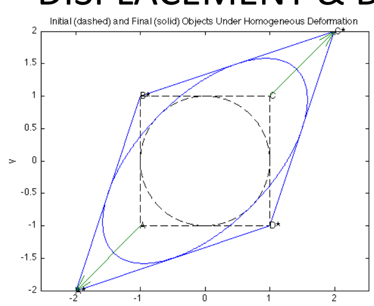
```
>> [vju,dju]=eig(Ju)
vju =
    0    1
    1    0
dju =
  -0.5000         0
        0    1.0000
```

Principal stretch (L_f/L_0) magnitudes are eigenvalues of F-matrix

Principal strain ($\Delta L/L_0$) magnitudes are eigenvalues of J_u-matrix

$\varepsilon = S - 1$

14. HOMOGENEOUS FINITE STRAIN: DISPLACEMENT & DEFORMATION GRADIENTS



$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Position transformations (matrix form)	$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ Displacement gradient tensor J_u

```
>> [vf,df]=eig(F)
vf =
 -0.7071    0.7071
  0.7071    0.7071
df =
    1    0
    0    2
```

```
>> [vju,dju]=eig(Ju)
vju =
 -0.7071    0.7071
  0.7071    0.7071
dju =
    0    0
    0    1
```

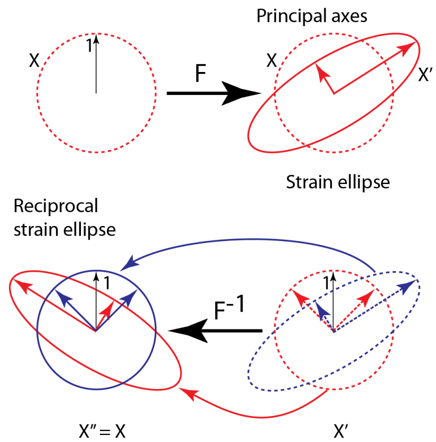
Principal stretch (L_f/L_0) magnitudes are eigenvalues of F-matrix

Principal strain ($\Delta L/L_0$) magnitudes are eigenvalues of J_u-matrix

$\varepsilon = S - 1$

10. PRINCIPAL STRAINS

V Principal strains for rotational deformation
- A [F] transforms unit circle into strain ellipse
- B $[F]^{-1}$ transforms strain ellipse into unit circle
- C $[F]^{-1}$ also transforms unit circle into reciprocal strain ellipse
- D The rotation describes the angular difference between the retro-deformed strain axes (solid blue) and the axes of the reciprocal strain ellipse (red in lower row)



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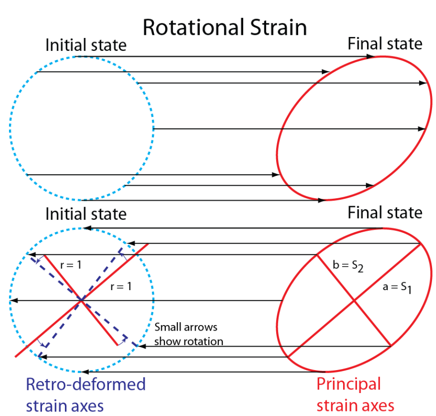
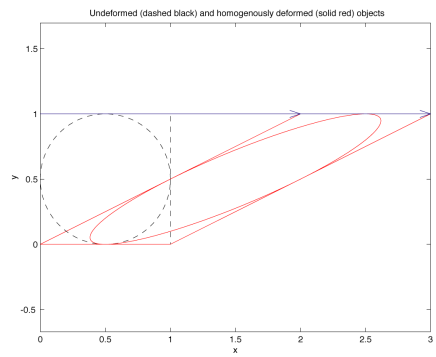
9. Strain and Eigenvectors

IX Symmetric and non-symmetric matrices
- B If [F] is not symmetric, then vectors that F operates on that *<u>don't</u>* rotate (e.g., a horizontal vector at right) are not necessarily the longest and shortest



10. PRINCIPAL STRAINS

IV Principal strains for rotational deformation
- A Unit circles deform into strain ellipses/ellipsoids
- B F-matrix is non-symmetric
- C Eigenvalues of F-matrix are directions of lines that do not rotate (in example to right, a line parallel to the x-axis), but they don't give the axes of the strain ellipse
- D Eigenvectors of F-matrix are not perpendicular
- E Principal strain axes do rotate during deformation



$$F = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

10. PRINCIPAL STRAINS

IV Principal strains for rotational deformation
- F How can we convert a non-symmetric matrix F into a symmetric matrix?
- G C = F^TF and B = FF^T are symmetric, as are sqrt(C) and sqrt(B).

