Eigenvectors, Eigenvalues, and Finite Strain

GG303, 2013

"Lab 9"

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

I Main Topics

- A Elementary linear algebra relations
- B Equations for an ellipse
- C Equation of homogeneous deformation
- D Eigenvalue/eigenvector equation
- E Solutions for symmetric homogeneous deformation matrices
- F Solutions for general homogeneous deformation matrices
- G Rotations in homogeneous deformation

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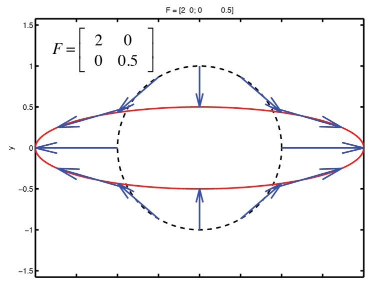
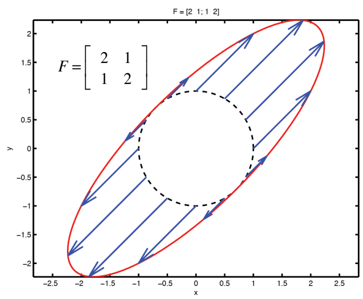
<u>Examples of 2D homogeneous deformation</u>

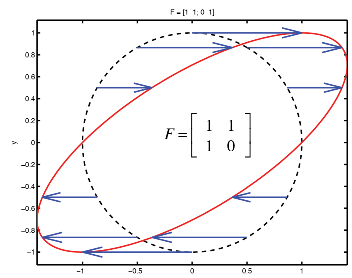
Note that the symmetry of the displacement fields (or lack thereof) in the examples corresponds to the symmetry (or lack thereof) in the deformation gradient matrix [F].

What is a simple way to describe homogeneous deformation that is geometrically meaningful?

What is the geologic relevance?







9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

II Elementary linear algebra relations

A Inverse $[A]^{-1}$ of a real matrix A

1 $[A][A]^{-1} = [A]^{-1}[A] = [I]$,
where [I] = identity matrix (e.g., $[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

2 [A] and $[A]^{-1}$ must be square nxn matrices

3 Inverse $[A]^{-1}$ of a 2x2 matrix

$$[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad [A]^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4 Inverse $[A]^{-1}$ of a 3x3 matrix also requires determinant |A| to be non-zero

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II Elementary linear algebra relations

B Determinant |A| of a real matrix A

1 A number that provides scaling information on a square matrix

2 Determinant of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ |A| = ad - bc$$

Akin to:
Cross product (an area)
Scalar triple product (a volume)

3 Determinant of a 3x3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \ |A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

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II Elementary linear algebra relations

C Transpose $For\ [A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, [A]^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

D Transpose of a matrix product

$$If\ [A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} and\ [B] = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, then\ [A]^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} and\ [B]^T = \begin{bmatrix} e & g \\ f & h \end{bmatrix}$$

$$[A][B] = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}, [[A][B]]^T = \begin{bmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{bmatrix}$$

$$[B]^T[A]^T = \begin{bmatrix} ea+gb & ec+gd \\ fa+hb & fc+hd \end{bmatrix} = [[A][B]]^T$$

This is true for any real nxn matrices

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II Elementary linear algebra relations

E Representation of a dot product using matrix multiplication and the matrix transpose

$$\vec{\mathbf{a}} \bullet \vec{\mathbf{b}} = \langle a_x, a_y, a_z \rangle \bullet \langle b_x, b_y, b_z \rangle = a_x b_x + a_y b_y + a_z b_z$$

$$= \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [a]^T [b]$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

III Equations for an ellipse

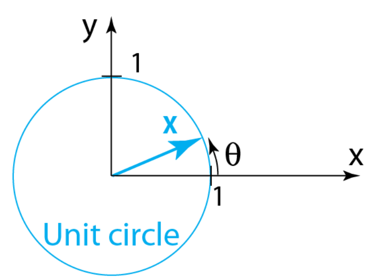
A Equation of a unit circle

1 $x^2 + y^2 = \vec{\mathbf{X}} \bullet \vec{\mathbf{X}} = 1$

2 $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [X]^T [X] = 1$

3 $x = \cos\theta$

$y = \sin\theta$



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III Equations for an ellipse

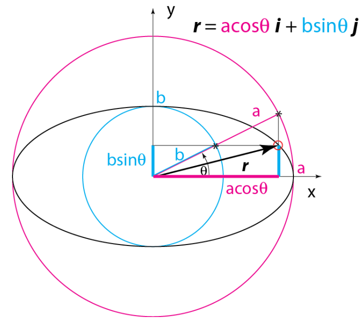
B Ellipse centered at (0,0), aligned along x,y axes

1 Standard form

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

2 General form

$$Ax^2 + Dy^2 + F = 0$$



3 Matrix form

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} Ax \\ Dy \end{bmatrix} = -F$$

A, D, and F are constants here, not matrices

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} A/-F & 0 \\ 0 & D/-F \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1$$

$$[X]^T[\text{Matrix of constants}][X] = 1$$

9/15/15 GG303 9

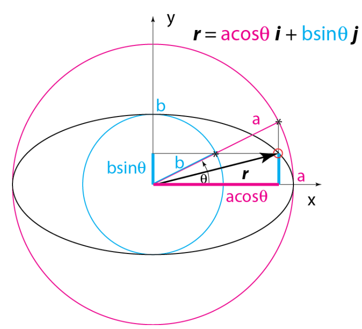
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

III Equations for an ellipse

B Ellipse centered at (0,0), aligned along x,y axes

4 Parametric form

$$x = a\cos\theta$$
$$y = b\sin\theta$$



5 Vector form

$$\vec{\mathbf{r}} = a\cos\theta\vec{\mathbf{i}} + b\sin\theta\vec{\mathbf{j}}$$

9/15/15 GG303 10

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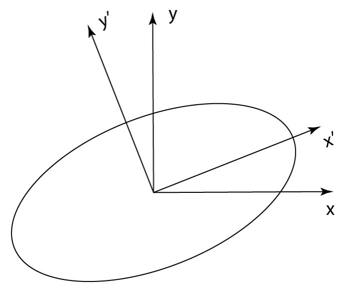
III Equations for an ellipse

