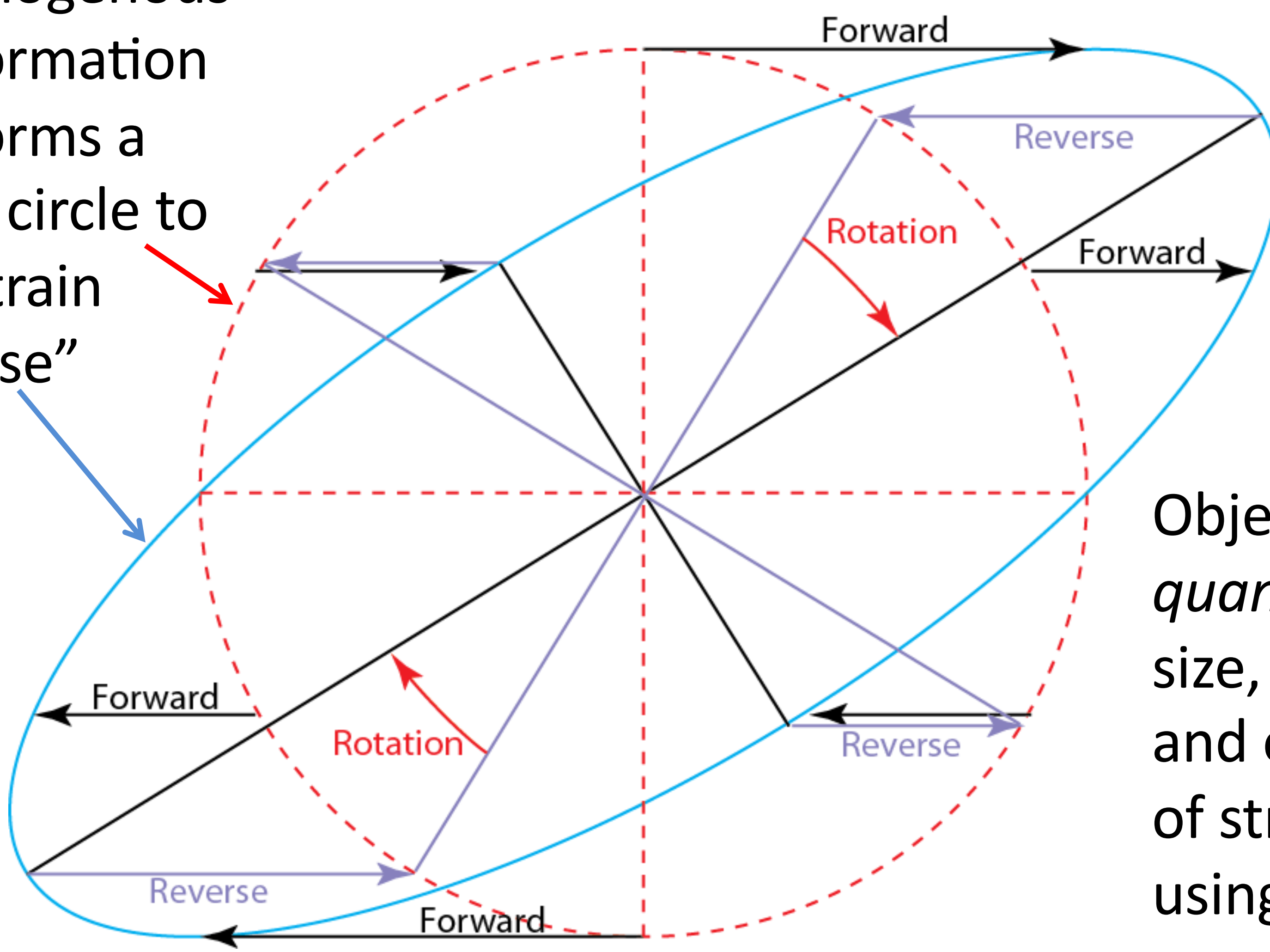
Eigenvectors, Eigenvalues, and Finite Strain

Strained Conglomerate
Sierra Nevada, California

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Homogenous deformation deforms a unit circle to a “strain ellipse”



Objective: To *quantify* the size, shape, and orientation of strain ellipse using its axes

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

I Main Topics

A Equations for ellipses

B Rotations in homogeneous deformation

C Eigenvectors and eigenvalues

D Solutions for general homogeneous deformation matrices

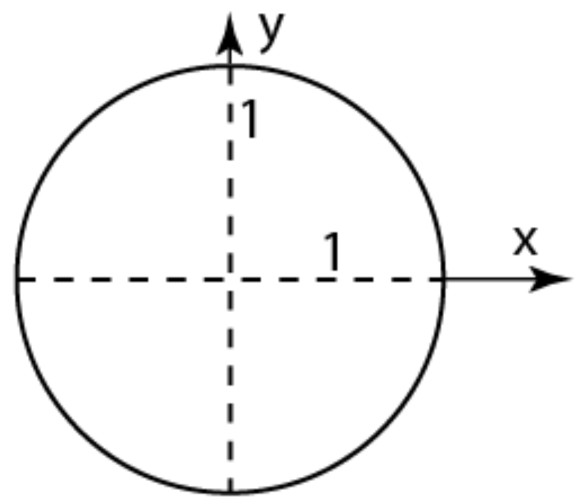
E Key results

F Appendices (1, 2, 3,4)

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

II Equations of ellipses

A Equation of a unit circle centered at the origin



1 $$x^2 + y^2 = 1$$

2 $$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1x + 0y \\ 0x + 1y \end{bmatrix} = 1$$

3 $$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

Symmetric

4 $$[X]^T [F][X] = 1$$

Here, [F] is the identity matrix [I]. So position vectors that define a unit circle transform to those same position vectors because [X'] = [F][X].

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

II Equations of ellipses

B Equation of an ellipse centered at the origin with its axes along the x- and y- axes

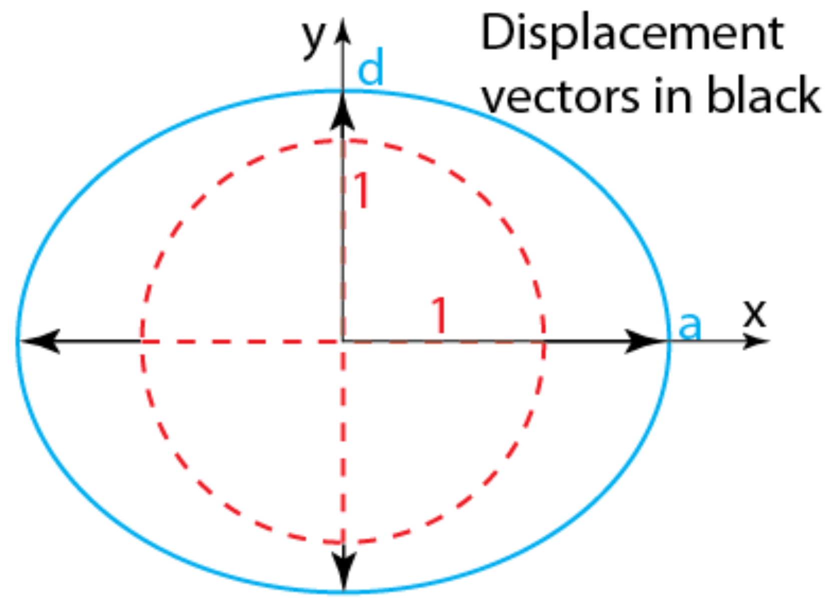
1 $$ax^2 + 0xy + dy^2 = 1$$

2 $$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax + 0y \\ 0x + dy \end{bmatrix} = 1$$

3 $$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

Symmetric

4 $$[X]^T [F][X] = 1$$



Position vectors that define a unit circle transform to position vectors that define an ellipse because [X'] = [F][X].

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

II Equations of ellipses

C "Symmetric" equation of an ellipse centered at the origin

$$Example: F = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

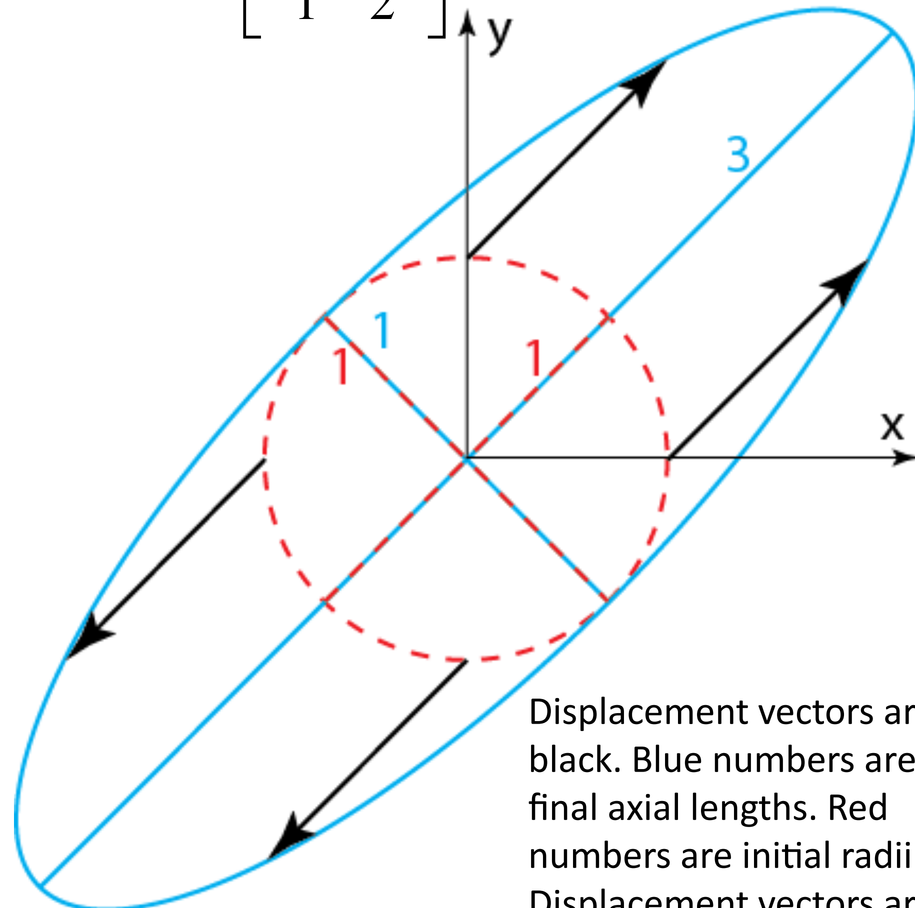
1 $$ax^2 + 2bxy + dy^2 = 1$$

2 $$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax + by \\ bx + dy \end{bmatrix} = 1$$

3 $$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

Symmetric

4 $$[X]^T [F][X] = 1$$



Displacement vectors are in black. Blue numbers are final axial lengths. Red numbers are initial radii. Displacement vectors are symmetric about axes of ellipse.