$F^TF = C$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$$

$FF^T = B$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$$

9. Strain and Eigenvectors

- IX Symmetric and non-symmetric matrices
  - C Treatment for non-symmetric F
    1. Start with the definition of *quadratic* elongation Q
    2. Express using dot products
    3. Clear the denominator

$$\frac{L_f^2}{L_0^2} = Q$$

$$\frac{\vec{X}' \bullet \vec{X}'}{\vec{X} \bullet \vec{X}} = Q$$

$$\vec{X}' \bullet \vec{X}' = \left(\vec{X} \bullet \vec{X}\right) Q$$

9. Strain and Eigenvectors

- IX Symmetric and non-symmetric matrices
  - C Treatment for non-symmetric F
    4. Replace X' with [FX]
    5. Re-arrange both sides
    6. Both sides of this equation lead off with $[X]^T$, which cannot be a zero vector, so it can be dropped from both sides to yield an eigenvector equation
    7. The eigenvalues of $[F^T F]$ are the principal quadratic elongations

$$\vec{X}' \bullet \vec{X}' = \left(\vec{X} \bullet \vec{X}\right) Q$$

$$[FX]^T [FX] = [X]^T [X] Q$$

$$[X]^T [F]^T [FX] = [X]^T Q [X]$$

$$\left[F^T F\right][X] = Q[X]$$

$$"[A][X] = \lambda[X]"$$

9. Strain and Eigenvectors

IX Symmetric and non-symmetric matrices

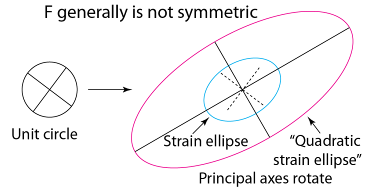
D General Case

1. Since $[F^TF]$ is symmetric, the principal quadratic elongations are perpendicular
2. The principal stretches are the square roots of the principal quadratic elongations (Q); they are the square roots of the eigenvalues of $[F^TF]$
3. The directions of the principal quadratic elongations (Q) and principal stretches (S) are the same; see Lab 9 lecture, slide 38
4. The axes of principal strain rotate

$$\left[F^T F\right][X] = Q[X]$$

$$Q = \frac{L_f^2}{L_0^2};\ S = \frac{L_f}{L_0} \Rightarrow \sqrt{Q} = S$$

F generally is not symmetric



9. Strain and Eigenvectors

IX Symmetric and non-symmetric matrices

E Special Case: [F] is symmetric

1. $[F^TF] = [F^2]$ because $F = F^T$
2. The principal stretches (S) again are the square roots of the principal quadratic elongations (Q) (i.e., the square roots of the eigenvalues of $[F^2]$)
3. The principal stretches (S) also are the eigenvalues of [F], directly
4. The directions of the principal stretches (S) are the eigenvectors of [F], and of $[F^TF] = [F^2]$!
5. The principal axes do not rotate

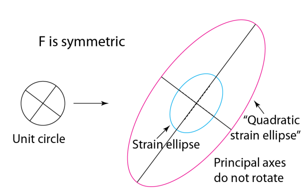
$$\left[F^T F\right][X] = Q[X]$$

$$\left[F^2\right][X] = Q[X]$$

$$Q = \frac{L_f^2}{L_0^2};\ S = \frac{L_f}{L_0} \Rightarrow \sqrt{Q} = S$$

$$[F][X] = S[X]$$

F is symmetric



10. PRINCIPAL STRAINS

IV Principal strains for rotational deformation

H <u>Polar decomposition theorem</u>

[F] can be decomposed into the product of a rotation matrix [R] and an irrotational ***symmetric*** stretch matrix ([U] or [V]); this is stated here without a proof (see Malvern, 1969, p. 174)

I [F] = [R][U]

Stretch by [U], followed by a rotation [R]

J [F] = [V][R]

rotation by [R], followed by a stretch by [V]

10. PRINCIPAL STRAINS

IV Principal strains for rotational deformation (cont).