C Ellipse centered at (0,0), arbitrary orientation



1 General form

$$Ax^2 + (B+C)xy + Dy^2 + F = 0$$

provided 4AD > $(B+C)^2$

2 Matrix form

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} Ax+By \\ Cx+Dy \end{bmatrix} = -F$$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} A/-F & B/-F \\ C/-F & D/-F \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1$$

$$[X]^T[\text{Matrix of constants}][X] = 1$$

A, B, C, D, and F are constants here, not matrices

9/15/15 GG303 11

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

III Equations for an ellipse

D Position vector for an ellipse

$$\vec{\mathbf{r}} = a\cos\theta\vec{\mathbf{i}} + b\sin\theta\vec{\mathbf{j}}$$

E Derivative of position vector for an ellipse (d**r**/dθ)

$$\frac{d\vec{\mathbf{r}}}{d\theta} = -a\sin\theta\vec{\mathbf{i}} + b\cos\theta\vec{\mathbf{j}}$$

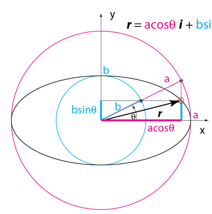
F Dot product of **r** and d**r**/dθ

$$\vec{\mathbf{r}} \bullet \frac{d\vec{\mathbf{r}}}{d\theta} = (b^2 - a^2)\sin\theta\cos\theta$$

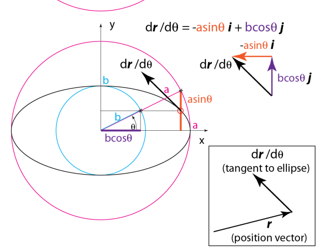
G The position vector and its tangent are perpendicular if and only if

1 a=b, or
2 θ = 0°, or ← Along axes
3 θ = 90° ← of ellipse

We will use these results shortly



So the axes of an ellipse/ellipsoid are perpendicular, and the tangents to an ellipse/ellipsoid at the ends of the axes are perpendicular. Those tangents parallel the axes.



$\mathbf{r} \cdot d\mathbf{r}/d\theta = -a^2\sin\theta\cos\theta + b^2\sin\theta\cos\theta = (b^2-a^2)\sin\theta\cos\theta$
$\mathbf{r} \cdot d\mathbf{r}/d\theta = 0$ if a=b, θ = 0°, or θ = 90°

9/15/15 GG303 12

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Equation of homogeneous deformation

A [X’] = [F][X]

B 2D

$$\begin{bmatrix} dx' \\ dy' \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} F_{x'x} & F_{x'y} \\ F_{y'x} & F_{y'y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

C 3D

$$\begin{bmatrix} dx' \\ dy' \\ dz' \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \Rightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} F_{x'x} & F_{x'y} & F_{x'z} \\ F_{y'x} & F_{y'y} & F_{y'z} \\ F_{z'x} & F_{z'y} & F_{z'z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

For homogeneous strain, the derivatives are uniform (constants) , and dx, dy can be small or large

9/15/15 GG303 13

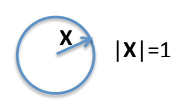
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Equation of homogeneous deformation [X’] = [F][X]

D Critical matter: Understanding the geometry of the deformation

E In homogeneous deformation, a unit circle transforms to an ellipse (and a sphere to an ellipsoid)

F Proof



$$[X]^T[X]=1$$

$$[X']=[F][X]$$

Now solve for [X]

$$[F]^{-1}[X']=[F]^{-1}[F][X]=[I][X]=[X]$$

$$[X]=[F]^{-1}[X']$$

Now solve for $[X]^T$

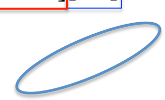
$$[X]^T=\left[[F]^{-1}[X']\right]^T=[X']^T\left[[F]^{-1}\right]^T$$

Now substitute for $[X]^T$ and [X] in first equation

$$[X]^T[X]=[X']^T\left[[F]^{-1}\right]^T[F]^{-1}[X']=1$$

$$[X']^T[\text{Symmetric matrix}][X']=1$$

Equation of ellipse
See slide 11



9/15/15 GG303 14

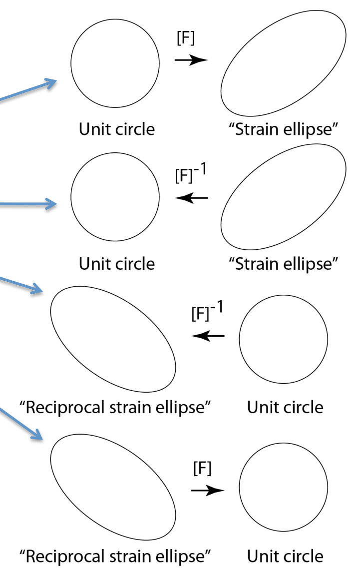
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Equation of homogeneous deformation [X'] = [F][X]

Geometric meanings of [F], $[F]^{-1}$

- G [F] transforms a unit circle to a "strain ellipse"
- H "Strain ellipse" geometrically represents [F][X]
- I $[F]^{-1}$ transforms a "strain ellipse" back to a unit circle
- J $[F]^{-1}$ transforms a unit circle to a "reciprocal strain ellipse"
- K [F] transforms a "reciprocal strain ellipse" back to a unit circle
- L "Reciprocal strain" ellipse geometrically represents $[F]^{-1}[X]$

$$[F]=\begin{bmatrix} a & b \\ c & d \end{bmatrix};\ [F]^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$



9/15/15 GG303 15

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues

A The eigenvalue matrix equation [A][X] = λ[X]

1. [A] is a (known) square matrix (nxn)
2. [X] is a non-zero directional <u>eigenvector</u> (nx1)
3. λ is a number, an <u>eigenvalue</u>
4. λ[X] is a vector (nx1) parallel to [X]
5. [A][X] is a vector (nx1) parallel to [X]

9/15/15 GG303 16

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

A The eigenvalue matrix equation [A][X] = λ[X] (cont.)

6 The vectors [[A][X]], λ[X], and [X] share the same direction if [X] is an eigenvector

7 If [X] is a unit vector, λ is the length of [A][X]

8 Eigenvectors [X_i] have corresponding eigenvalues [λ_i], and vice-versa

9 In Matlab, [vec,val] = eig(A), finds eigenvectors (vec) and eigenvalues (val)

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

B Examples

1 Identity matrix [I]

[A] [X] = λ [X]

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix}$$

All vectors in the x,y-plane maintain their orientation when operated on by the identity matrix, so all vectors are eigenvectors of [I], and all vectors maintain their length, so all eigenvalues of [I] equal 1. The eigenvectors are not uniquely determined but could be chosen to be perpendicular.