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II Equations of ellipses

D General equation of an ellipse centered at the origin

$$Example: F = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

1 $$ax^2 + (b+c)xy + dy^2 = 1$$

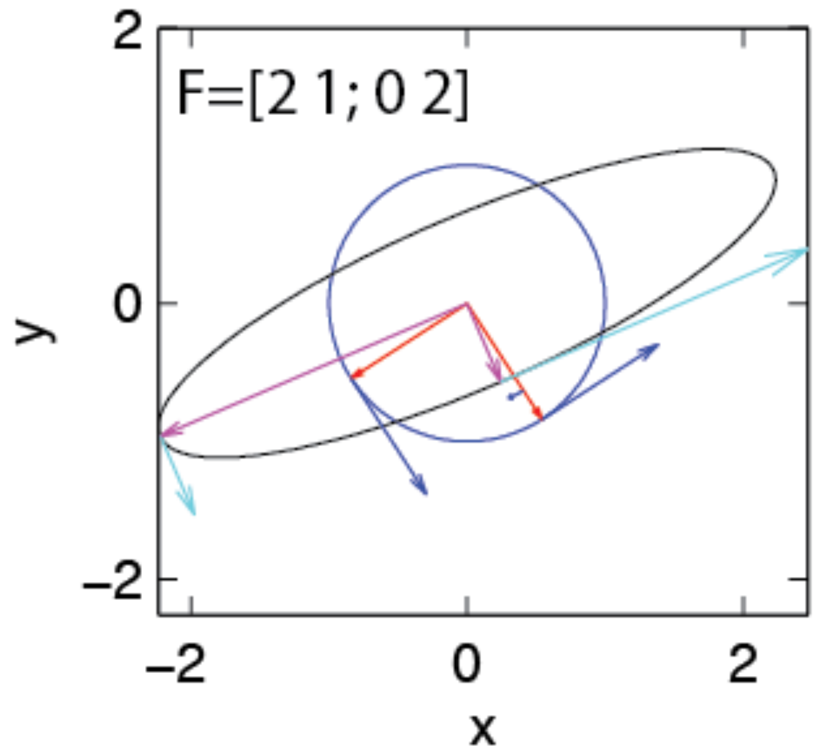
2 $$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = 1$$

3 $$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

Not symmetric if $b \neq c$ →

4 $$[X]^T [F][X] = 1$$

Unit circle and Strain ellipse
Curved arrow shows rotation angle



Vectors along axes of ellipse transform back to perpendicular vectors along axes of unit circle

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III Rotations in homogenous deformation

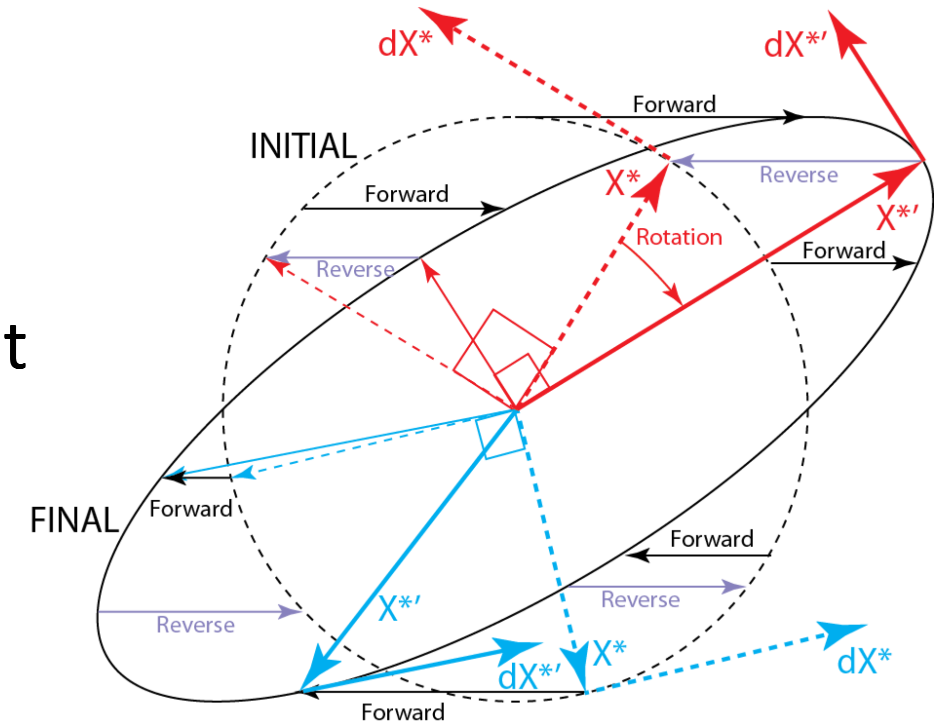
A Let [X] be the set of all position vectors that define a unit circle

B Let [X'] be the set of all position vectors that define an ellipse described by a homogenous deformation at a point

C [X'] = [F][X] (Forward def.)

D $[X] = [F^{-1}][X']$ (Reverse def.)

E The matrices [F] and $[F^{-1}]$ contain <u>constants</u>



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

III Rotations in homogenous deformation (cont.)

F The differential tangent vectors [dX'] and [dX] come from differentiating [X'] = [F][X] and [X] = $[F^{-1}]$[X'] , respectively.

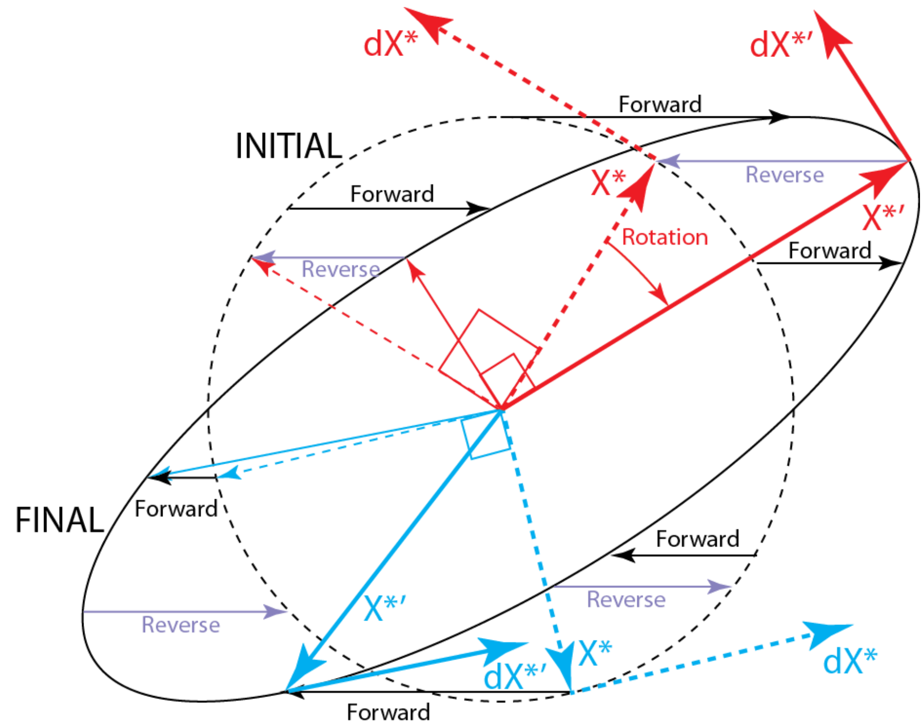
G [dX'] = [F][dX] (Forward def.)

H [dX] = $[F^{-1}]$[dX'] (Reverse def.)

I [F] transforms [X] to [X'], and [dX] to [dX']

J $[F^{-1}]$ transforms [X'] to [X], and [dX'] to [dX]

K Position vectors are paired to corresponding tangents



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

III Rotations in homogenous deformation (cont.)

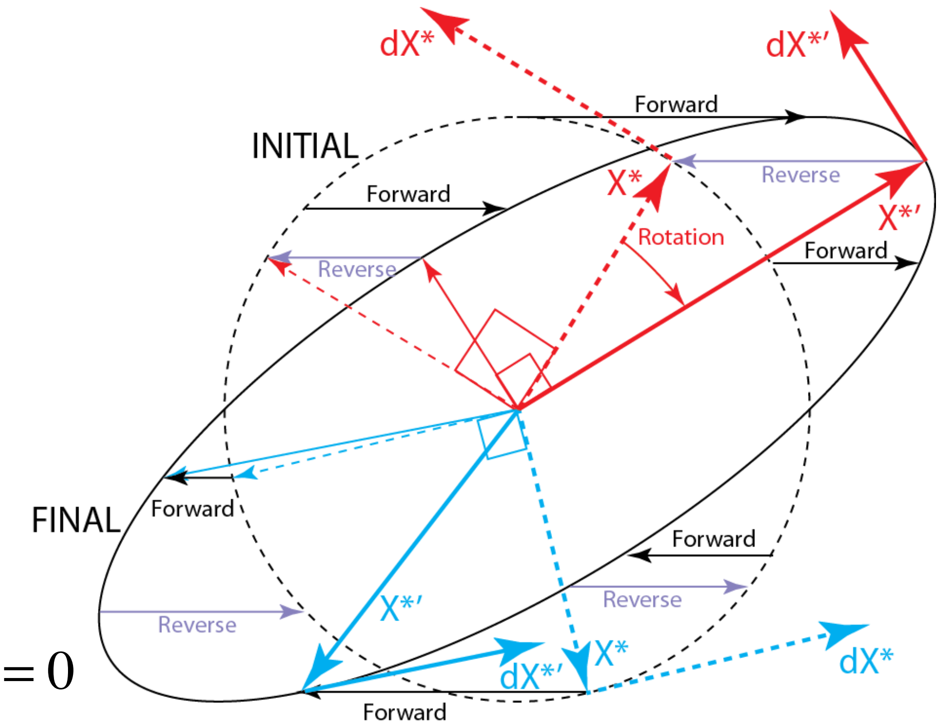
L Where a non-zero position vector and its tangent are perpendicular, the position vector achieves its greatest and smallest (squared) lengths, as shown below

M $Q' = \vec{X}' \bullet \vec{X}' = [X']^T [X']$

N Maxima and minima of (squared) lengths occur where dQ' = 0

O $dQ' = d\left(\vec{X}' \bullet \vec{X}'\right) = \vec{X}' \bullet d\vec{X}' + d\vec{X}' \bullet \vec{X}' = 0$

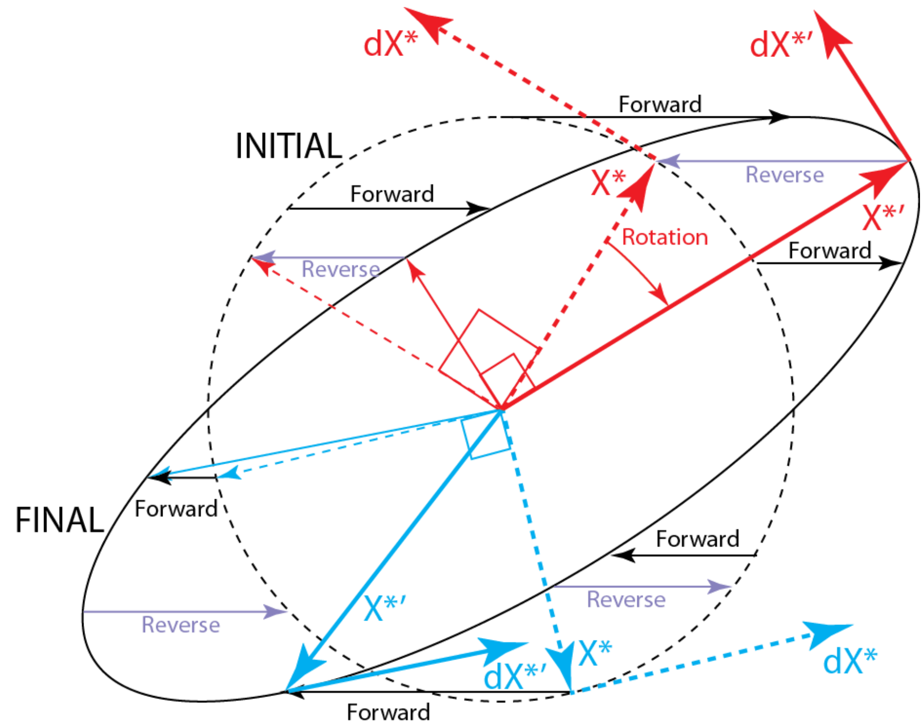
P $2\left(\vec{X}' \bullet d\vec{X}'\right) = 0 \Rightarrow \boxed{\left(\vec{X}' \bullet d\vec{X}'\right) = 0}$



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

III Rotations in homogenous deformation (cont.)

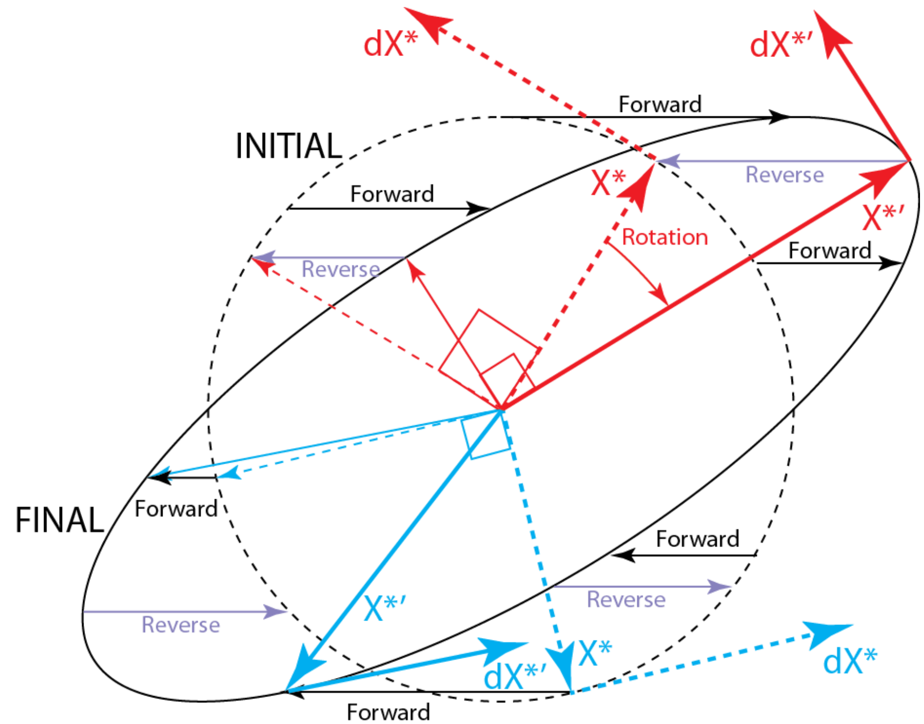
- Q The tangent vector perpendicular to the longest position vector parallels the shortest position vector (which lies along the semi-minor axis), and vice-versa.
- R Similar reasoning applies to the corresponding unit circle.