$[C] = [F^T]\ [F] = [RU]^T[RU]$

$[C] = [U^T]\ [R^T]\ [R]\ [U]$

$[C] = [U^T\ [I]\ [U]$

$[C] = [U]\ [U]=[U^2]$

[U] = [C]^(1/2)

$[B] = [F]\ [F^T] = [VR]\ [VR]^T$

$[B] = [V]\ [R\,]\ [R]^T\ [V]^T$

$[B] = [V]\ [I]\ [V]^T$

$[B] = [V]\ [V]=[V^2]$

[V] = [B] ^(1/2)

10. PRINCIPAL STRAINS

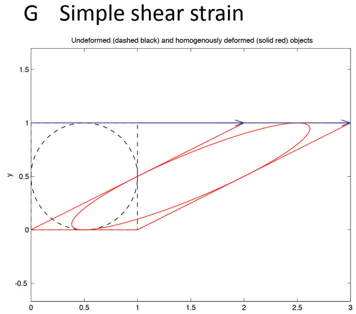
IV Principal strains for rotational deformation (cont).

- Since F = RU, the rotation is then found as R = FU $^{-1}$, or since F = VR, R = V $^{-1}$F
- Unit eigenvectors of C match unit eigenvectors of U (see wikiuniversity)
- Unit eigenvectors of B match unit eigenvectors of V (see wikiuniversity)
- Square roots of eigenvalues of C give the stretch of the axes described by U (see wikiuniversity)
- Square roots of eigenvalues of B give the stretch of the axes described by V (see wikiuniversity)

10. PRINCIPAL STRAINS

V Examples

G Simple shear strain



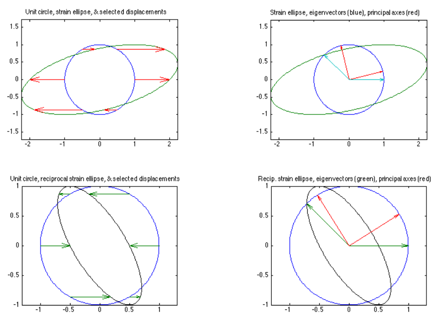
$$F = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

```
>> F = [1 2;0 1]
F =
  1    2
 0     1
>> C=F'*F
C =
1     2
2     5
>> B=F*F'
B =
 5     2
 2     1
```

```
>> [vc,dc]=eig(C)
vc =
 -0.9239   0.3827
  0.3827   0.9239
dc =
0.1716       0
  0          5.8284
>> [vb,db]=eig(B)
vb =
 0.3827  -0.9239
 -0.9239  -0.3827
db =
0.1716       0
0            5.8284
```

10. PRINCIPAL STRAINS

Example with vr_decomp
F = VR

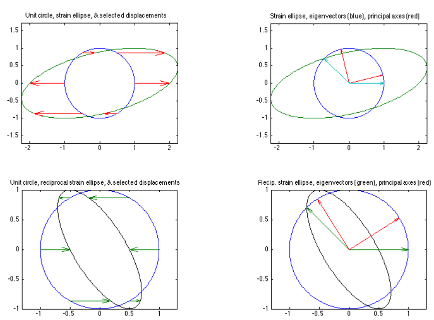


V=sqrtm(F*F')
Eigenvectors of [V] (red, upper right) give directions of axes of strain ellipse.
Eigenvectors of [V^{-1}] (red, lower right) give directions of axes of reciprocal strain ellipse.

```
>>[V,R,eigvecV,eigvalV,eigvecF,eigvalF,eigvecV_] =
      vr_decomp([2,1;0,1]);
Deformation gradient matrix [F] = [V][R]
   2    1
   0    1
Symmetric component [V]
   2.2136    0.3162
   0.3162    0.9487
Rotation matrix [R]
   0.9487    0.3162
  -0.3162    0.9487
Eigenvalues of [V]
   2.2882         0
        0    0.8740
Eigenvectors of [V] (in columns)
   0.9732   -0.2298
   0.2298    0.9732
Eigenvalues of inverse of [V]
   2.2882         0
        0    0.8740
Eigenvectors of inverse of [V]
   0.9732   -0.2298
   0.2298    0.9732
```