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

B Examples (cont.)

2 A matrix for rotations in the x,y plane

[A] [X] = λ [X]

$$\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

All non-zero real vectors rotate; a 2D rotation matrix has no real eigenvectors and hence no real eigenvalues

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

B Examples (cont.)

3 A 3D rotation matrix

a The only unit vector that is not rotated is along the axis of rotation

b The real eigenvector of a 3D rotation matrix gives the orientation of the axis of rotation

c A rotation does not change the length of vectors, so the real eigenvalue equals 1

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

B Examples (cont.)

Eigenvalues

Eigenvectors

4 $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

$$A\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} = 2\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$A\begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} = -2\begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

B Examples (cont.)

Eigenvalues

Eigenvectors

5 $A = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$

$$A\begin{bmatrix} -3\sqrt{0.1} \\ -\sqrt{0.1} \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -3\sqrt{0.1} \\ -\sqrt{0.1} \end{bmatrix} = \begin{bmatrix} -30\sqrt{0.1} \\ -10\sqrt{0.1} \end{bmatrix} = 10\begin{bmatrix} -3\sqrt{0.1} \\ -\sqrt{0.1} \end{bmatrix}$$

$$A\begin{bmatrix} \sqrt{0.1} \\ -3\sqrt{0.1} \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{0.1} \\ -3\sqrt{0.1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} \sqrt{0.1} \\ -3\sqrt{0.1} \end{bmatrix}$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

E Geometric meanings of the real matrix equation [A][X] = [B] = 0

1 |A| ≠ 0 ;

a $[A]^{-1}$ exists

b Describes two lines (or 3 planes) that intersect at the origin

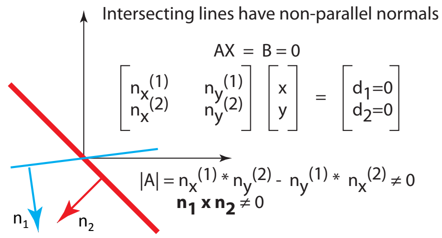
c X has a unique solution

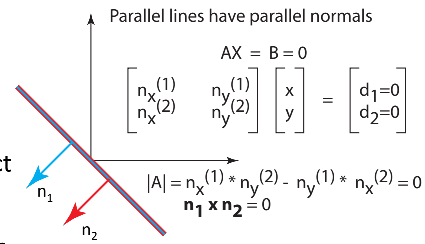
2 |A| = 0 ;

a $[A]^{-1}$ does not exist

b Describes two co-linear lines that that pass through the origin (or three planes that intersect in a line or a plane through the origin)

c [X] has no unique solution





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V Eigenvectors and eigenvalues (cont.)

F Alternative form of an eigenvalue equation

1 [A][X]=λ[X]

Subtracting λ[IX] = λ[X] from both sides yields:

2 [A-Iλ][X]=0 (same form as [$\mathscr{A}$][X]=0)

G Solution conditions and connections with determinants

1 Unique trivial solution of [X] = 0 if and only if |A-Iλ|≠0

2 Eigenvector solutions ([X] ≠ 0) if and only if |A-Iλ|=0

* See previous slide

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

- H Characteristic equation: |A-Iλ|=0
  - 1 The roots of the characteristic equation are the eigenvalues

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

- H Characteristic equation: |A-Iλ|=0 (cont.)
  - 2 Eigenvalues of a general 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

a $|A - I\lambda| = \begin{vmatrix} a - I & b \\ c & d - \lambda \end{vmatrix} = 0$

b $(a-\lambda)(d-\lambda) - bc = 0$

c $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$

d $\lambda_1, \lambda_2 = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

(a+d) = tr(A)
(ad-bc) = |A|

$\lambda_1 + \lambda_2 = tr(A)$
$\lambda_1 \lambda_2 = |A|$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

I To solve for eigenvectors, substitute eigenvalues back into AX= lX and solve for X

J See notes of lecture 19 for details of analytic solution for eigenvectors of 2D matrices

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

K Matlab solution: [vec,val] = eig(M)

1. M = matrix to solve for
2. vec = matrix of unit eigenvectors (in columns)
3. val = matrix of eigenvalues (in columns)

L Example:>> [vec,val]=eig([2 2;2 2])

```
vec =
   -0.7071   0.7071
    0.7071   0.7071
val =
    0   0
    0   4
```

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Solutions for symmetric matrices

A Eigenvalues of a ***symmetric*** 2x2 matrix

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

1 $$\lambda_1, \lambda_2 = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - b^2)}}{2}$$

2 $$\lambda_1, \lambda_2 = \frac{(a+d) \pm \sqrt{(a+2ad+d)^2 - 4ad + 4b^2}}{2}$$

3 $$\lambda_1, \lambda_2 = \frac{(a+d) \pm \sqrt{(a-2ad+d)^2 + 4b^2}}{2}$$

4 $$\lambda_1, \lambda_2 = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4b^2}}{2}$$

Replace "c" by "b" in eqns. Of slide 26

Radical term cannot be negative; it is the sum of two squares. Eigenvalues are real.

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Solutions for symmetric matrices (cont.)

B Any distinct eigenvectors ($\mathbf{X_1}$, $\mathbf{X_2}$) of a ***symmetric*** nxn matrix are perpendicular ($\mathbf{X_1} \bullet \mathbf{X_2} = 0$)

1a $AX_1 = \lambda_1 X_1$ 1b $AX_2 = \lambda_2 X_2$

AX_1 parallels X_1, AX_2 parallels X_2 (property of eigenvectors)

Dotting AX_1 by X_2 and AX_2 by X_1 can test whether X_1 and X_2 are orthogonal.

2a $\mathbf{X_2} \bullet AX_1 = \mathbf{X_2} \bullet \lambda_1 \mathbf{X_1} = \lambda_1 (\mathbf{X_2} \bullet \mathbf{X_1})$

2b $\mathbf{X_1} \bullet AX_2 = \mathbf{X_1} \bullet \lambda_2 \mathbf{X_2} = \lambda_2 (\mathbf{X_1} \bullet \mathbf{X_2})$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

B' Distinct eigenvectors ($\mathbf{X_1}, \mathbf{X_2}$) of a symmetric 2x2 matrix are perpendicular ($\mathbf{X_1} \bullet \mathbf{X_2} = 0$) (cont.)