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

III Rotations in homogenous deformation (cont.)

- For the unit circle, all initial position vectors are radial vectors, and each initial tangent vector is perpendicular to the associated radial position vector. The red initial vector pair [X*, dX*] and the blue initial vector pair [X*, dX*] both show this.



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

III Rotations in homogenous deformation (cont.)

T All the final position-tangent vector pairs for the ellipse have corresponding initial position-tangent vector pairs for the unit circle (and vice-versa).

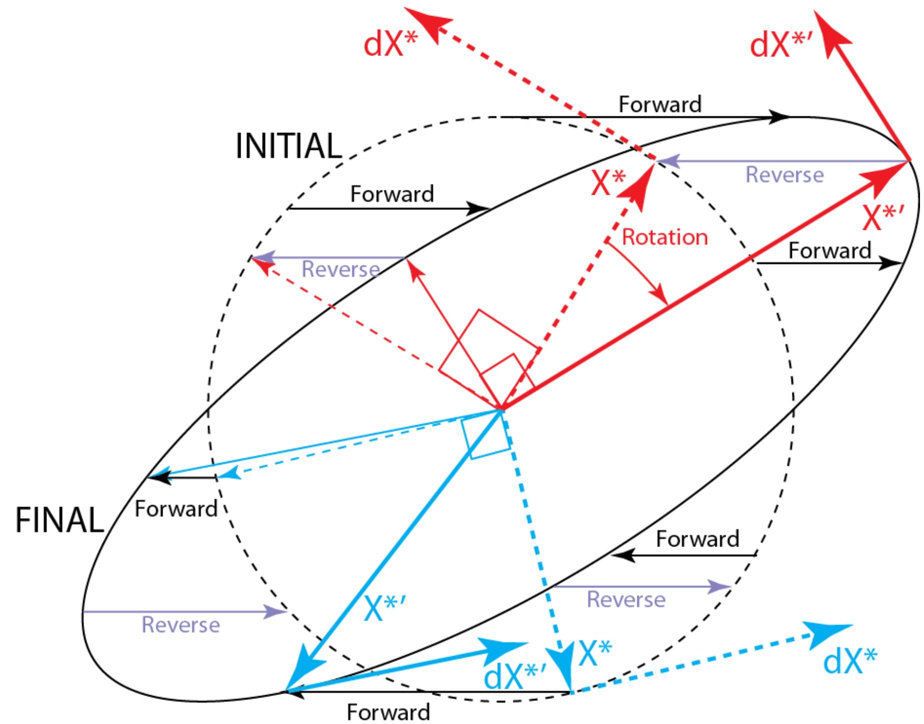
U Every position-tangent vector pair for the unit circle contains perpendicular vectors.

V *Only* the position-tangent vector pair for the ellipse that parallel the major and minor axes (i.e., the red pair [X*', dX*']) are perpendicular.

W "Retro-transforming" [X*', dX*'] by [F^{-1}] yields the initial red pair of perpendicular vectors [X*, dX*].

X Conversely, the forward transformation of the red pair of initial perpendicular vectors [X*, dX*] using [F] yields the final perpendicular vectors pair [X*', dX*'].

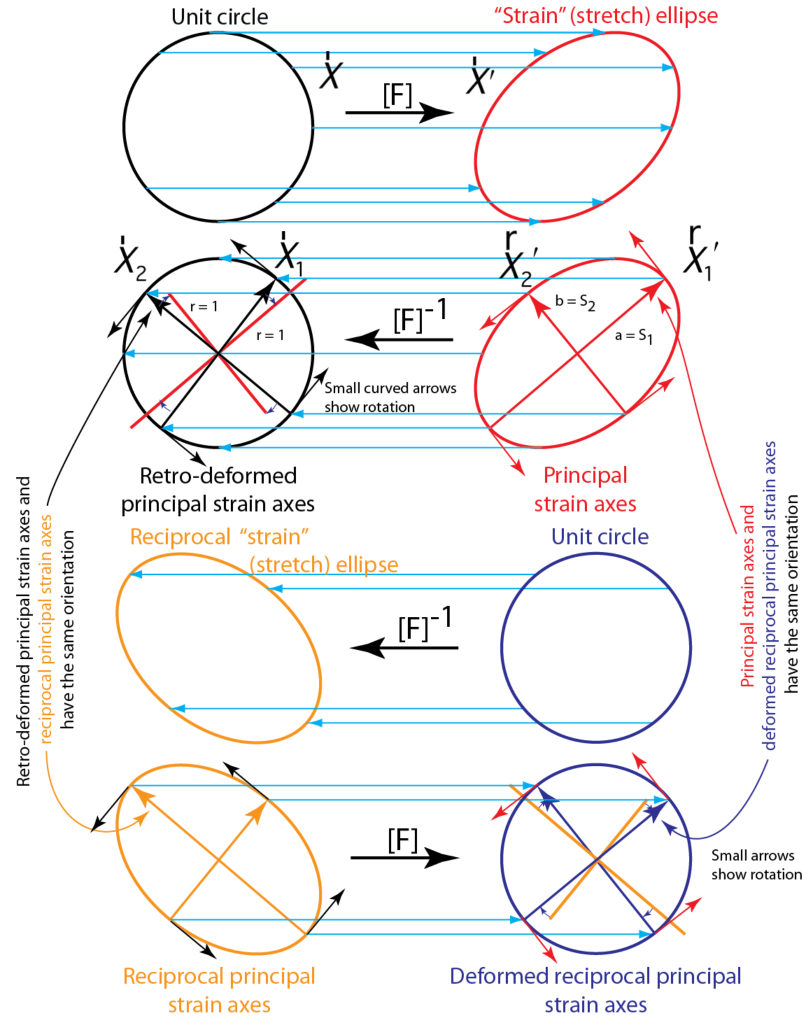
Y The transformation from [X*, dX*] to [X*', dX*'] involves a rotation, and that is how the rotation is defined.



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

III Rotations in homogenous deformation (cont.)

- The longest (X_1') and shortest (X_2') position vectors of the ellipse are perpendicular, along the red axes of the ellipse, and parallel the tangents.
- The corresponding retro-transformed vectors ($[X_1] = [F]^{-1}[X_1']$, and $[X_2] = [F]^{-1}[X_2']$) (along the black axes) are perpendicular unit vectors that maintain the 90° angle between the principal directions.
- The angle of rotation is defined as the angle between the perpendicular pair {$\mathbf{X}_1$ and $\mathbf{X}_2$} along the black axes of the unit circle and the perpendicular principal pair {$\mathbf{X}_1'$, $\mathbf{X}_2'$} along the red axes of the ellipse.
- These results extend to three dimensions if all three sections along the principal axes of the "strain" (stretch) ellipsoid are considered.
- See Appendix 4 for more examples.



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Eigenvectors and eigenvalues
(used to obtain stretches and rotations)

A The eigenvalue matrix equation $[A][X] = \lambda[X]$

1 [A] is a (known) square matrix (nxn)

2 [X] is a non-zero directional <u>eigenvector</u> (nx1)

3 λ is a number, an <u>eigenvalue</u>

4 $\lambda[X]$ is a vector (nx1) parallel to [X]

5 [A][X] is a vector (nx1) parallel to [X]

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

A The eigenvalue matrix equation [A][X] = λ[X] (cont.)

6 The vectors [[A][X]], λ[X], and [X] share the same <u>direction</u> if [X] is an eigenvector

7 If [X] is a unit vector, λ is the <u>length</u> of [A][X]

8 Eigenvectors [X_i] have corresponding eigenvalues [λ_i], and vice-versa

9 In Matlab, [vec,val] = eig(A), finds eigenvectors (vec) and eigenvalues (val)

IV Eigenvectors and eigenvalues

B Example: Mathematical meaning of [A][X]=λ[X]

Two eigenvalues

Two eigenvectors

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A\begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \underline{1}\begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$A\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} \\ 3\sqrt{2} \end{bmatrix} = \underline{3}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

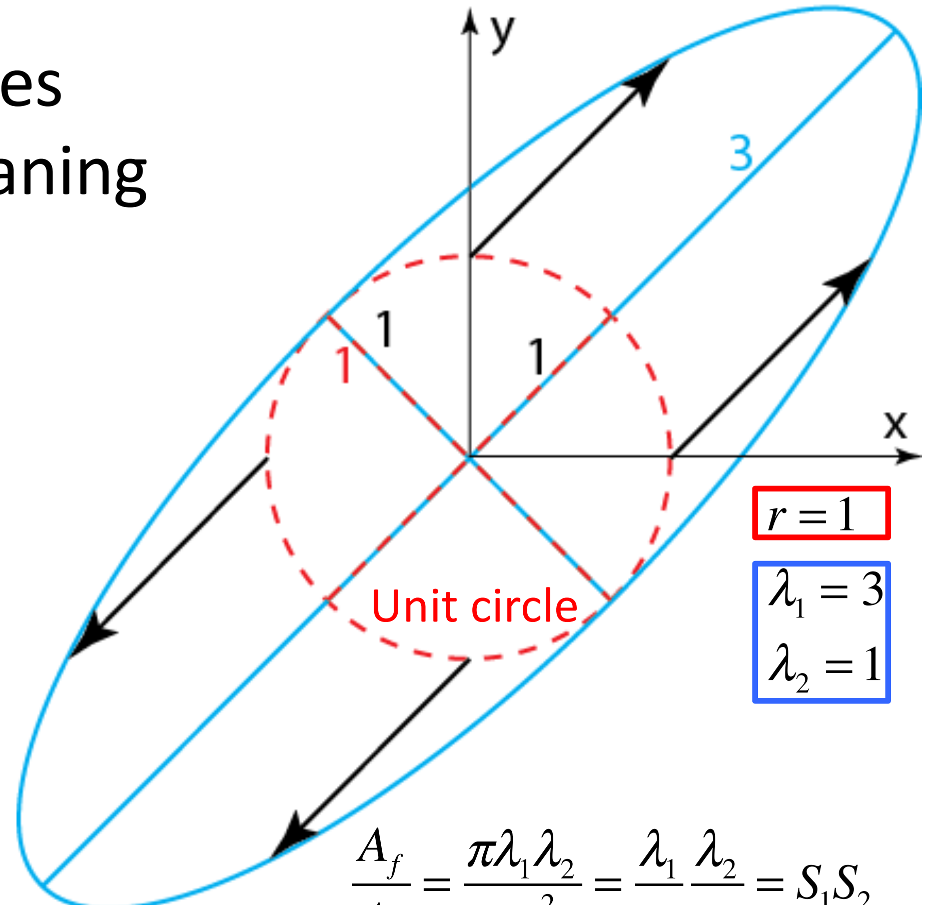
IV Eigenvectors and eigenvalues

C Example: Geometric meaning of [A][X]=λ[X]

$$X' = FX$$

$$F = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Eigenvectors of *symmetric* F give directions of the principal stretches
- Eigenvalues of *symmetric* F (i.e., λ_1, λ_2) are magnitudes of the principal stretches S_1 and S_2



$$\frac{A_f}{A_0} = \frac{\pi\lambda_1\lambda_2}{\pi r^2} = \frac{\lambda_1}{r}\frac{\lambda_2}{r} = S_1S_2$$

$$\Delta = \frac{A_f - A_0}{A_0} = \frac{A_f}{A_0} - \frac{A_0}{A_0} = S_1S_2 - 1$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Eigenvectors and eigenvalues

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

D Example: Matlab solution of [A][X]=λ[X]

```
>> A = [2 1; 1 2]
A =
    2    1
    1    2
>> [vec,val] = eig(A)
vec =
  -0.7071   0.7071
   0.7071   0.7071
val =
    1    0
    0    3
>> theta1 = atan2(vec(2,2),vec(2,1))*180/pi
theta1 =
   45
>> theta2 = atan2(vec(1,2),vec(1,1))*180/pi
theta2 =
   135
```