10. PRINCIPAL STRAINS

Example with vr_decomp
F = VR



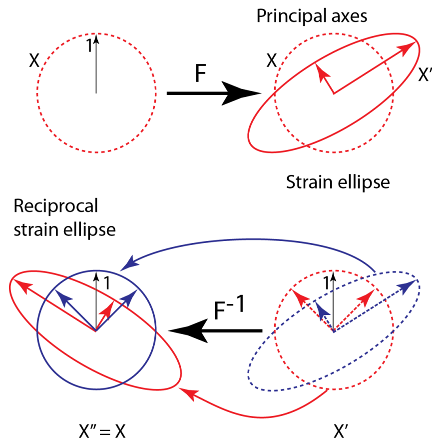
V=sqrtm(F*F')
Eigenvectors of [V] (red, upper right) give directions of axes of strain ellipse.
Eigenvectors of [V^{-1}] (red, lower right) give directions of axes of reciprocal strain ellipse.

```
>>[V,R,eigvecV,eigvalV,eigvecF,eigvalF,eigvecV_] =
      vr_decomp([2,1;0,1]);
Deformation gradient matrix [F] = [V][R]
   2    1
   0    1
Symmetric component [V]
   2.2136    0.3162
   0.3162    0.9487
Rotation matrix [R]
   0.9487    0.3162
  -0.3162    0.9487
Eigenvalues of [F]
   2    0
   0    1
Eigenvectors of [F] (in columns)
   1.0000   -0.7071
        0    0.7071
```

10. PRINCIPAL STRAINS

V Key points for geologic applications

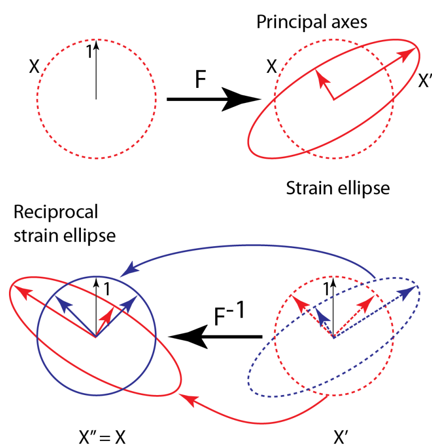
- A Here we know the [F] and $[J_u]$ matrices; in nature we usually don't
- B Here we know the ***initial*** and final positions of points; in nature we usually don't
- C To apply finite strain theory to rocks, we make assumptions about initial shapes and/or positions ***at a point.***
- D Important to know these assumptions
- E In a forward model, we know the conditions/assumptions



10. PRINCIPAL STRAINS

V Key points for geologic applications

- F In infinitesimal strain theory, the strain matrix (see Lecture 15, slide 11) is symmetric, the eigenvalues give the maximum and minimum extensions, and the eigenvalues give the directions of the axes of the strain ellipse (and the reciprocal strain ellipse).
- G Valid if displacement derivatives << 1.

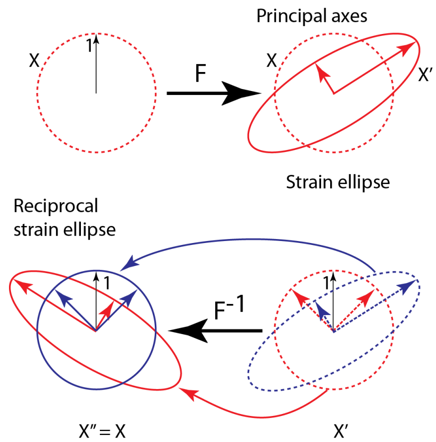


10. PRINCIPAL STRAINS

VI Recap

A Strain deals with changes in the size and/or shape of bodies or pieces of bodies

B The size/shape changes are manifest in changes in the lengths of line segments and the angles between line segments



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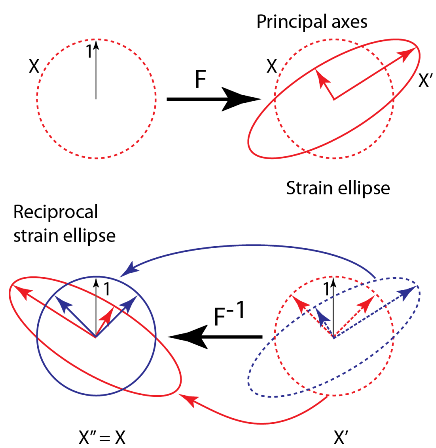
10. PRINCIPAL STRAINS

VI Recap

C Strain is defined at a point using dimensionless derivatives but in practice is measured over finite regions (volumes)

D If the displacement derivatives with respect to position are small, infinitesimal strain theory can be applied



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10. PRINCIPAL STRAINS

VI Recap

E The deformation gradient matrix [F] relates initial position vectors to final position vectors

F The displacement gradient matrix [J_u] relates initial position vectors to displacement vectors

G [F] and [J_u] are easily related

H At a point, [F] and [J_u] contain constants, so the components of [dX'] and [U] are linearly related to the components of [dX]

$$[dX'] = [F][dX]$$

$$\begin{bmatrix} dx' \\ dy' \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$[U] = [J_u][dX]$$