The material below shows $\mathbf{X}_1 \bullet AX_2 = \mathbf{X}_2 \bullet AX_1$ for the 2D case:

3a

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \bullet \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \bullet \begin{bmatrix} ax_2 + by_2 \\ bx_2 + dy_2 \end{bmatrix} = \begin{matrix} ax_1x_2 + bx_1y_2 \\ +by_1x_2 + dy_1y_2 \end{matrix}$$

3b

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \bullet \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \bullet \begin{bmatrix} ax_1 + by_1 \\ bx_1 + dy_1 \end{bmatrix} = \begin{matrix} ax_1x_2 + by_1x_2 \\ +bx_1y_2 + dy_1y_2 \end{matrix}$$

The sums on the right sides are scalars, but the ordering of the terms in the sums look like the elements of transposed matrices

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

B" Distinct eigenvectors ($\mathbf{X_1}, \mathbf{X_2}$) of a symmetric 3x3 matrix are perpendicular ($\mathbf{X_1} \bullet \mathbf{X_2} = 0$) (cont.)

The material below shows $\mathbf{X}_1 \bullet AX_2 = \mathbf{X}_2 \bullet AX_1$ for the 3D case:

3c

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \bullet \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \bullet \begin{bmatrix} ax_2 + by_2 + cz_2 \\ bx_2 + dy_2 + ez_2 \\ cx_2 + ey_2 + fz_2 \end{bmatrix} = \begin{matrix} ax_1x_2 + bx_1y_2 + cx_1z_2 \\ +by_1x_2 + dy_1y_2 + ey_1z_2 \\ +cz_1x_2 + ez_1y_2 + fz_1z_2 \end{matrix}$$

3d

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \bullet \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \bullet \begin{bmatrix} ax_1 + by_1 + cz_1 \\ bx_1 + dy_1 + ez_1 \\ cx_1 + ey_1 + fz_1 \end{bmatrix} = \begin{matrix} ax_1x_2 + by_1x_2 + cz_1x_2 \\ +bx_1y_2 + dy_1y_2 + ez_1y_2 \\ +cx_1z_2 + ey_1z_2 + fz_1z_2 \end{matrix}$$

Again, the sums on the right sides are scalars, but the ordering of the terms in the sums look like the elements of transposed matrices

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

B'" Distinct eigenvectors ($\mathbf{X}_1$, $\mathbf{X}_2$) of a symmetric nxn matrix are perpendicular ($\mathbf{X}_1 \bullet \mathbf{X}_2 = 0$) (cont.)

The 2D and 3D results suggest matrix transposes could test whether $\mathbf{X}_1 \bullet A X_2 = \mathbf{X}_2 \bullet A X_1$ in general

$$\mathbf{X}_1 \bullet \mathbf{A}\mathbf{X}_2 = [X_1]^T[A][X_2]$$

Are these equal?

$$\mathbf{X}_2 \bullet \mathbf{A}\mathbf{X}_1 = [X_2]^T[A][X_1] = \left[[X_2]^T[A][X_1]\right]^T$$

The transpose of a scalar is the same scalar

$$= \left[[A][X_1]\right]^T \left[[X_2]^T\right]^T$$

This step and the next invoke $[BC]^T = [C]^T[B]^T$

$$= [X_1]^T[A]^T\left[[X_2]^T\right]^T$$

$$= [X_1]^T[A]^T\left[[X_2]\right]$$

$$\left[[X_2]^T\right]^T = [X_2]$$

$$= [X_1]^T[A]\left[[X_2]\right]$$

If [A] is symmetric, $[A]^T = [A]$

Yes!

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

B Distinct eigenvectors ($\mathbf{X}_1$, $\mathbf{X}_2$) of a symmetric nxn matrix are perpendicular (cont.)

Since the left sides of (2a) and (2b) are equal, the right sides must be equal too. Hence,

4 $\lambda_1 (\mathbf{X}_2 \bullet \mathbf{X}_1) = \lambda_2 (\mathbf{X}_1 \bullet \mathbf{X}_2)$

Now subtract the right side of (4) from the left

5 $(\lambda_1 - \lambda_2)(\mathbf{X}_2 \bullet \mathbf{X}_1) = 0$

- The eigenvalues generally are different, so $\lambda_1 - \lambda_2 \neq 0$.
- This means for (5) to hold that $\mathbf{X}_2 \bullet \mathbf{X}_1 = 0$.
- The eigenvectors ($\mathbf{X}_1$, $\mathbf{X}_2$) of a symmetric nxn matrix are perpendicular (or can be chosen to be perpendicular)
- We can pick reference frames with orthogonal axes to simplify problems and gain insight into their solutions

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Solutions for symmetric matrices (cont.)

C Maximum and minimum squared lengths

Set derivative of squared lengths to zero

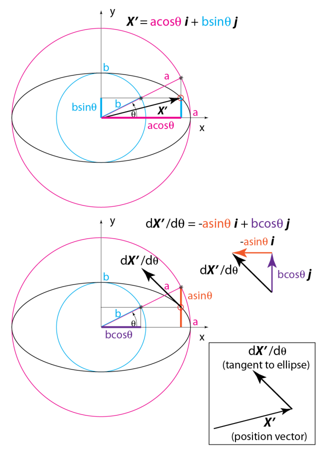
$$\vec{\mathbf{X}}' \bullet \vec{\mathbf{X}}' = (A\mathbf{X}) \bullet (A\mathbf{X}) = L_f^{\,2}$$

$$\frac{d\left(\vec{\mathbf{X}}' \bullet \vec{\mathbf{X}}'\right)}{d\theta} = \vec{\mathbf{X}}' \bullet \frac{d\vec{\mathbf{X}}'}{d\theta} + \frac{d\vec{\mathbf{X}}'}{d\theta} \bullet \vec{\mathbf{X}}' = 0$$

$$2\left(\vec{\mathbf{X}}' \bullet \frac{d\vec{\mathbf{X}}'}{d\theta}\right) = 0$$

$$\left(\vec{\mathbf{X}}' \bullet \frac{d\vec{\mathbf{X}}'}{d\theta}\right) = 0$$

D Position vectors (**X’**) with maximum and minimum (squared) lengths are those that are perpendicular to tangent vectors (d**X’**) along ellipse



$\mathbf{X}' \cdot d\mathbf{X}'/d\theta = -a^2\sin\theta\cos\theta + b^2\sin\theta\cos\theta = (b^2 - a^2)\sin\theta\cos\theta$

$\mathbf{X}' \cdot d\mathbf{X}'/d\theta = 0$ if $a=b$, $\theta = 0^\circ$, or $\theta = 90^\circ$

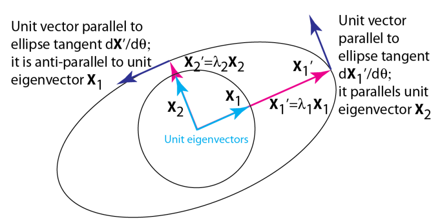
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Solutions for symmetric matrices (cont.)