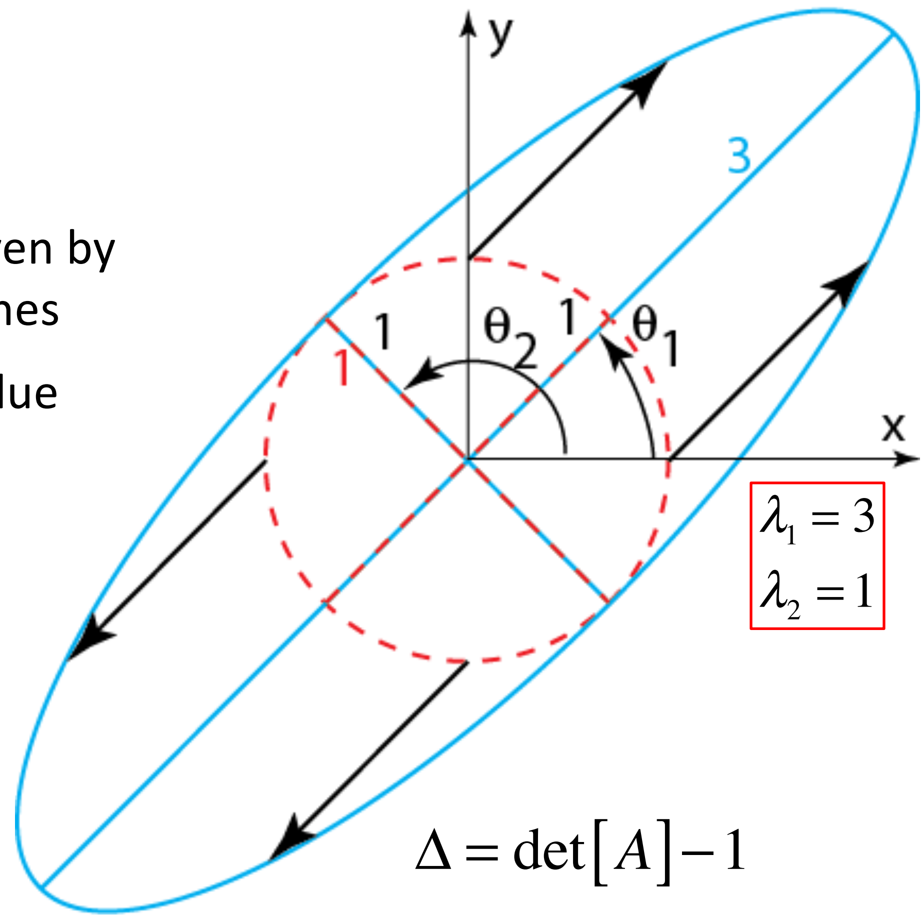
Eigenvectors [X] given by their direction cosines

Eigenvector/eigenvalue pairs

Eigenvalues (λ)

Angle between x-axis and largest eigenvector

Angle between x-axis And smallest eigenvector



$\lambda_1 = 3$

$\lambda_2 = 1$

$$\Delta = \det[A] - 1$$

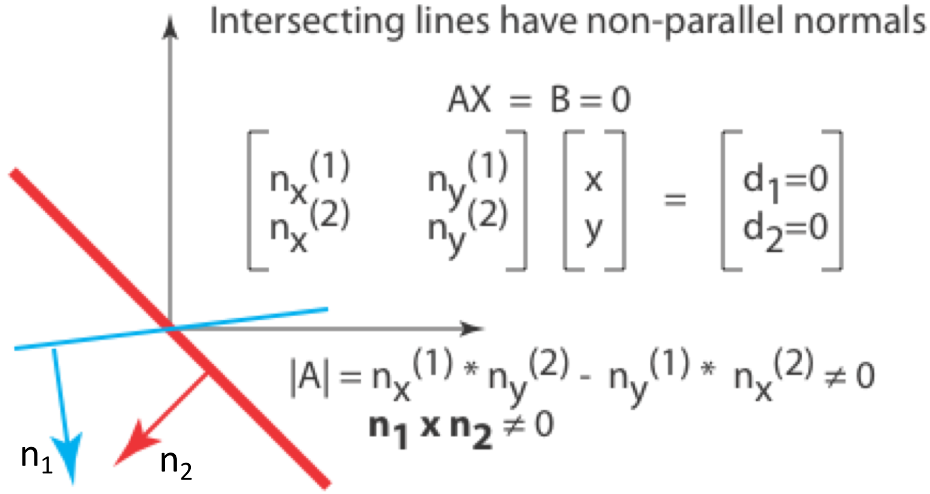
$$Here, \Delta = 3 - 1 = 2$$

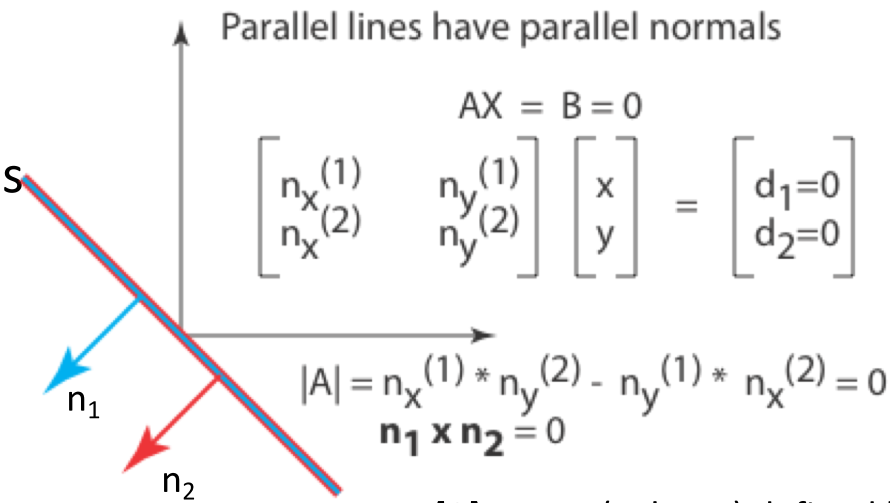
* Matlab in 2016 does not order eigenvalues from largest to smallest

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Eigenvectors and eigenvalues (cont.)

- E Geometric meanings of the real matrix equation [A][X] = [B] = 0
  - 1 $|A| \neq 0$;
    - a $[A]^{-1}$ exists
    - b Describes two lines (or 3 planes) that intersect at the origin
    - c X has a unique solution
  - 2 $|A| = 0$;
    - a $[A]^{-1}$ does not exist
    - b Describes two co-linear lines that that pass through the origin (or three planes that intersect in a line or in a plane through the origin)
    - c [X] has no unique solution; can have multiple solutions





Det[A] = area (volume) defined by parallelogram (parallelepiped) based on unit normals

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Eigenvectors and eigenvalues (cont.)

D Alternative form of an eigenvalue equation

1 $[A][X]=\lambda[X]$

Subtracting $I\lambda[X] = \lambda[IX] = \lambda[X]$ from both sides yields:

2 $[A-I\lambda][X]=0$ (same form as $[\mathcal{A}][X]=0$)

E Solution conditions and connections with determinants

1 Unique trivial solution of $[X] = 0$ if and only if $|A-I\lambda| \neq 0$

2 Multiple eigenvector solutions ($[X] \neq 0$) if and only if $|A-I\lambda|=0$

* See previous slide

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Eigenvectors and eigenvalues (cont.)

F Characteristic equation: $|A-I\lambda|=0$

1 The roots of the characteristic equation are the eigenvalues (λ)

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Eigenvectors and eigenvalues (cont.)

F Characteristic equation: |A-Iλ|=0 (cont.)

2 Eigenvalues of a general 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

a $$|A - I\lambda| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

b $$(a-\lambda)(d-\lambda) - bc = 0$$

c $$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

d $$\lambda_1, \lambda_2 = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

(a+d) = tr(A)
(ad-bc) = |A|

$$\lambda_1 + \lambda_2 = tr(A)$$
$$\lambda_1 \lambda_2 = |A|$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Eigenvectors and eigenvalues (cont.)

- G To solve for eigenvectors, substitute eigenvalues back into AX= lX and solve for X (see Appendix 1)
- H Eigenvectors of real symmetric matrices are perpendicular (for distinct eigenvalues); see Appendix 3

* All these points are important

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Solutions for general homogeneous deformation matrices

A Eigenvalues

1 Start with the definition of *quadratic* elongation Q, which is a scalar

$$\frac{L_f^{\ 2}}{L_0^{\ 2}} = Q$$

2 Express using dot products

$$\frac{\vec{X}' \bullet \vec{X}'}{\vec{X} \bullet \vec{X}} = Q$$

3 Clear the denominator. Dot products and Q are scalars.

$$\vec{X}' \bullet \vec{X}' = \left(\vec{X} \bullet \vec{X}\right) Q$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Solutions for general homogeneous deformation matrices

A Eigenvalues

4 Replace X' with [FX]

5 Re-arrange both sides

6 Both sides of this equation lead off with $[X]^T$, which cannot be a zero vector, so it can be dropped from both sides to yield an eigenvector equation

7 $[F^TF]$ is symmetric: $[F^TF]^T=[F^TF]$

8 The eigenvalues of $[F^TF]$ are the principal quadratic elongations $Q = (L_f/L_0)^2$

9 The eigenvalues of $[F^TF]^{1/2}$ are the principal stretches $S = (L_f/L_0)$

$$\vec{X}' \bullet \vec{X}' = \left(\vec{X} \bullet \vec{X}\right) Q$$

$$\left[\underset{nxn}{[F]}\underset{nx1}{[X]}\right]^T \left[\underset{nxn}{[F]}\underset{nx1}{[X]}\right] = \left[\underset{nx1}{X}\right]^T \underset{nx1}{[X]}\underset{1x1}{Q}$$

$$\left[\underset{nx1}{X}\right]^T \left[\underset{nxn}{F}\right]^T \left[\underset{nxn}{F}\ \underset{nx1}{X}\right] = \left[\underset{nx1}{X}\right]^T \underset{1x1}{Q} \left[\underset{nx1}{X}\right]$$

$$\left[\underset{nxn}{F^T}\ \underset{nxn}{F}\right]\left[\underset{nx1}{X}\right] = Q\left[\underset{nx1}{X}\right]$$

$$"[A][X] = \lambda[X]"$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Solutions for general homogeneous deformation matrices

B Special Case: [F] is symmetric

1 $[F^TF] = [F^2]$ because $F = F^T$

2 The principal stretches (S) again are the square roots of the principal quadratic elongations (Q) (i.e., the square roots of the eigenvalues of $[F^2]$)

3 The principal stretches (S) also are the eigenvalues of [F], directly

4 The directions of the principal stretches (S) are the eigenvectors of [F], and of $[F^TF] = [F^2]$!

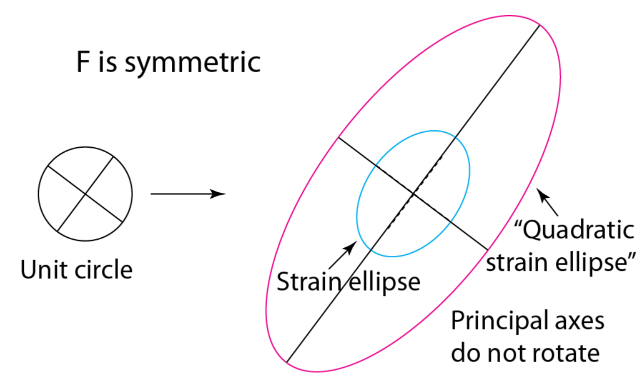
5 The axes of the principal (greatest and least) strain do not rotate

$$\left[F^T F\right][X] = Q[X]$$

$$\left[F^2\right][X] = Q[X]$$

$$Q = \frac{L_f^{\ 2}}{L_0^{\ 2}};\ S = \frac{L_f}{L_0} \Rightarrow \sqrt{Q} = S$$

$$[F][X] = S[X]$$

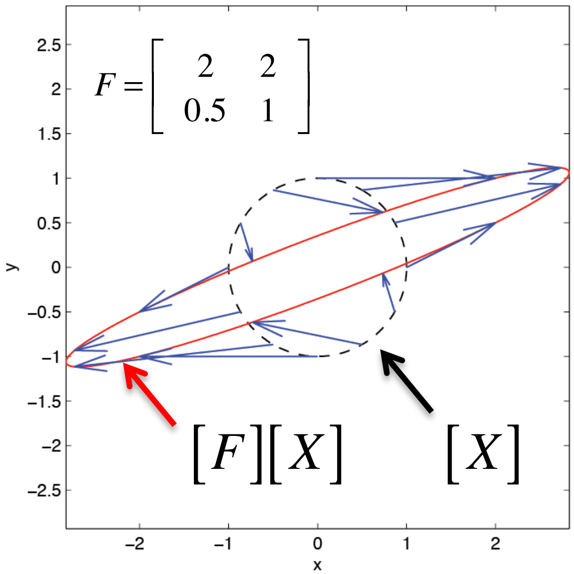


9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Example 1

[F] = [R][U]