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$[U] = [dX'] - [dX] = [F][dX] - [I][dX]$$

$$[J_u] = [F] - [I]$$

10. PRINCIPAL STRAINS

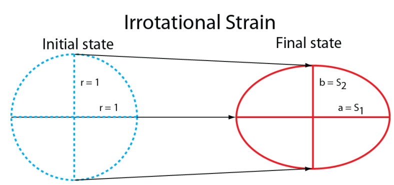
VI Recap

I If the [F] matrix is symmetric, then orthogonal vectors [dX] can be found that do not rotate when operated on by [F], so [F][dX] = λ[dX] . Those vectors are called eigenvectors and they give the directions of the principal stretches at a point ($S = L_f/L_o$)

J The stretches (λ) of these eigenvectors are called eigenvalues. They provide maximum and minimum stretches.

$$[dX'] = [F][dX]$$

$$[F][dX] = \lambda[dX]$$



Note: The principal stretch directions are not always "horizontal and "vertical".

10. PRINCIPAL STRAINS

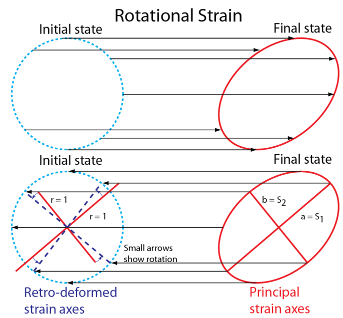
VI Recap

K Even if [F] is not symmetric, $[F^T][F]$ is.

L The principal stretches at a point can be found by taking the square roots of the principal quadratic strains (i.e., the square roots of the eigenvalues of $[F^T][F]$)

$$\left[F^T F\right][dX] = Q[dX]$$

$$S = \lambda = \sqrt{Q}$$



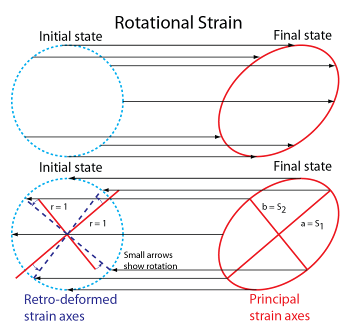
10. PRINCIPAL STRAINS

VI Recap

M The symmetric stretch matrix $[V] = \left[FF^T\right]^{1/2}$

N The rotation matrix [R] describes the difference in orientation $[R] = \left[V^{-1}\right][F]$

O If [F] is symmetric, then $[V]=[F]$, $[V^{-1}] = [F^{-1}]$, and $[R] = [F^{-1}]\,[F] = [I]$



$$[F] = [V][R]$$

$$\left[V^{-1}\right][F] = \left[V^{-1}\right][V][R] = [R]$$

10. PRINCIPAL STRAINS

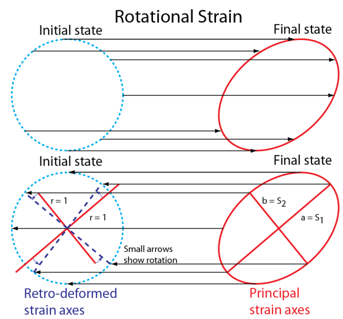
VI Recap

P The symmetric stretch matrix $[U]=\left[F^T F\right]^{1/2}$

Q The rotation matrix [R] describes the difference in orientation

$$[R]=[F]\left[U^{-1}\right]$$

R If [F] is symmetric, then [U]=[F], $[U^{-1}]$ = $[F^{-1}]$, and [R] = [F] $[F^{-1}]$ = [I]



$$[F]=[R][U]$$

$$[F]\left[U^{-1}\right]=[R][U]\left[U^{-1}\right]=[R]$$

10. PRINCIPAL STRAINS

VI Recap

S The stress matrix, by Cauchy's formula, converts the normal vector to a plane to the traction vector acting on the plane.

T This transformation has the form [A][X] = λ[X]

U The stress matrix is symmetric, so its eigenvalues are the principal stresses, and its eigenvalues are the principal values

V The principal stress directions can be used to predict the sense of slip on faults and the orientation of dikes and joints (normal to the most tensile stress)

$$\tau_j = n_i\sigma_{ij} = \sigma_{ij}n_i \Rightarrow [\tau]=[\sigma][n]$$

$$\begin{bmatrix}\tau_x \\ \tau_y\end{bmatrix} = \begin{bmatrix}\sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}\begin{bmatrix}n_x \\ n_y\end{bmatrix}$$

$$[\tau]=|\vec{\tau}|[n]$$

$$|\vec{\tau}|[n]=[\sigma][n]$$

The components of a vector come from projecting the vector onto the coordinate axes