E A**X**=λ**X**

F Since eigenvectors of symmetric matrices are mutually perpendicular, so too are the parallel transformed vectors λ**X**

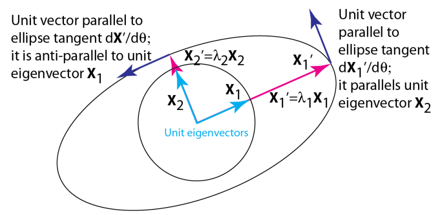
G At the point identified by the transformed vector λ**X**, the other eigenvector(s) is (are) perpendicular and hence must parallel d**X**’ and be tangent to the ellipse



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Solutions for symmetric matrices (cont.)

- H Recall that position vectors (**X'**) with maximum and minimum (squared) lengths are those that are perpendicular to tangent vectors (d**X'**) along ellipse. Hence, the smallest and largest transformed vectors λ**X** for a symmetric matrix give the minimum and maximum distances to an ellipse from its center and the directions of the ellipse axes.
- I The λ values are the principal stretches associated with a symmetric [F] matrix
- J These conclusions extend to three dimensions and ellipsoids



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VII Solutions for general homogeneous deformation matrices

- A Eigenvalues
  1. Start with the definition of *quadratic* elongation Q, which is a scalar
  2. Express using dot products
  3. Clear the denominator. Dot products and Q are scalars.

$$\frac{L_f^2}{L_0^2} = Q$$

$$\frac{\vec{X}' \bullet \vec{X}'}{\vec{X} \bullet \vec{X}} = Q$$

$$\vec{X}' \bullet \vec{X}' = \left(\vec{X} \bullet \vec{X}\right) Q$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VII Solutions for general homogeneous deformation matrices

A Eigenvalues

4 Replace X’ with [FX]

5 Re-arrange both sides

6 Both sides of this equation lead off with $[X]^T$, which cannot be a zero vector, so it can be dropped from both sides to yield an eigenvector equation

7 $[F^TF]$ is symmetric: $[F^TF]^T=[F^TF]$

8 The eigenvalues of $[F^TF]$ are the principal quadratic elongations $Q = (L_f/L_0)^2$

9 The eigenvalues of $[F^TF]^{1/2}$ are the principal stretches $S = (L_f/L_0)$

$$\vec{X}'\bullet\vec{X}' = \left(\vec{X}\bullet\vec{X}\right)Q$$

$$\Big[\underset{nxn}{[F]}\underset{nx1}{[X]}\Big]^T\Big[\underset{nxn}{[F]}\underset{nx1}{[X]}\Big] = \Big[\underset{nx1}{X}\Big]^T\underset{nx1}{[X]}\underset{1x1}{Q}$$

$$\Big[\underset{nx1}{X}\Big]^T\Big[\underset{nxn}{F}\Big]^T\Big[\underset{nxn\,nx1}{F\,X}\Big] = \Big[\underset{nx1}{X}\Big]^T\underset{1x1}{Q}\Big[\underset{nx1}{X}\Big]$$

$$\Big[\underset{nxn}{F^T}\ \underset{nxn}{F}\Big]\Big[\underset{nx1}{X}\Big] = Q\Big[\underset{nx1}{X}\Big]$$

$$"[A][X]=\lambda[X]"$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VII Solutions for general homogeneous deformation matrices

B Special Case: [F] is symmetric

1 $[F^TF] = [F^2]$ because $F = F^T$

2 The principal stretches (S) again are the square roots of the principal quadratic elongations (Q) (i.e., the square roots of the eigenvalues of $[F^2]$)

3 The principal stretches (S) also are the eigenvalues of [F], directly

4 The directions of the principal stretches (S) are the eigenvectors of [F], and of $[F^TF] = [F^2]$!

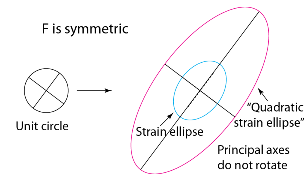
5 The axes of the principal (greatest and least) strain do not rotate

$$\left[F^TF\right][X]=Q[X]$$

$$\left[F^2\right][X]=Q[X]$$

$$Q=\frac{L_f^2}{L_0^2};\ S=\frac{L_f}{L_0}\Rightarrow\sqrt{Q}=S$$

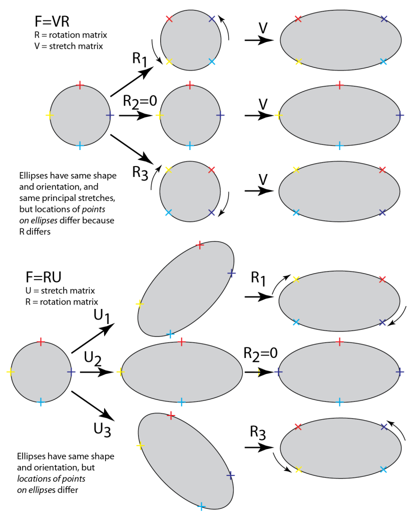
$$[F][X]=S[X]$$



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

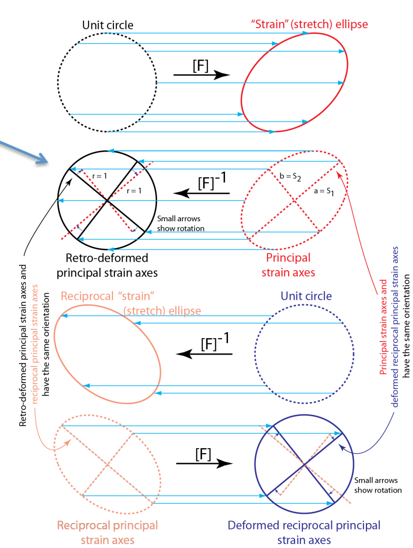
- A Just getting the size and shape of the "strain" (stretch) ellipse is not enough. Need to consider points on the ellipse
- B F=VR (which "R"?)
  - 1 R = rotation matrix
  - 2 V = stretch matrix
- C F=RU (which "U"? "R"?)
  - 1 U = stretch matrix
  - 2 R = rotation matrix
- D The choices narrow if the stretch matrices are symmetric