By the polar decomposition theorem, F can be formed by a stretch and a rotation



$$[X'] = [F][X];\ [F] = [R][U]$$

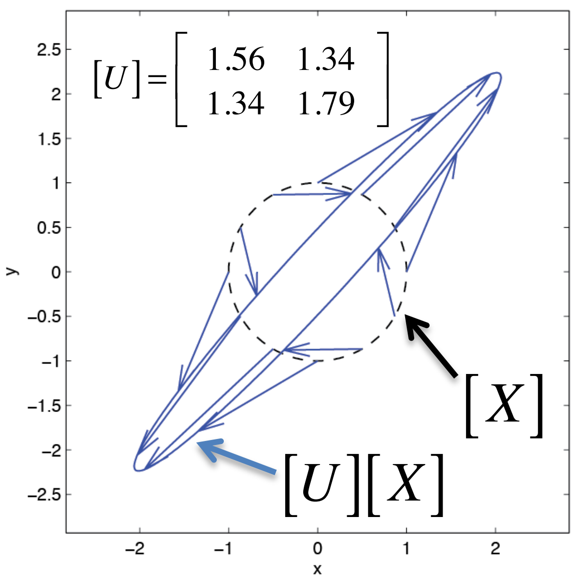
$$[F] = \begin{bmatrix} 2 & 2 \\ 0.5 & 1 \end{bmatrix};\ [F]^T = \begin{bmatrix} 2 & 0.5 \\ 2 & 1 \end{bmatrix}$$

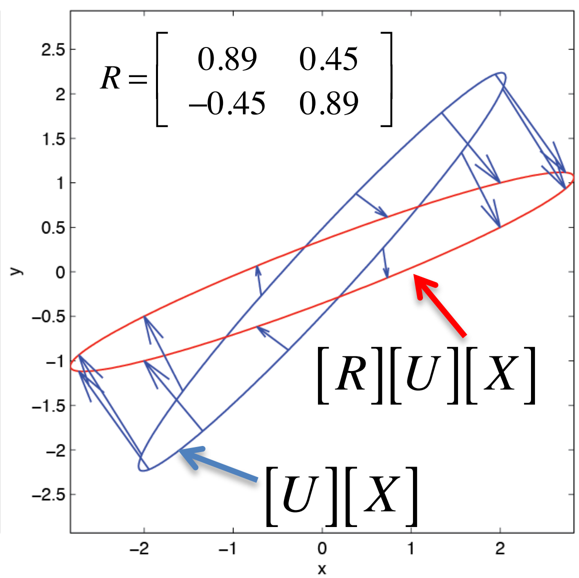
$$[U] = \left[[F]^T[F]\right]^{1/2} = \begin{bmatrix} 4.25 & 4.5 \\ 4.5 & 5 \end{bmatrix}^{1/2} = \begin{bmatrix} 1.56 & 1.34 \\ 1.34 & 1.79 \end{bmatrix}$$

$$[R] = [F][U]^{-1} = \begin{bmatrix} 2 & 2 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} 1.79 & -1.34 \\ -1.34 & 1.56 \end{bmatrix} = \begin{bmatrix} 0.89 & 0.45 \\ -0.45 & 0.89 \end{bmatrix}$$

Eigenvalues of [U] give principal stretch magnitudes

First, symmetrically stretch the unit circle using [U]





Eigenvectors of [U] are along axes of blue ellipses. <u>Rotated</u> eigenvectors of [U] give principal stretch directions

Second, rotate the ellipse (*<u>not</u>* the reference frame) using [R]

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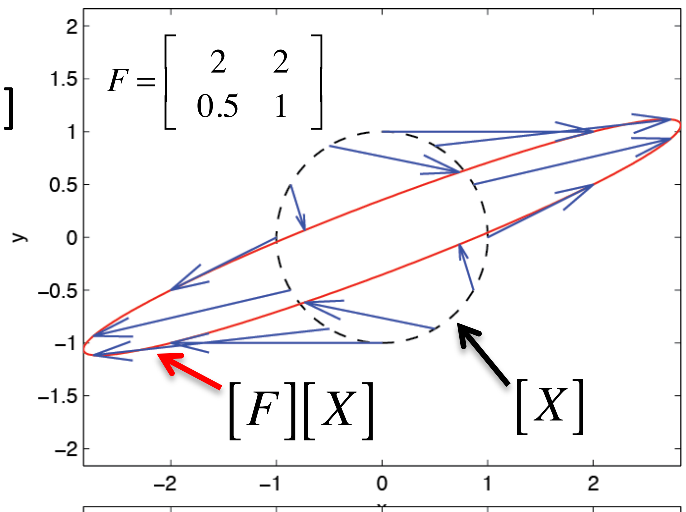
Example 2

[F] = [V] [R]

F also can be formed by a rotation and a stretch



$[X'] = [F][X]; \; [F] = [V][R]$

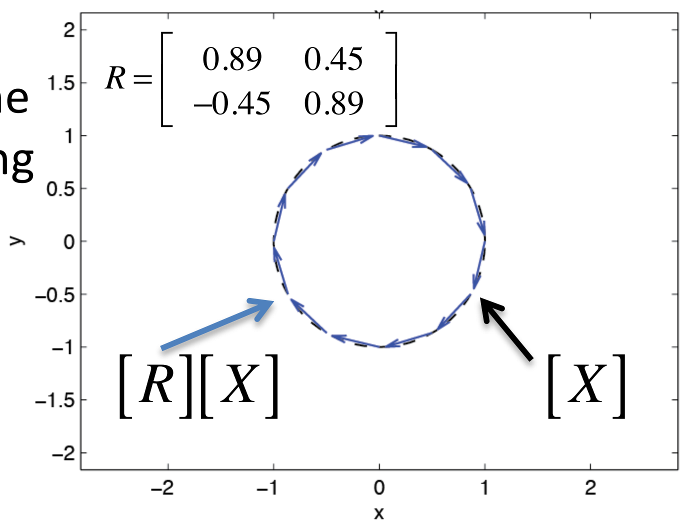
$[F] = \begin{bmatrix} 2 & 2 \\ 0.5 & 1 \end{bmatrix}; \; [F]^T = \begin{bmatrix} 2 & 0.5 \\ 2 & 1 \end{bmatrix}$

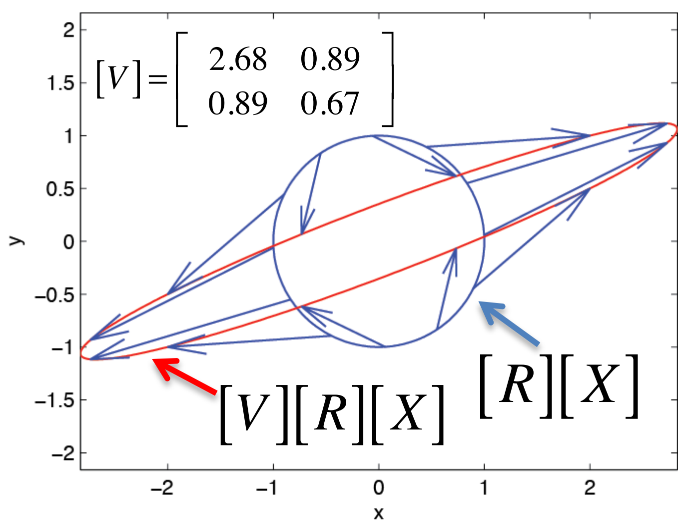
$[V] = \left[[F][F]^T\right]^{1/2} = \begin{bmatrix} 8 & 3 \\ 3 & 1.5 \end{bmatrix}^{1/2} = \begin{bmatrix} 2.68 & 0.89 \\ 0.89 & 0.67 \end{bmatrix}$

$[R] = [V]^{-1}[F] = \begin{bmatrix} 0.67 & -0.89 \\ -0.89 & 2.68 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 0.89 & 0.45 \\ -0.45 & 0.89 \end{bmatrix}$

Eigenvalues of [V] also give principal stretch magnitudes

First, rotate the unit circle using [R]

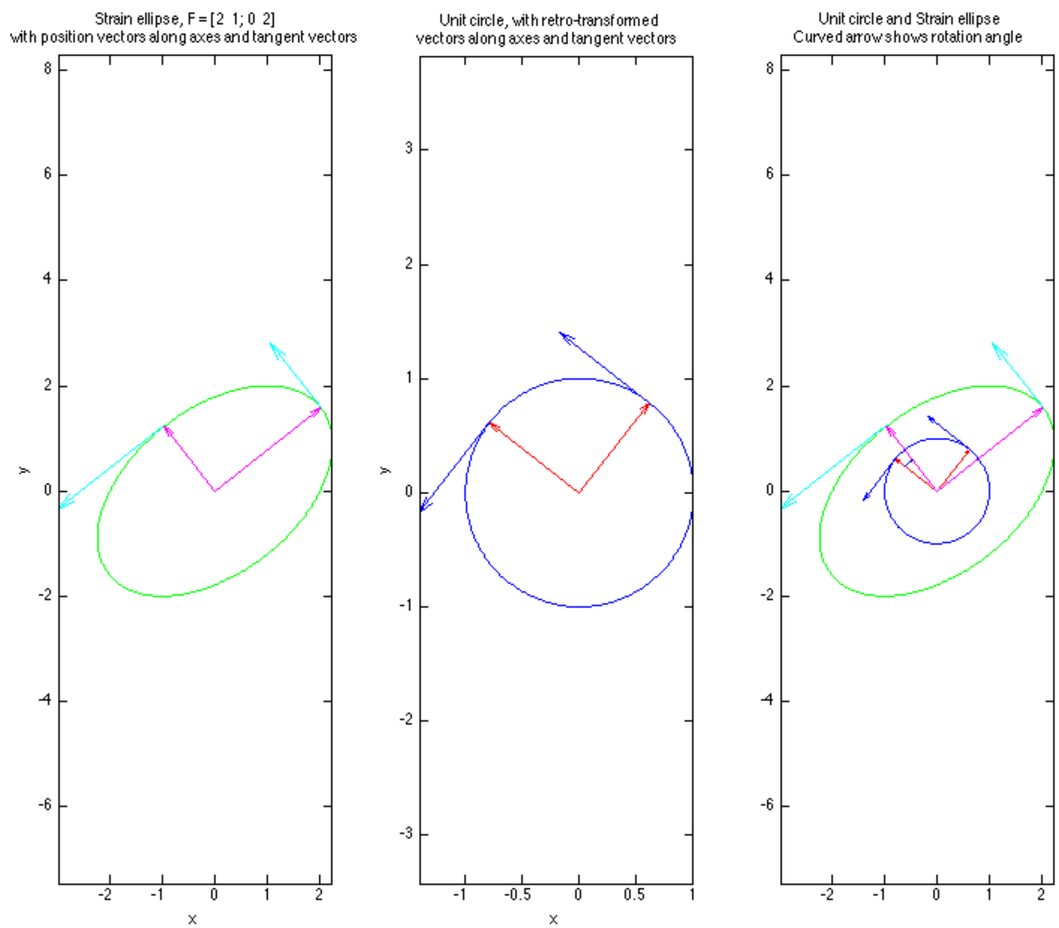




Unrotated eigenvectors of [V] give principal stretch directions directly

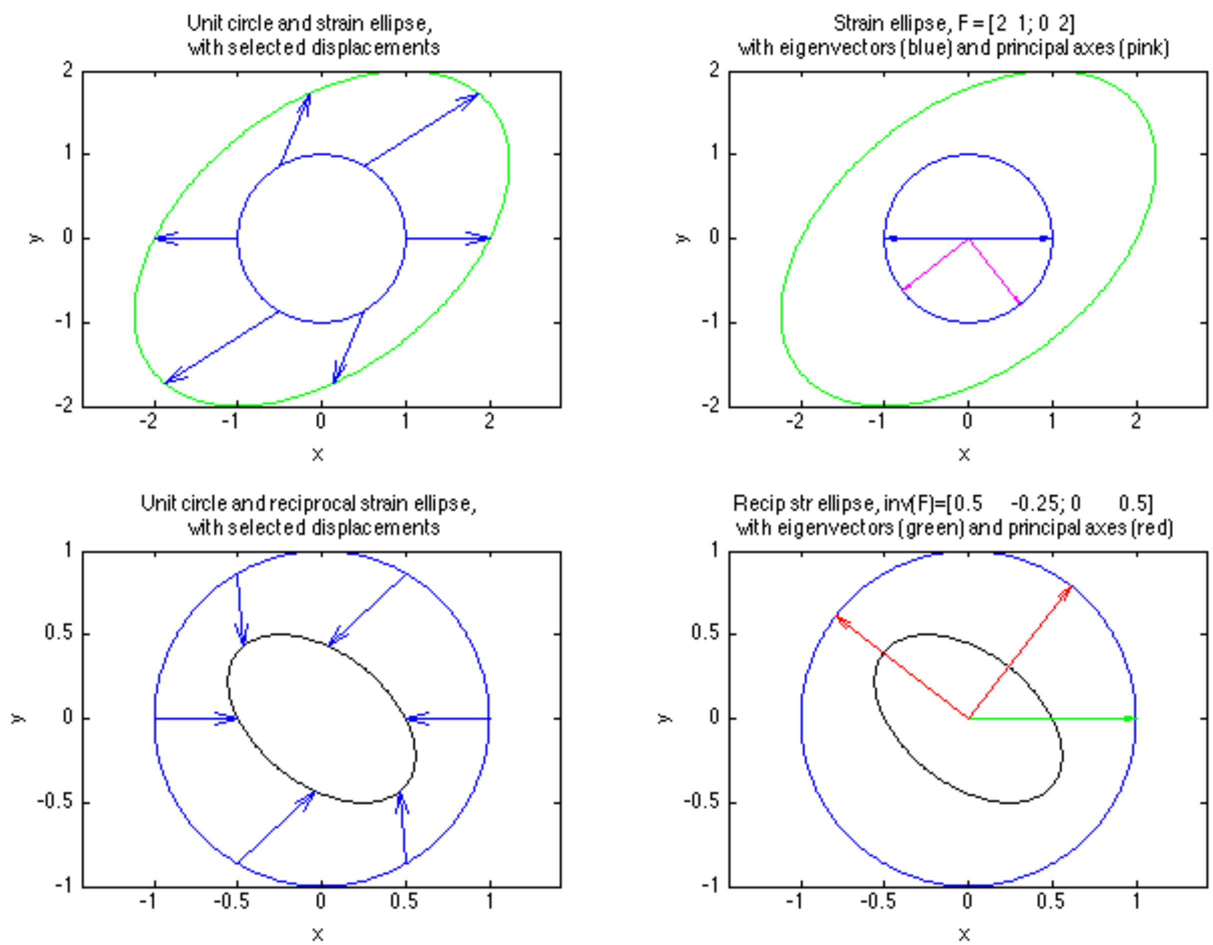
Second, stretch the rotated unit circle symmetrically using [V]

Example



Strain ellipse, F = [2 1; 0 2]
with position vectors along axes and tangent vectors
Unit circle, with retro-transformed
vectors along axes and tangent vectors
Unit circle and Strain ellipse
Curved arrow shows rotation angle

Example



Unit circle and strain ellipse,
with selected displacements
Strain ellipse, F = [2 1; 0 2]
with eigenvectors (blue) and principal axes (pink)
Unit circle and reciprocal strain ellipse,
with selected displacements
Recip str ellipse, inv(F)=[0.5 -0.25; 0 0.5]
with eigenvectors (green) and principal axes (red)
x
y

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Key results

- A For symmetric F matrices ($F = F^T$)
  - 1 Eigenvectors of F give directions of principal stretches
  - 2 Eigenvectors of F are perpendicular
  - 3 Eigenvalues of F give magnitudes of principal stretches
  - 4 Eigenvectors do not rotate
- B For non-symmetric F matrices ($F \neq F^T$)
  - 1 The directions of the principal stretches are given by rotated eigenvectors of $[F^TF]$
  - 2 Eigenvectors of $[F^TF]$ are perpendicular; eigenvectors of F are not
  - 3 Eigenvalues of $[F^TF]$ give magnitudes of principal quadratic elongations
  - 4 F can be decomposed into a symmetric stretch and rotation (or vice-versa)
    - a The stretch matrix $U =[F^TF]^{1/2}$
    - b The stretch matrix $V =[FF^T]^{1/2}$
  - 5 The rotation matrix $R = F[F^TF]^{1/2} = [FF^T]^{1/2} F$
- C Need to know initial locations and final locations, or F, to calculate strains
- D The F-matrix does not uniquely determine the displacement history: e.g., F=RU=VR

Appendix 1

Examples of long-hand solutions for eigenvalues and eigenvectors

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Characteristic equation: |A-Iλ|=0 $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Eigenvalues for symmetric [A]

tr(A) = (a+d) = 4

|A|=(ad-bc) = 3

a $|A-I\lambda|=\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}=\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}=0$

b $(a-\lambda)(d-\lambda)-bc=(2-\lambda)(2-\lambda)-(1)(1)=0$

c $\lambda^2-(a+d)\lambda+(ad-bc)=0$

d $\lambda_1,\lambda_2=\dfrac{(a+d)\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$

$$=\frac{(2+2)\pm\sqrt{(2+2)^2-4(2\times 2-1\times 1)}}{2}=2\pm 1$$

e $\lambda_1=3, \lambda_2=1$

$$tr(A)=\lambda_1+\lambda_2=4$$
$$|A|=\lambda_1\lambda_2=3$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Eigenvalue equation: AX=λX

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Eigenvectors for symmetric [A]

Direction cosines of first eigenvector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 + \beta_1 \\ \alpha_1 + 2\beta_1 \end{bmatrix} = 3\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \Rightarrow \beta_1 = \alpha_1$$

λ_1

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 2\alpha_2 + \beta_2 \\ \alpha_2 + 2\beta_2 \end{bmatrix} = 1\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \Rightarrow \beta_2 = -\alpha_2$$

λ_2

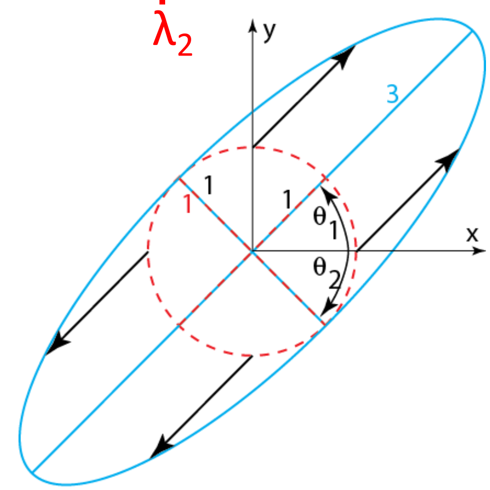
Direction cosines of first eigenvector

$$\theta_1 = \tan^{-1}\frac{\beta_1}{\alpha_1} = \tan^{-1}\frac{\alpha_1}{\alpha_1} = \tan^{-1}\frac{1}{1} = 45^\circ$$

Angle for eigenvector 1

$$\theta_2 = \tan^{-1}\frac{\beta_2}{\alpha_2} = \tan^{-1}\frac{-\alpha_2}{\alpha_2} = \tan^{-1}\frac{-1}{1} = -45^\circ$$

Angle for eigenvector 2



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Characteristic equation: |A-Iλ|=0 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

Eigenvalues for non-symmetric [A]

tr(A) = (a+d) = 4

|A|=(ad-bc) = 4

a $|A - I\lambda| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0$

b $(a-\lambda)(d-\lambda) - bc = (2-\lambda)(2-\lambda) - (0)(1) = 0$

c $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$

d $\lambda_1, \lambda_2 = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

$$= \frac{(2+2) \pm \sqrt{(2+2)^2 - 4(2\times 2 - 0\times 1)}}{2} = 2 \pm 0$$

e $\lambda_1 = 2, \lambda_2 = 0$

$tr(A) = \lambda_1 + \lambda_2 = 4$

$|A| = \lambda_1 \lambda_2 = 4$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Eigenvalue equation: AX=λX

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

Eigenvectors for non-symmetric [A]

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 \\ \alpha_1 + 2\beta_1 \end{bmatrix} = 2\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \Rightarrow \alpha_1 = 0$$

λ_1

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 2\alpha_2 \\ \alpha_2 + 2\beta_2 \end{bmatrix} = 2\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \Rightarrow \alpha_2 = 0$$

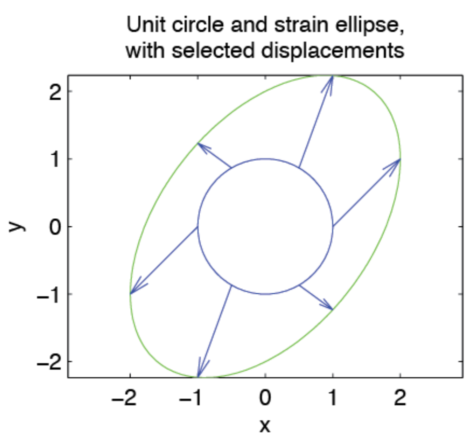
λ_2

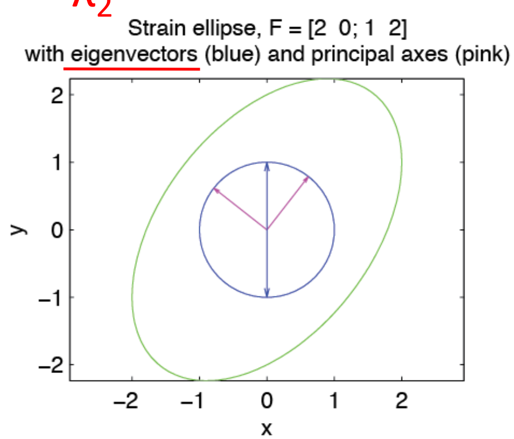
Angle for eigenvector 1

$$\theta_1 = \tan^{-1}\frac{\beta_1}{\alpha_1} = \tan^{-1}\frac{\beta_1}{0} = \tan^{-1}\infty = \pm 90^\circ$$

$$\theta_2 = \tan^{-1}\frac{\beta_2}{\alpha_2} = \tan^{-1}\frac{\beta_2}{0} = \tan^{-1}\infty = \pm 90^\circ$$

Angle for eigenvector 2





9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Appendix 2

Proof that the vectors $\lambda\mathbf{X}$ are the longest and shortest vectors from the center of an ellipse to its perimeter

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Eigenvectors of a symmetric matrix

A Maximum and minimum squared lengths

Set derivative of squared lengths to zero to find orientation of maxima and minimum distance from origin to ellipse

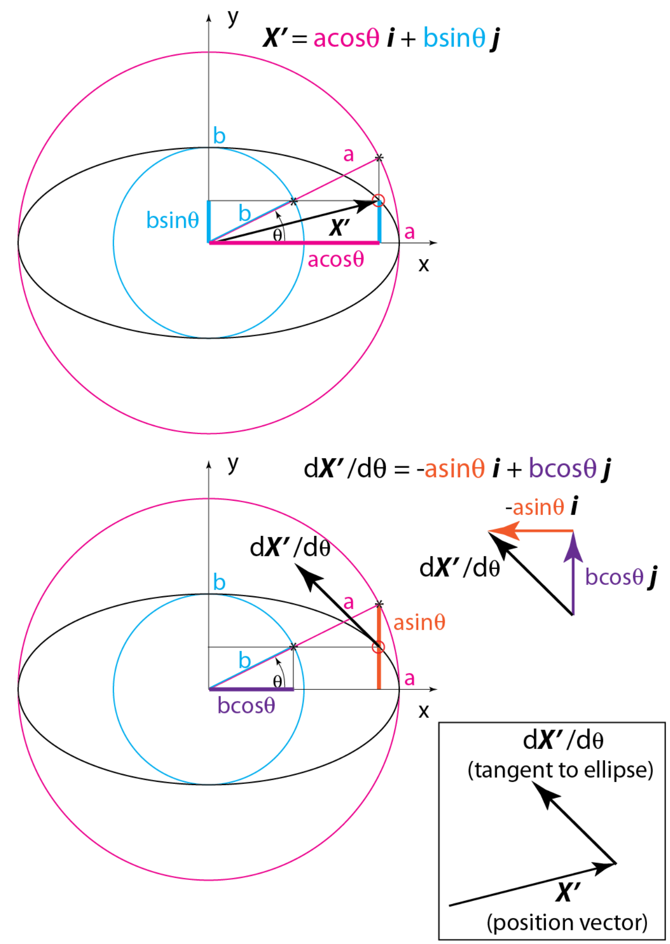
$$\vec{\mathbf{X}}' \bullet \vec{\mathbf{X}}' = L_f^{\,2}$$

$$\frac{d\left(\vec{\mathbf{X}}' \bullet \vec{\mathbf{X}}'\right)}{d\theta} = \vec{\mathbf{X}}' \bullet \frac{d\vec{\mathbf{X}}'}{d\theta} + \frac{d\vec{\mathbf{X}}'}{d\theta} \bullet \vec{\mathbf{X}}' = 0$$

$$2\left(\vec{\mathbf{X}}' \bullet \frac{d\vec{\mathbf{X}}'}{d\theta}\right) = 0$$

$$\left(\vec{\mathbf{X}}' \bullet \frac{d\vec{\mathbf{X}}'}{d\theta}\right) = 0$$

B Position vectors (**X'**) with maximum and minimum (squared) lengths are those that are perpendicular to tangent vectors (d**X'**) along ellipse



$X' \cdot dX'/d\theta = -a^2\sin\theta\cos\theta + b^2\sin\theta\cos\theta = (b^2 - a^2)\sin\theta\cos\theta$