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

- E If an ellipse is transformed to a unit circle, the axes of the ellipse are transformed too.
- F In the diagram, the axes of the ellipses do not maintain their orientation when the ellipse is transformed back to a unit circle
- G If F is not symmetric, the axes of the red ellipse and the retro-deformed (black) axes will have a different *absolute* orientation
- H <u>The transformation from the the retro-deformed (black) axes to the the orientation of the principal axes gives the rotation of the axes</u>

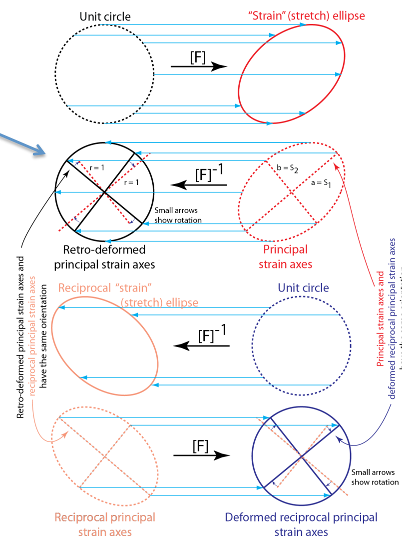


9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

I We know how to find the principal stretch magnitudes: they are the square roots of the eigenvalues of the symmetric matrix [$[F^T][F]$]

J The eigenvectors of [$[F^T][F]$] give the some of the information needed to find the direction of the principal stretch axes. The rotation describes the orientation difference between the principal strain (stretch) axes and their retro-deformed counterparts



9/15/15 GG303 43

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

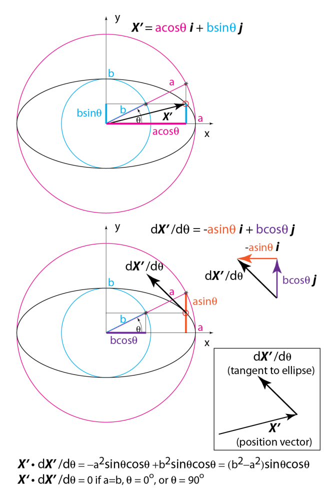
K To find the rotation of the principal axes, start with the parametric equation for an ellipse and its tangent, and the requirement that the position vectors for the semi-axes of the ellipse are perpendicular to the tangent

$$\vec{X}' = \left(a\cos\theta + b\sin\theta\right)\vec{i} + \left(c\cos\theta + d\sin\theta\right)\vec{j}$$

$$\frac{d\vec{X}'}{d\theta} = \left(-a\sin\theta + b\cos\theta\right)\vec{i} + \left(-c\sin\theta + d\cos\theta\right)\vec{j}$$

$$\vec{X}' \bullet \frac{d\vec{X}'}{d\theta} = 0$$

Recall the θ gives the orientation of a unit vector that is used to define a unit circle: x = cosθ; y = sinθ



9/15/15 GG303 44

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

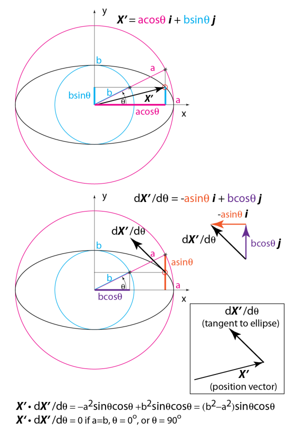
Now solve for θ

$$\vec{X}' = (a\cos\theta + b\sin\theta)\vec{i} + (c\cos\theta + d\sin\theta)\vec{j}$$

$$\frac{d\vec{X}'}{d\theta} = (-a\sin\theta + b\cos\theta)\vec{i} + (-c\sin\theta + d\cos\theta)\vec{j}$$

$$\vec{X}' \bullet \frac{d\vec{X}'}{d\theta} = 0$$

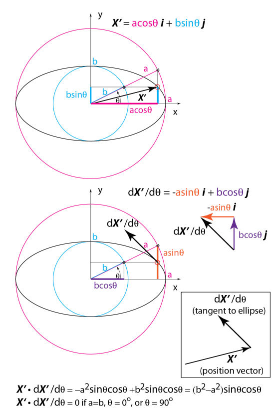
$$= -a^2\sin\theta\cos\theta + ab\cos^2\theta - ab\sin^2\theta + b^2\sin\theta\cos\theta$$
$$- c^2\sin\theta\cos\theta + cd\cos^2\theta - cd\sin^2\theta + d^2\sin\theta\cos\theta$$
$$= -\left(a^2 - b^2 + c^2 - d^2\right)\sin\theta\cos\theta + (ab+cd)\cos^2\theta - (ab+cd)\sin^2\theta$$
$$= -\left(a^2 - b^2 + c^2 - d^2\right)\sin\theta\cos\theta + (ab+cd)\left(\cos^2\theta - \sin^2\theta\right)$$
$$= \frac{-\left(a^2 - b^2 + c^2 - d^2\right)}{2}\sin 2\theta + (ab+cd)\cos 2\theta$$
$$= \frac{\left(a^2 - b^2 + c^2 - d^2\right)}{2}\sin(-2\theta) + (ab+cd)\cos(-2\theta) = 0$$



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

Continuing….



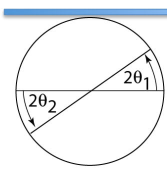
$$\frac{\left(a^2 - b^2 + c^2 - d^2\right)}{2}\sin(-2\theta) + (ab+cd)\cos(-2\theta) = 0$$

$$\tan(-2\theta) = \frac{-2(ab+cd)}{a^2 - b^2 - c^2 - d^2}$$

$$\theta_1 = \frac{1}{2}\tan^{-1}\left(\frac{2(ab+cd)}{a^2 - b^2 - c^2 - d^2}\right), \theta_2 = \frac{1}{2}\tan^{-1}\left(\frac{2(ab+cd)}{a^2 - b^2 - c^2 - d^2}\right) \pm 90^\circ$$

So θ_1 and θ_2 are 90° apart

Recall that two angles that differ by 180° have the same tangent



So the unit vectors that are transformed to give the perpendicular principal axes of the strain ellipse are themselves perpendicular.
The angle between the those perpendicular unit vectors and the corresponding vectors along the axes of the principal strains is the angle of rotation.