$X' \cdot dX'/d\theta = 0$ if $a=b$, $\theta = 0^\circ$, or $\theta = 90^\circ$

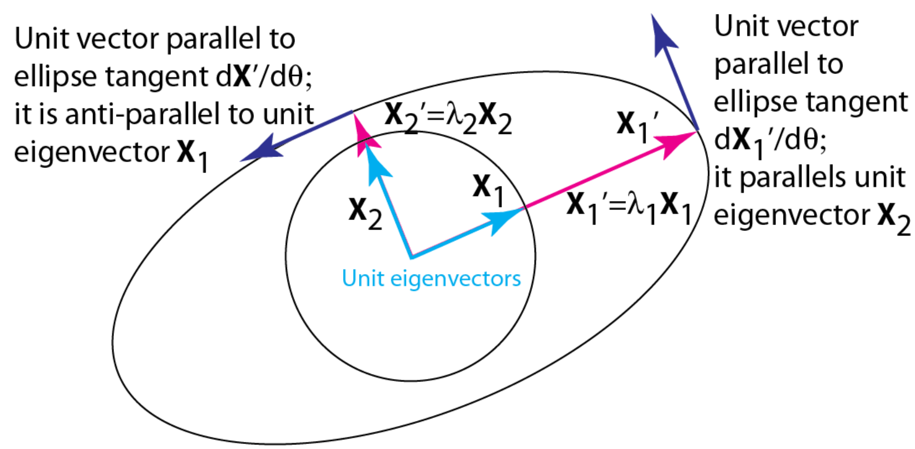
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Eigenvectors of a symmetric matrix

C $A\mathbf{X}=\lambda\mathbf{X}$

D Since eigenvectors **X** of symmetric matrices are mutually perpendicular, so too are the transformed vectors $\lambda\mathbf{X}$

E At the point identified by the transformed vector $\lambda\mathbf{X}$, the perpendicular eigenvector(s) must parallel d**X**' and be tangent to the ellipse



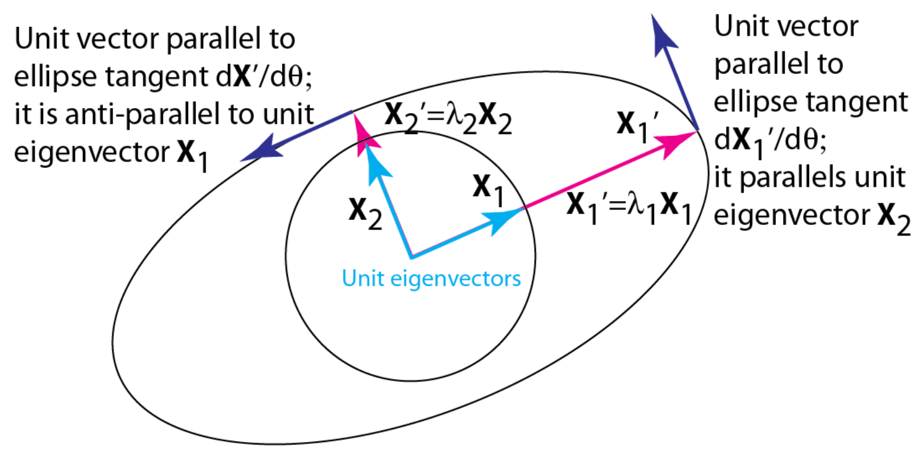
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Eigenvectors of a symmetric matrix

F Recall that position vectors (**X’**) with maximum and minimum (squared) lengths are those that are perpendicular to tangent vectors (d**X’**) along ellipse. Hence, the smallest and largest transformed vectors λ**X** give the minimum and maximum distances to an ellipse from its center.

G The λ values are the principal stretches

H These conclusions extend to three dimensions and ellipsoids



Appendix 3

Proof that distinct eigenvectors of a real symmetric matrix $A=A^T$ are perpendicular

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

1a $AX_1 = \lambda_1 X_1$ 1b $AX_2 = \lambda_2 X_2$

Eigenvectors X_1 and X_2 parallel AX_1 and AX_2, respectively

Dotting AX_1 by X_2 and AX_2 by X_1 can test whether X_1 and X_2 are orthogonal.

2a $\mathbf{X}_2 \bullet A\mathbf{X}_1 = \mathbf{X}_2 \bullet \lambda_1 \mathbf{X}_1 = \lambda_1 (\mathbf{X}_2 \bullet \mathbf{X}_1)$

2b $\mathbf{X}_1 \bullet A\mathbf{X}_2 = \mathbf{X}_1 \bullet \lambda_2 \mathbf{X}_2 = \lambda_2 (\mathbf{X}_1 \bullet \mathbf{X}_2)$

If $A=A^T$, then the left sides of (2a) and (2b) are equal:

3 $\mathbf{X}_2 \bullet A\mathbf{X}_1 = A\mathbf{X}_1 \bullet \mathbf{X}_2 = [A\mathbf{X}_1]^T[\mathbf{X}_2] = [[\mathbf{X}_1]^T[A]^T][\mathbf{X}_2]$

$= [\mathbf{X}_1]^T[A][\mathbf{X}_2] = [\mathbf{X}_1]^T[[A][\mathbf{X}_2]] = \mathbf{X}_1 \bullet A\mathbf{X}_2$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Since the left sides of (2a) and (2b) are equal, the right sides must be equal too. Hence,

4 $\lambda_1 (\mathbf{X}_2 \bullet \mathbf{X}_1) = \lambda_2 (\mathbf{X}_1 \bullet \mathbf{X}_2)$

Now subtract the right side of (4) from the left

5 $(\lambda_1 - \lambda_2)(\mathbf{X}_2 \bullet \mathbf{X}_1) = 0$

- The eigenvalues generally are different, so $\lambda_1 - \lambda_2 \neq 0$.
- For (5) to hold, then $\mathbf{X}_2 \bullet \mathbf{X}_1 = 0$.
- Therefore, the eigenvectors ($\mathbf{X}_1$, $\mathbf{X}_2$) of a real symmetric 2x2 matrix are perpendicular where eigenvalues are distinct
- The eigenvectors can be *chosen* to be perpendicular if the eigenvactors are the same

Appendix 4

Rotations in homogenous deformation: An algebraic perspective

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Rotations in homogeneous deformation

A Just getting the size and shape of the "strain" (stretch) ellipse is not enough if [F] is not symmetric. Need to consider how points on the ellipse transform

B Can do this through a combination of stretches and rotations

1 F=VR (which "R"?)

a V = symmetric stretch matrix

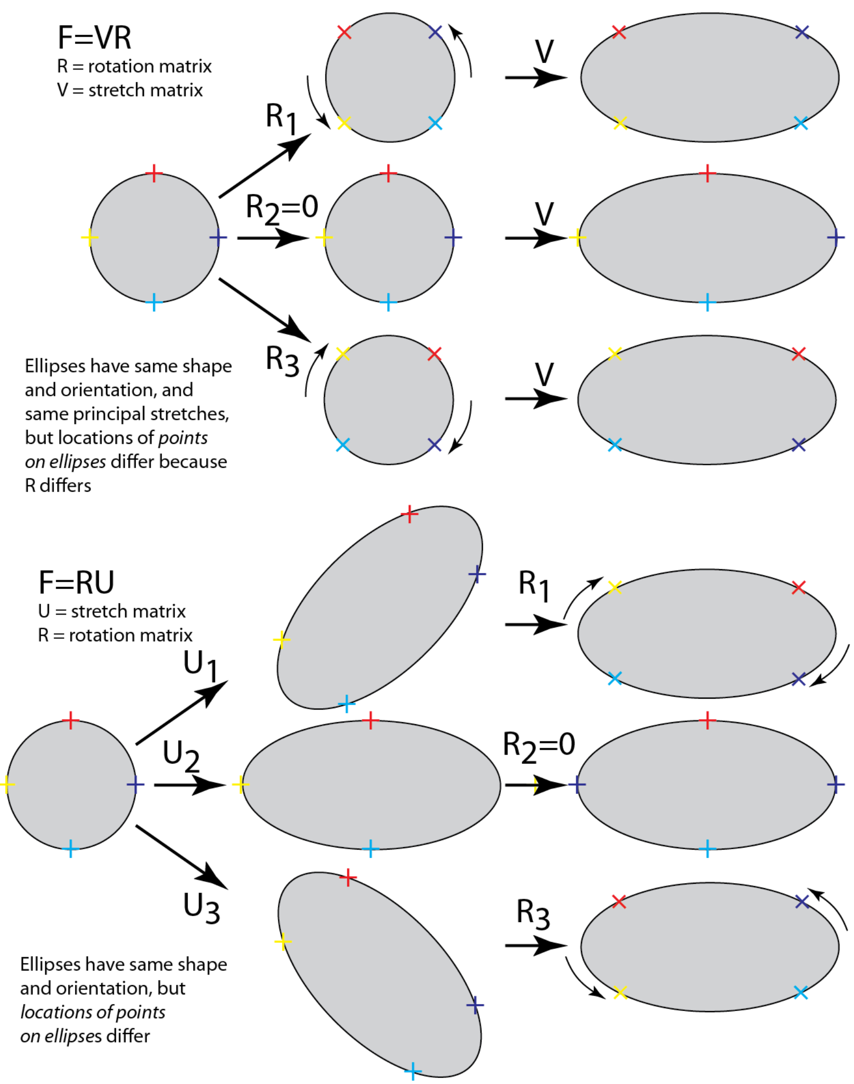
b R = rotation matrix

2 F=RU (which "U"? "R"?)

a R = rotation matrix

b U = symmetric stretch matrix

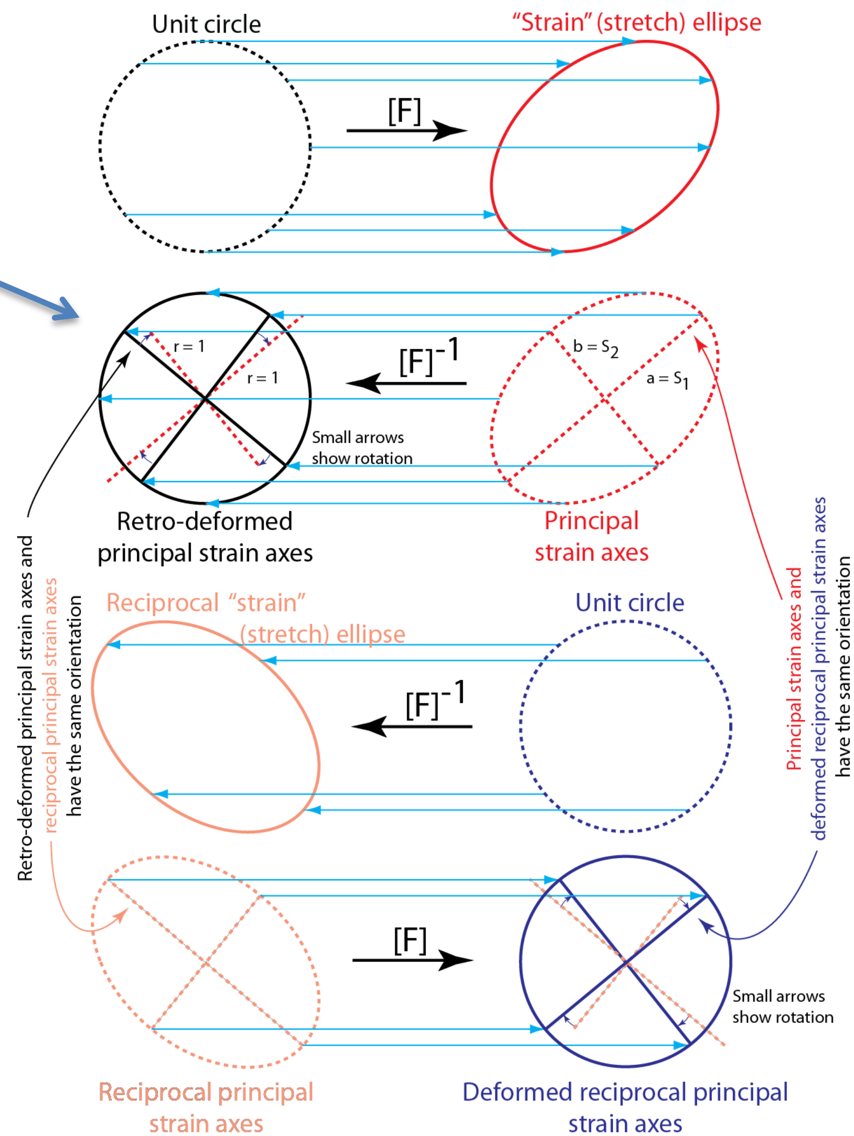
3 The choices become unique for symmetric stretch matrices



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Rotations in homogeneous deformation

- C If an ellipse is transformed to a unit circle, the axes of the ellipse are transformed too.
- D In general, the axes of the ellipses do not maintain their orientation when the ellipse is transformed back to a unit circle
- E If F is not symmetric, the axes of the red ellipse and the retro-deformed (black) axes will have a different *absolute* orientation
- F <u>The transformation from the the retro-deformed (black) axes to the the orientation of the principal axes gives the rotation of the axes.</u>

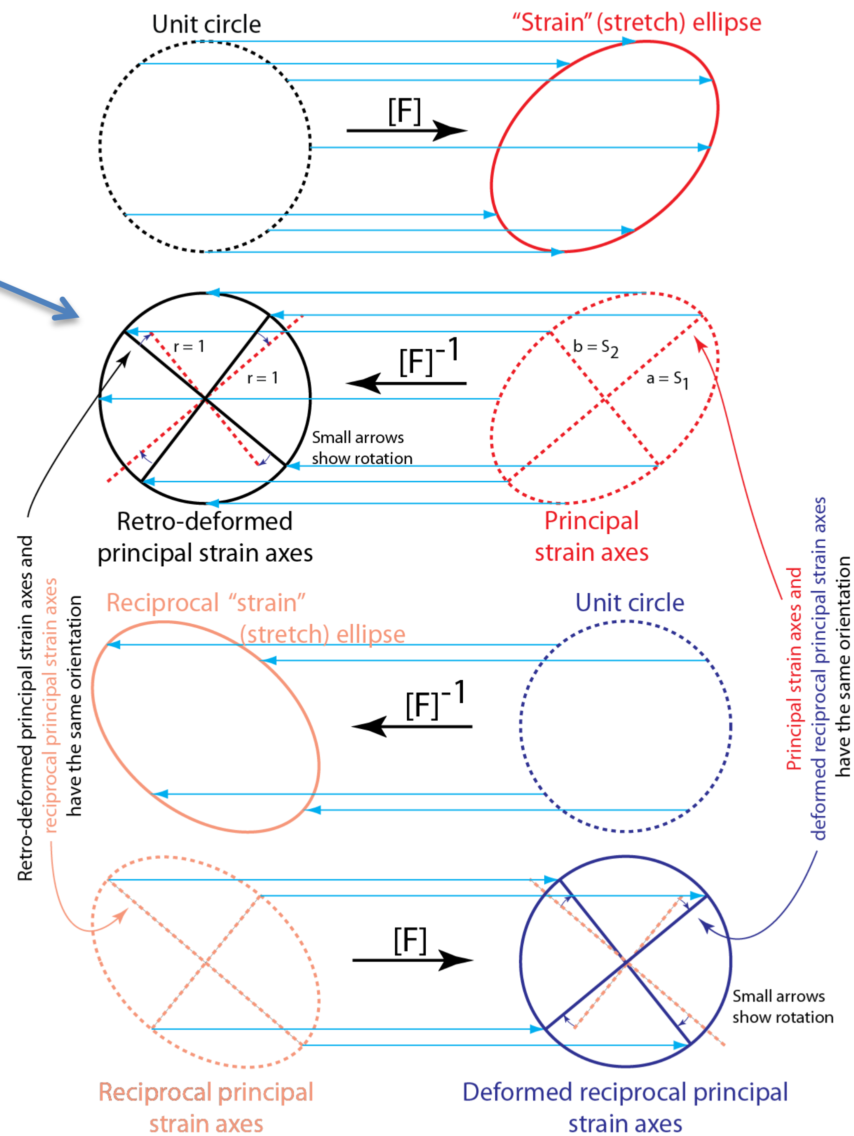


9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Rotations in homogeneous deformation

G We know how to find the principal stretch magnitudes: they are the square roots of the eigenvalues of the symmetric matrix [$[F^T][F]$]

H The eigenvectors of [$[F^T][F]$] give some of the information needed to find the direction of the principal stretch axes. The rotation describes the orientation difference between the (red) principal strain (stretch) axes and their (black) retro-deformed counterparts



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Rotations in homogeneous deformation

I To find the rotation of the principal axes, start with the parametric equation for an ellipse and its tangent, and the requirement that the position vectors for the semi-axes of the ellipse are perpendicular to the tangent

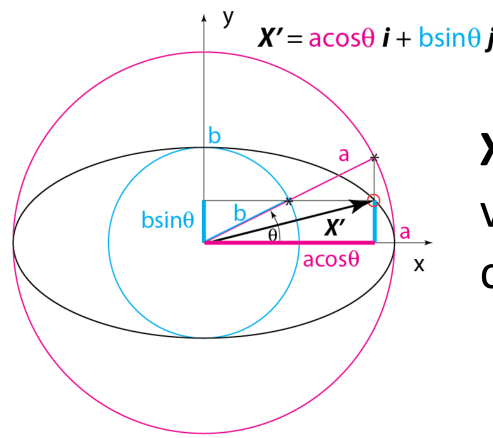
Let θ give the orientation of **X**, where **X** transforms to **X'**.

$$\vec{X}' = (a\cos\theta + b\sin\theta)\vec{i} + (c\cos\theta + d\sin\theta)\vec{j}$$

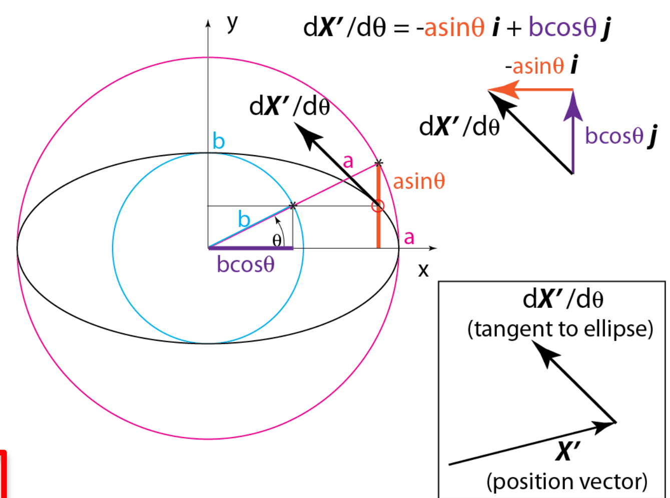
$$\frac{d\vec{X}'}{d\theta} = (-a\sin\theta + b\cos\theta)\vec{i} + (-c\sin\theta + d\cos\theta)\vec{j}$$

$$\vec{X}' \bullet \frac{d\vec{X}'}{d\theta} = 0$$

What value of θ will yield a vector X such that X' will be perpendicular to the tangent of the ellipse?



X is a position vector for a unit circle. [X'] = [F][X].



$X' \cdot dX'/d\theta = -a^2\sin\theta\cos\theta + b^2\sin\theta\cos\theta = (b^2-a^2)\sin\theta\cos\theta$

$X' \cdot dX'/d\theta = 0$ if a=b, $\theta = 0°$, or $\theta = 90°$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Rotations in homogeneous deformation

Now solve for θ satisfying

$X' \bullet dX'/d\theta = 0$

$$\vec{X}' = (a\cos\theta + b\sin\theta)\vec{i} + (c\cos\theta + d\sin\theta)\vec{j}$$

$$\frac{d\vec{X}'}{d\theta} = (-a\sin\theta + b\cos\theta)\vec{i} + (-c\sin\theta + d\cos\theta)\vec{j}$$

$$\vec{X}' \bullet \frac{d\vec{X}'}{d\theta} = 0$$

$$= -a^2\sin\theta\cos\theta + ab\cos^2\theta - ab\sin^2\theta + b^2\sin\theta\cos\theta$$

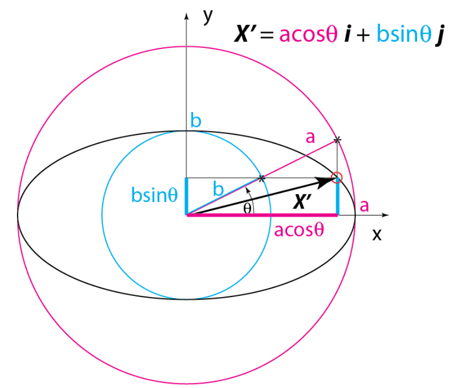
$$- c^2\sin\theta\cos\theta + cd\cos^2\theta - cd\sin^2\theta + d^2\sin\theta\cos\theta$$

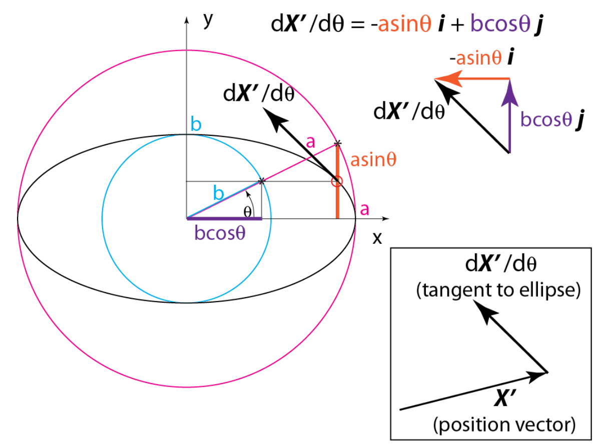
$$= -(a^2 - b^2 + c^2 - d^2)\sin\theta\cos\theta + (ab+cd)\cos^2\theta - (ab+cd)\sin^2\theta$$

$$= -(a^2 - b^2 + c^2 - d^2)\sin\theta\cos\theta + (ab+cd)(\cos^2\theta - \sin^2\theta)$$

$$= \frac{-(a^2 - b^2 + c^2 - d^2)}{2}\sin 2\theta + (ab+cd)\cos 2\theta$$

$$= \frac{(a^2 - b^2 + c^2 - d^2)}{2}\sin(-2\theta) + (ab+cd)\cos(-2\theta) = 0$$





$X' \bullet dX'/d\theta = -a^2\sin\theta\cos\theta + b^2\sin\theta\cos\theta = (b^2 - a^2)\sin\theta\cos\theta$

$X' \bullet dX'/d\theta = 0$ if a=b, θ = 0°, or θ = 90°

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

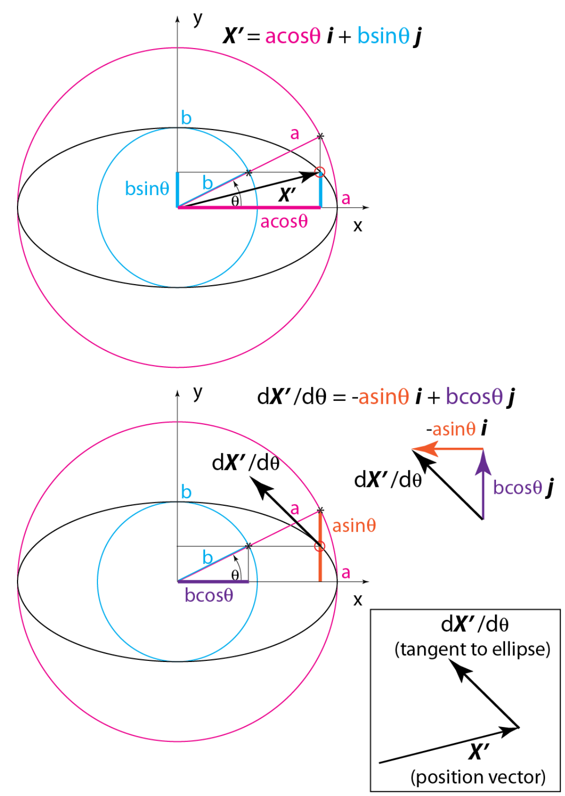
VI Rotations in homogeneous deformation (Cont.)

$$\frac{\left(a^2-b^2+c^2-d^2\right)}{2}\sin(-2\theta)+(ab+cd)\cos(-2\theta)=0$$

$$\tan(-2\theta)=\frac{-2(ab+cd)}{a^2-b^2+c^2-d^2}$$

$$\theta_1=\frac{1}{2}\tan^{-1}\left(\frac{2(ab+cd)}{a^2-b^2+c^2-d^2}\right),\ \theta_2=\frac{1}{2}\tan^{-1}\left(\frac{2(ab+cd)}{a^2-b^2+c^2-d^2}\right)\pm 90^\circ$$

So θ_1 and θ_2 are 90° apart. So $\mathbf{X}_1$ and $\mathbf{X}_2$ that transform to $\mathbf{X}_1'$ and $\mathbf{X}_2'$ are perpendicular.



$\mathbf{X}'\cdot d\mathbf{X}'/d\theta = -a^2\sin\theta\cos\theta + b^2\sin\theta\cos\theta = (b^2-a^2)\sin\theta\cos\theta$

$\mathbf{X}'\cdot d\mathbf{X}'/d\theta = 0$ if a=b, $\theta = 0^\circ$, or $\theta = 90^\circ$

Recall that two angles that differ by 180° have the same tangent