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

The longest and shortest values of X' are the perpendicular vectors along the axes of the ellipse, which have the following orientations:

$$\left[X_1'\right]=[F]\left[X(\theta_1)\right]$$

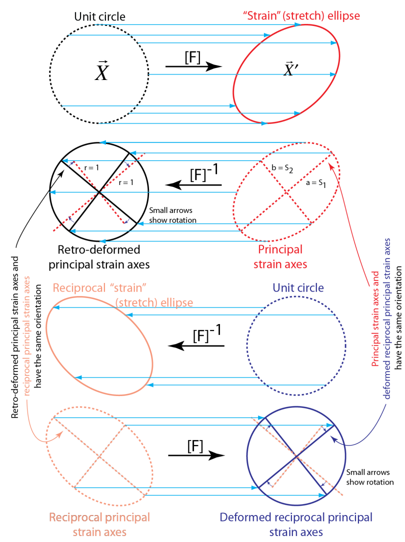
$$\left[X_2'\right]=[F]\left[X(\theta_2)\right]$$

The corresponding back-transformed vectors are:

$$\left[F^{-1}\right]\left[X_1'\right]=\left[F^{-1}\right][F]\left[X(\theta_1)\right]=\left[X(\theta_1)\right]$$

$$\left[F^{-1}\right]\left[X_2'\right]=\left[F^{-1}\right][F]\left[X(\theta_2)\right]=\left[X(\theta_2)\right]$$

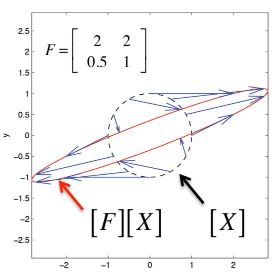
The back-transformed vectors (along the black axes) are just unit vectors in the directions of θ_1 and θ_2, respectively. This means the back-transformed vectors maintain the 90° angle between the principal directions. The angle of rotation is defined as the angle between the perpendicular pair {**X**(θ_1) and **X**(θ_2)} along the black axes of the unit circle and the perpendicular principal pair {**X'**(θ_1) and **X'**(θ_2)} along the red axes of the ellipse. These results carry over to three dimensions if all three sections along the principal axes of the "strain" (stretch) ellipse are considered.



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Example 1

[F] = [R][U]



$$[X']=[F][X];\ [F]=[R][U]$$

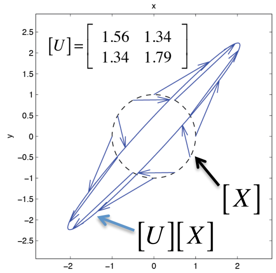
$$[F]=\begin{bmatrix}2 & 2\\ 0.5 & 1\end{bmatrix};\ [F]^T=\begin{bmatrix}2 & 0.5\\ 2 & 1\end{bmatrix}$$

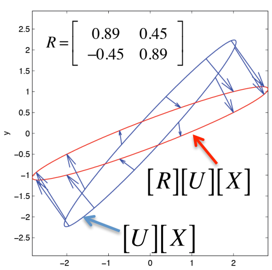
$$[U]=\left[[F]^T[F]\right]^{1/2}=\begin{bmatrix}4.25 & 4.5\\ 4.5 & 5\end{bmatrix}^{1/2}=\begin{bmatrix}1.56 & 1.34\\ 1.34 & 1.79\end{bmatrix}$$

$$[R]=[F][U]^{-1}=\begin{bmatrix}2 & 2\\ 0.5 & 1\end{bmatrix}\begin{bmatrix}1.79 & -1.34\\ -1.34 & 1.56\end{bmatrix}=\begin{bmatrix}0.89 & 0.45\\ -0.45 & 0.89\end{bmatrix}$$

Eigenvalues of [U] give principal stretch magnitudes

First, symmetrically stretch the unit circle using [U]





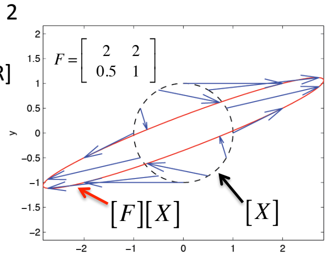
Eigenvectors of [U] are along axes of blue ellipses. <u>Rotated</u> eigenvectors of [U] give principal stretch directions

Second, rotate the ellipse (<u>*not*</u> the reference frame) using [R]

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Example 2

[F] = [V] [R]



$[X']=[F][X];\ [F]=[V][R]$

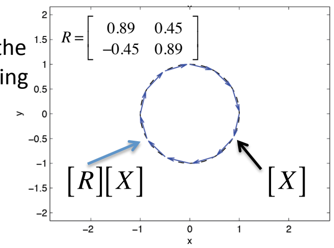
$[F]=\begin{bmatrix} 2 & 2 \\ 0.5 & 1 \end{bmatrix};\ [F]^T=\begin{bmatrix} 2 & 0.5 \\ 2 & 1 \end{bmatrix}$

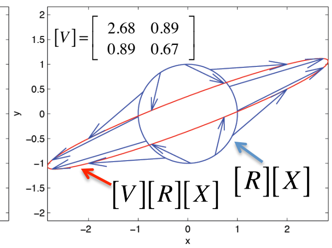
$[V]=\left[[F][F]^T\right]^{1/2}=\begin{bmatrix} 8 & 3 \\ 3 & 1.5 \end{bmatrix}^{1/2}=\begin{bmatrix} 2.68 & 0.89 \\ 0.89 & 0.67 \end{bmatrix}$

$[R]=[V]^{-1}[F]=\begin{bmatrix} 0.67 & -0.89 \\ -0.89 & 2.68 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0.5 & 1 \end{bmatrix}=\begin{bmatrix} 0.89 & 0.45 \\ -0.45 & 0.89 \end{bmatrix}$

Eigenvalues of [V] also give principal stretch magnitudes

First, rotate the unit circle using [R]





Unrotated eigenvectors of [V] give principal stretch directions directly

Second, stretch the rotated unit circle symmetrically using [V]

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

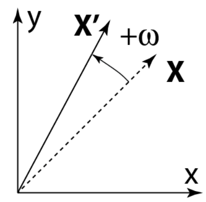
- Decomposition of F = VR by method of Ramsay and Huber (for 2D). Consider the effect of an irrotational (symmetric) strain [V] that follows a pure rotation [R] of an *object* (not a rigid rotation of the reference frame)

$$F=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}=VR$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

- Key fact about rotation matrices: $[R]^{-1} = [R]^T$

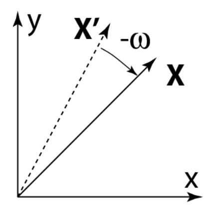


$$[X'] = [R][X]$$

$$R(\omega) = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}$$

$$R^{-1} = R(-\omega) = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}$$

$$R^T = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}$$



$$[X] = [R]^{-1}[X']$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

- Key fact about rotation matrices: $[R]^{-1} = [R]^T$
- 3D treatment: rotating a reference frame does not change the length of a vector, so **X•X=X'•X'**. This also leads to $[R]^{-1} = [R]^T$:

$$[X'] = [R][X]$$

$$\vec{X} \bullet \vec{X} = \vec{X}' \bullet \vec{X}'$$

$$\vec{X} \bullet \vec{X} = [X]^T [X] = [X]^T [I][X]$$

$$\vec{X}' \bullet \vec{X}' = [[R][X]]^T [[R][X]]$$

$$= [X]^T [R]^T [R][X]$$

$$[R]^T [R] = [I], but$$

$$[R]^{-1}[R] = [I]$$

$$\therefore [R]^T = [R]^{-1}$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

1 $F=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}=VR$

2 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} A\cos\omega+B\sin\omega & -A\sin\omega+B\cos\omega \\ B\cos\omega+D\sin\omega & -B\sin\omega+D\cos\omega \end{bmatrix}$

By inspection, c-b = (A+D)sinω, and a+d = (A+D)cosω

3 $\frac{c-b}{a+d}=\tan\omega$ If c=b, then F is symmetric and ω=0!

From 3 one can obtain ω and hence R. $[R]=\begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

Post-multiplying both sides of (1) by $[R]^{-1} = R^T$ yields V, the symmetric "part" of F.

$F = VR \rightarrow F[R]^{-1} = VR[R]^{-1} = VR[R]^T = V$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}^{-1}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}=\begin{bmatrix} A & B \\ B & D \end{bmatrix}=V$$

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IX Closing comments
1. Our solutions so far depend on knowing the displacement field.
2. With satellite imaging we can get an approximate value for the displacement field at the surface of the Earth for current deformations
3. Evaluating strains for past deformations require certain assumptions about initial sizes and shapes of bodies, the original locations of point, and/or the displacement field.
4. Alternative approach: formulation and solution of boundary value problems to solve for the displacement and strain fields.
5. The deformation gradient matrix F has strain and rotation intertwined; the two can be separated using matrix multiplication. In the infinitesimal strain matrix [ε], the rotation is already separated.
6. References

a. Ramsay, J.G., and Huber, M.I., 1983, The techniques of modern structural geology, volume 1: strain analysis: Academic Press, London, 307 p. (See equations of section 5, p. 291).
b. Ramsay, J.G., and Lisle, M.I., 1983, The techniques of modern structural geology, volume 3: applications of continuum mechanics in structural geology: Academic Press, London, 307 p. (See especially sessions 33 and 36).
c. Malvern, L.E., 1969, Introduction to the mechanics of a continuous medium: Prentice-Hall, Englewood Cliffs, New Jersey, 713 p. (See equations 4.6.1, 4.6.3 a, 4.6.3b on p. 172-174).)

Appendices

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

- FF^T and F^TF yield the same quadratic elongations [Q]; they have the same eigenvalues

$\frac{\vec{X}'\bullet\vec{X}'}{\vec{X}\bullet\vec{X}}=Q$ Start with definition of Q

* $\vec{X}'\bullet\vec{X}'=Q\vec{X}\bullet\vec{X}$ Denominator cleared

$[X]=[F^{-1}][X']$ Formula for recip. strain ellipse

$[X']^T[X']=Q[[F^{-1}][X']]^T[[F^{-1}][X']]$ In * replace [X] by $[F^{-1}X']$

$[X']^T[X']=Q[[X']^T[F^{-1}]^T][[F^{-1}][X']]$ With $[F^{-1}X']^T$ expanded

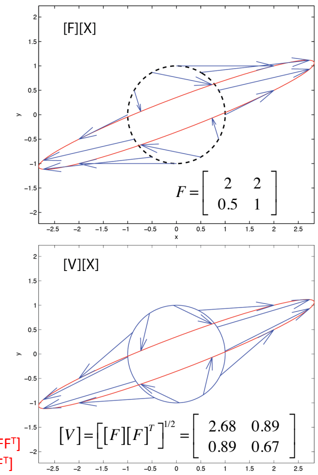
* $[X']=Q[[F^{-1}]^T][[F^{-1}][X']]$ After $[X']^T$ is dropped from front

$[X']=Q[[F^T]^{-1}][[F^{-1}][X']]$ After replacing $[F^{-1}]^T$ by $[F^T]^{-1}$

$[F^T][X']=[F^T]Q[[F^T]^{-1}][[F^{-1}][X']]=Q[[F^{-1}][X']]$

$[F][F^T][X']=[F]Q[[F^{-1}][X']]=Q[X']$ X' is an eigenvector of $[FF^T]$; Q is an eigenvalue of $[FF^T]$

Eigenvalue equation



9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

- FF^T and $[F^{-1}]^T[F^{-1}]$ have the same eigenvectors

* $[X']=Q[[F^{-1}]^T][[F^{-1}][X']]$ Start with * of previous page

$\frac{1}{Q}[X']=[[F^{-1}]^T][[F^{-1}][X']]$ Divide both sides by Q

$[[F^{-1}]^T][[F^{-1}][X']]=\frac{1}{Q}[X']$ After switching left and right sides

Eigenvalue equation

X' is an eigenvector of $[[F^{-1}]^T[F^{-1}]]$
1/Q is an eigenvalue of $[[F^{-1}]^T[F^{-1}]]$

So X' is an eigenvector of both $[FF^T]$ and $[F^{-1}]^T[F^{-1}]$ have the same eigenvectors [X'], although their eigenvalues are reciprocals. Now, eigenvector [X] ($[F]^T[F]$ [X] = Q[X]) is associated with the quadratic elongations (see red axes), and the last equation above has the same form, with $[F^{-1}]$ replacing [F] and 1/Q replacing Q. This means eigenvector [X'] is associated with the reciprocal quadratic elongations (see orange axes).